Exercise Sheet #4

Fabian Schubert <fschubert@itp.uni-frankfurt.de>
Oren Neumann <neumann@itp.uni-frankfurt.de>

Problem 1 (*Preferential Attachment & Network Growth*) 10 Pts

Suppose papers are published more or less at a constant rate, and that the number of references in a paper is essentially constant in time, at an average of m. Let's assume that the probability of a paper to be cited by a new paper (that is, that a new vertex j in the network of papers will link to an older node i) is proportional to the number of papers citing said paper,

$$\Pi_i \propto k_i + C \ ,$$

for some constant C. Consider also that initially there were m_0 papers without references.

1. Derive an equation for the evolution of the degree k_i and find the degree distribution of the network.
2. Is it a scale free network? If so, what is the exponent?
3. Phenomenologically, what causes the difference with the Barabasi-Albert model?

Problem 2 (*Probability Generating Functions for the Matching Problem*) 10 Pts

Suppose you have a deck of N cards, numbered from $1...N$. You shuffle the cards so that they are in completely random order and put them down in a line. A *match* denotes the case where a card with some number n is in the nth position in the line, and we shall denote this by $I_n = 1$, while $I_n = 0$ means no match. Furthermore, $p_N(k)$ is the probability of finding k matches in a deck of size N. It is straightforward to see that the following relation holds:

$$p_N(k) = p_{N+1}\left(k+1 | I_{N+1} = 1\right) \tag{1}$$

where $p(x|y)$ denotes the conditional probability of x given y.

1. Using this relation, show that the probability generating function $G_N(x)$ of $p_N(k)$ satisfies $G_N(x) = \frac{\mathrm{d}}{\mathrm{d}x} G_{N+1}(x)$.
   - Use Bayes' rule to invert the conditional probability: $p(x|y) = p(y|x)p(x)/p(y)$.

- Next, think of expressions for $p_{N+1}(I_{N+1} = 1)$ and $p_{N+1}(I_{N+1} = 1|k+1)$ to simplify the equation.
- Use this equation to arrive at the given recursive relation for $G_N(x)$.

2. The chance of a no-match for a set of size N is $p_N(M_N = 0) = \sum_{i=0}^{N} (-1)^i / i!$. Using this formula and the previous recursive relation, show that

$$p_N(M_N = k) = \frac{1}{k!} \sum_{i=0}^{N-k} \frac{(-1)^i}{i!} \,. \tag{2}$$