Exercise Sheet #6

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Problem 1 (*Generalised Liénard System*) 10 Pts

Consider the following prototype system:

$$\ddot{x} = f\left(V(x)\right)\dot{x} - \nabla V(x)\ , \tag{1}$$

where $f(V)$ is a generalised friction term, which depends explicitly on the mechanical potential function $V(x)$.

(a) Analyse the stability of the fixpoints, considering $V(x) = x^3/3 - x^2/2$ and $f(V) = \mu - V$.

(b) Defining the total enery as $E = y^2/2 + V(x)$, show that the energy uptake/dissipation can be controlled by the $f(V)$ friction term.

(c) Show that the fixpoints of (1), corresponding to local extrema of any general potential function $V(x)$, undergo a Hopf bifurcation when dissipation changes to antidissipation in their neighborhood. *Hint:* the real parts of the eigenvalues should change their sign.

Problem 2 (*Triangle Map*) 10 Pts

Consider the Map

$$f(x) = \begin{cases} rx & \text{if } 0 \le x < 1/2 \\ r - rx & \text{if } 1/2 \le x \le 1 \end{cases}\ , \tag{2}$$

for $0 \le x \le 1$ and the parameter $0 \le r \le 2$.

(a) Plot the function.

(b) Look for fixed points and cycles (up to length 3).

(c) Derive the analytic expression for the maximal Lyapunov exponent, defined by

$$\lambda_{\max} = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\mathrm{d}f^{(n)}}{\mathrm{d}x} \right|\ , \text{ where } f^{(n)}(x) = f\left(f^{(n-1)}(x)\right)\ . \tag{3}$$

Hint: use the chain rule.

(d) For which range of r does the triangle map exhibit chaos?

Problem 3 (*Sawtooth Map (Optional!)*) 0 Pts

A slight variation to the triangle map is given by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x \leq 1 \end{cases}, \tag{4}$$

for $0 \leq x \leq 1$.

(a) Derive the analytic expression for the maximal Lyapunov exponent.

(b) You should find a positive Lyapunov exponent, indicating that the map is chaotic. Implement the map in a programming language of your choice and run it for at least 50-100 iterations. What effect do you observe when you plot the evolution of x? Can you give an explanation?