Exercise Sheet #2

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Problem 1 (*Graph properties*) 10 Pts

Calculate the following properties for each of the graphs in Fig. 1:

(a) the adjacency matrix (2 Pts)

(b) the spectrum of the graphs (2 Pts)

(c) the normalized graph Laplacian and (2 Pts)

(d) the spectrum of the normalized graph Laplacian. (2 Pts)

What is special about the graph in Fig. 1 (*ii*) and what changes if one connects the nodes 1 and 4? (2 Pts)

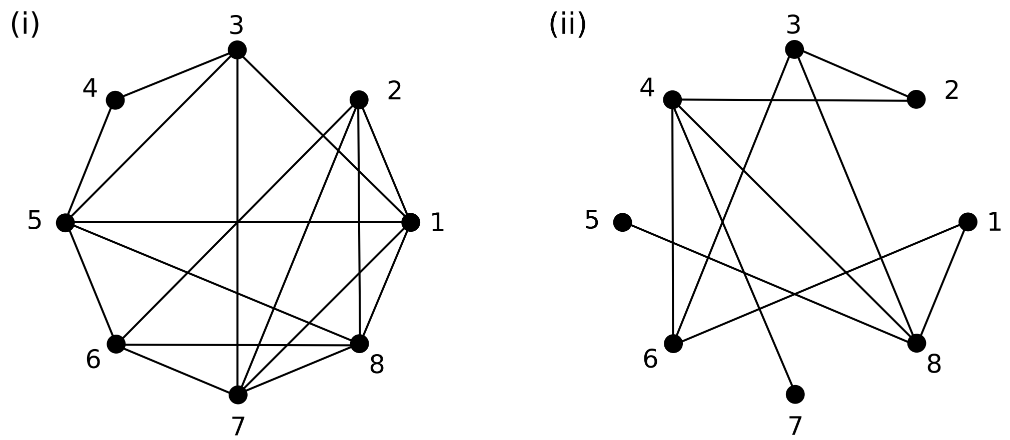

Figure 1: Random graphs with N=8 vertices and bidirectional edges.

Problem 2 (*Eigenvalue centrality*) 6 Pts

There are many possible measures for the relevance of a node in the network, i.e. the *centrality* of a node.

Let us define x_i as the "eigenvalue centrality" of node i as a value directly proportional to the sum of the centrality of its first neighbors, that is:

$$x_i \propto \sum_{j \in \mathcal{N}_i^{(1)}} x_j \,, \qquad (1)$$

where $\mathcal{N}_j^{(1)}$ is the group of first-neighbors of i. Using the adjacency matrix A, reduce this expression to an eigenvalue problem:

$$A\mathbf{x} = \lambda\mathbf{x}\,, \tag{2}$$

where $\mathbf{x}$ is a vector with components x_i and λ is the corresponding eigenvalue of A. Apply this measure to identify the three most "relevant" nodes in the network shown in Fig. 1 (i).

Hint: Only the eigenvector with the largest eigenvalue results in the desired centrality properties. Numerical solutions are accepted.

Problem 3 (*Green's function*) 4 Pts

The spectral density $\rho(\lambda)$ of a graph G with N nodes and adjacency matrix A is

$$\rho(\lambda) = \frac{1}{N}\sum_{j=0}^{N}\delta(\lambda-\lambda_j)\,, \tag{3}$$

where $\{\lambda_j\}_{0\le\lambda\le N}$ are the eigenvalues of A and δ denotes the Dirac delta distribution. Show that the spectral density ρ is related to the Green's function

$$G(\lambda) = \frac{1}{N}\,\mathrm{Tr}\left[(\lambda-A)^{-1}\right] = \frac{1}{N}\sum_{j=0}^{N}\frac{1}{\lambda-\lambda_j}\,, \tag{4}$$

where Tr denotes the trace of a matrix.

Problem 4 (*Approximation of continued fraction — optional*) 0 Pts

The Green's function of a random graph is given by a continued fraction

$$G(\lambda) = \frac{1}{\lambda-\frac{z}{\lambda-\frac{z-1}{\lambda-\frac{z-1}{\lambda-\dots}}}}\,. \tag{5}$$

Approximating the coordination number $z \approx z-1$ one can re-write Eq. (5) by

$$G(\lambda) = \frac{1}{\lambda - zG(\lambda)}\,, \tag{6}$$

which leads to the solution

$$G(\lambda) = \frac{\lambda}{2z} - \sqrt{\frac{\lambda^2}{4z^2}-\frac{1}{z}}\,. \tag{7}$$

Show that the continued fraction expression (5) approaches the solution (7) by performing a numerical simulation. How many steps do you need to take into account in order to make both expressions match? (0 Pts)