Problem 1 (Hausdorff Dimension) 8 Pts

Calculate the Hausdorff dimension of a straight line and of the Cantor set.

The Cantor ternary set is created by repeatedly deleting the open middle thirds of a set of line segments. First, one removes the open middle third \((1/3, 2/3)\) from the interval \([0, 1]\), leaving two line segments: \([0, 1/3]\) and \([2/3, 1]\). Then, the open middle third of each of these remaining segments is also removed, leaving four line segments: \([0, 1/9]\), \([2/9, 1/3]\), \([2/3, 7/9]\) and \([8/9, 1]\). Then the remaining four segments get their open middle third removed and the process is repeated infinitely.

Problem 2 (Lorenz system) 12 Pts

(a) Analyse the fixpoints of the Lorenz system given by

\[
\begin{align*}
\dot{x} &= s(y - x) \\
\dot{y} &= x(r - z) - y \\
\dot{z} &= xy - bz
\end{align*}
\]

and their stability as a function of the Rayleigh number \(r > 0\) with parameters \(s = 10\) and \(b = 8/3\) fixed. Make a sketch of the \(x(r)\) bifurcation diagram. (Hint: You may calculate the the eigenvalues numerically.)

(b) Show analytically that all fixed points are unstable for \(r > r_H = s(s + b + 3)/(s - b - 1)\). (Hint: You do not necessarily need a full solution for the eigenvalue spectra, since you are only looking for this particular transition point.)

(c) What kind of bifurcation does the system undergo at \(r = r_H\) and what kind of dynamics are present in the system for \(r > r_H\)?