Exercise Sheet #8

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Problem 1 (*Biased Random Walk*) 10 Pts

Consider a one dimensional random walker subject to a bias, such that it steps forward with probability:

$$p_+ = \frac{1}{2} + (\lambda - \alpha x)\,\Delta x$$

and steps back with probability:

$$p_- = \frac{1}{2} - (\lambda - \alpha x)\,\Delta x$$

1. Find the equation governing the position distribution of the walker in the case of continuous time and space. To do so, take the limit $\Delta t, \Delta x \to \infty$, such that $\frac{(\Delta x)^2}{2\Delta t} \to 1$.

2. Why does it make sense for the bias $(\lambda - \alpha x)\Delta x$ to scale with Δx? (Think of what happens when an infinitesimal step has a large bias).

3. The result you should get has the form of a Fokker-Planck equation that corresponds to the stochastic differential equation $\dot{x} = \lambda - \alpha x + \xi(t)$, where $\xi(t)$ is white noise with a flat power spectrum. We would like to show that for $\lambda = 0$ the auto-correlation function $r_{xx}(\tau) = \langle x(t)x(t+\tau)\rangle$ of this process is given by $r_{xx} = \frac{\exp(-\alpha|\tau|)}{2\alpha}$. To do so, use the following relations:

   - The *Wiener-Khinchin theorem* relates $r_{xx}(\tau)$ to the inverse Fourier transform F^{-1} of the power spectrum $S(\omega) = |F[x(t)]|^2$ by

   $$r_{xx}(\tau) = F^{-1}[S(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} S(\omega)\ . \qquad (1)$$

   - Use the fact that $F[\dot{x}(t)] = i\omega F[x(t)]$ and the aforementioned property of $|F[\xi(t)]|^2 = 1$.
   - You should arrive at a Fourier integral that can be solved by using the Residue theorem and choosing an appropriate path in the complex plane. (Note that this requires a case distinction between positive or negative τ, but it is fine to just do the calculations for $\tau > 0$.)

Problem 2 (*Van der Pol Oscillator*) 10 Pts

Consider the Van der Pol oscillator, governed by the equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \epsilon(1-x^2)y - x \end{aligned}$$

Estimate the period of the limit cycle in the limit $\epsilon \gg 1$.

To do this, follow the analysis made in the class. Then, approximate the period by using

$$T = \int_{\text{beginning}}^{\text{end}} \mathrm{d}t = \int_{\text{beginning}}^{\text{end}} (\cdots)\, \mathrm{d}x \ ,$$

where the differential $\mathrm{d}t$ can be found by looking at the equations governing the oscillator. The positive slow branch begins at $x_a = 2$ and ends at $x_b = 1$.

Hint: Change dt into a dx and integrate over x. Prove $\dot{Y} \approx f'(x)\frac{\mathrm{d}x}{\mathrm{d}t}$, where $f(x)$ was defined in the class.