Problem 1 \textit{(Jacobian and Lyapunov exponents)} \quad 5 \text{ Pts}

For the dynamical system defined by the equations:
\[
\frac{dx}{dt} = \alpha x + \beta y \\
\frac{dy}{dt} = \alpha y - \beta x
\]

1. Calculate the Jacobian matrix and deduce from it the Lyapunov exponents of the fixpoint.

2. Draw a trajectory map of the system for positive/negative values of $\alpha$ and $\beta$ (all four combinations). What is the connection between the values of the exponents and the behaviour of this system close to the fixpoint?

Problem 2 \textit{(Pitchfork bifurcation of limit cycles)} \quad 5 \text{ Pts}

Consider the following two-dimensional dynamical system in polar coordinates $r$, $\phi$ with a constant angular velocity $\omega = \text{const.}$ and $\Gamma = \text{const.}$:
\[
\dot{r} = r(r - \Gamma)[a - (r - \Gamma)^2] \\
\dot{\phi} = \omega
\]  \quad (1)

Prove that the system shows a pitchfork bifurcation of limit cycles in the radial part when varying the parameter $a$. Sketch the system in polar coordinates for all of the different cases.

Problem 3 \textit{(Heteroclinic orbit)} \quad 10 \text{ Pts}

Consider the two-dimensional dynamical system defined by:
\[
\dot{x} = 1 - x^2 \\
\dot{y} = xy + \epsilon(1 - x^2)^2
\]  \quad (2)

1. Find the fixpoints of the system, check their type and stability by calculating the eigenvalues of the Jacobian. Calculate the eigenvectors corresponding to the eigenvalues.
2. (Optional programming exercise) Solve the ODEs numerically and compute the stable and unstable manifolds of the saddle fixed points. Reproduce the phase plane plots shown in Fig. 1.6 in the CADS lectures notes by plotting a few representative trajectories in the \((x,y)\) plane, together with the stable and unstable manifolds.

Hint: For computing the stable and unstable manifolds, start your trajectories close from the saddle point \(x^* = (x^*, y^*)\) in the direction of the eigenvectors \(u_n, n = 1, 2\), using first forward \((dt > 0)\) and then backward \((dt < 0)\) integration: \(x(0) = x^* + \delta u_n\), with \(\delta = 10^{-6}\).