Exercise Sheet #6

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Problem 1  (Generalised Liénard System)  10 Pts

Consider the following prototype system:

\[ \ddot{x} = f(V(x)) \dot{x} - \nabla V(x), \tag{1} \]

where \( f(V) \) is a generalised friction term, which depends explicitly on the mechanical potential function \( V(x) \).

(a) Analyse the stability of the fixpoints, considering \( V(x) = x^3/3 - x^2/2 \)
and \( f(V) = \mu - V \).

(b) Defining the total energy as \( E = y^2/2 + V(x) \), show that the energy uptake/dissipation can be controlled by the \( f(V) \) friction term.

(c) Show that the fixpoints of (1), corresponding to local extrema of any general potential function \( V(x) \), undergo a Hopf bifurcation when dissipation changes to antidissipation in their neighborhood. Hint: the real parts of the eigenvalues should change their sign.

Problem 2  (Triangle Map)  10 Pts

Consider the Map

\[ f(x) = \begin{cases} \frac{rx}{2} & \text{if } 0 \leq x < 1/2 \\ r - rx & \text{if } 1/2 \leq x \leq 1 \end{cases}, \tag{2} \]

for \( 0 \leq x \leq 1 \) and the parameter \( 0 \leq r \leq 2 \).

(a) Plot the function.

(b) Look for fixed points and cycles (up to length 3).

(c) Derive the analytic expression for the maximal Lyapunov exponent, defined by

\[ \lambda_{\text{max}} = \lim_{n \to \infty} \frac{1}{n} \log \left| \frac{d f^{(n)}(x)}{dx} \right|, \text{ where } f^{(n)}(x) = f \left( f^{(n-1)}(x) \right). \tag{3} \]

Hint: use the chain rule.

(d) For which range of \( r \) does the triangle map exhibit chaos?
Problem 3 \textit{(Sawtooth Map (Optional!) )} \hspace{1cm} 0 \text{ Pts}

A slight variation to the triangle map is given by

\[
f(x) = \begin{cases} 
2x & \text{if } 0 \leq x < 1/2 \\
2x - 1 & \text{if } 1/2 \leq x \leq 1 
\end{cases},
\]

(4) for \( 0 \leq x \leq 1 \).

(a) Derive the analytic expression for the maximal Lyapunov exponent.

(b) You should find a positive Lyapunov exponent, indicating that the map is chaotic. Implement the map in a programming language of your choice and run it for at least 50-100 iterations. What effect do you observe when you plot the evolution of \( x \)? Can you give an explanation?