Problem 1 \((\text{Largest Eigenvalue of Bipartite Graphs})\) \hspace{1cm} 7 Pts

It was shown in the lecture that the largest eigenvalue of the normalized Laplacian of a complete bipartite graph is 2 and it was stated that \(\lambda_{N-1} = 2\) is true if and only if the graph is bipartite. However, this was not shown generally. One way to prove this statement is to use the fact that the Rayleigh quotient of a matrix, in this case \(L\), can be used to define the maximum eigenvalue by

\[
\lambda_{\text{max}} = \max_v \left\{ \frac{v^T L v}{\|v\|^2} \right\}
\]

where \(v\) is the corresponding eigenvector.

1. Show that in the case of the normalized Laplacian, this can be written as

\[
\max_v \left\{ \frac{\sum_{i \leftrightarrow j} (v_i - v_j)^2}{\sum_i v_i^2 k_i} \right\}
\]

where \(i \leftrightarrow j\) denotes the sum over all connected pairs of vertices (not counted twice).

2. Using the general relation \((x - y)^2 \leq 2 (x^2 + y^2)\ \forall x, y \in \mathbb{R}\), show that one finds \(\lambda_{\text{max}} \leq 2\).

3. When do we find equality and why does this correspond only to cases of bipartite graphs (thus completing the proof)?

Problem 2 \((\text{Preferential Attachment & Network Growth})\) \hspace{1cm} 6 Pts

Suppose papers are published more or less at a constant rate, and that the number of references in a paper is essentially constant in time, at an average of \(m\). Let’s assume that the probability of a paper to be cited by a new paper (that is, that a new vertex \(j\) in the network of papers will link to an older node \(i\)) is proportional to the number of papers citing said paper,

\[\Pi_i \propto k_i + C,\]

for some constant \(C\). Consider also that initially there were \(m_0\) papers without references.
1. Derive an equation for the evolution of the degree $k_i$ and find the degree distribution of the network.

2. Is it a scale free network? If so, what is the exponent?

3. Phenomenologically, what causes the difference with the Barabasi-Albert model?

**Problem 3 (Moments of Cluster Sizes) 7 Pts**

Let the generating function of the degree distribution of a random graph be given by

$$G_0(x) = p_1 x + p_3 x^3 = (1 - p_3)x + p_3 x^3,$$

that is, a node can either have degree one with probability $p_1 = 1 - p_3$ or a degree three with probability $p_3$. Furthermore, as discussed in the lecture we assume that the size of the graph is large enough so that we can ignore recurrent loops.

1. Using the self-consistency equation derived in the lecture, find the analytic expression for $H_1(x)$, that is, the generating function for the size of clusters to a single node with one additional incoming connection. You should get it as the solution to a quadratic equation, which has two branches. Using the additional normalization constraint $H_1(1) = 1$, make sure to specify which solution to choose for a given value of $p_3$.

2. Calculate the first and second derivative of your $H_1(x)$.

3. Identify the Percolation threshold for the parameter $p_3$. Verify that both first and second moments of the distribution associated with $H_1(x)$ diverge at this threshold.

4. (Optional) From $H_0(x) = xG_0(H_1(x))$ (the generating function for the size of clusters connected to the starting node), derive the general expressions for the first and second moments $\langle s \rangle$ and $\langle s^2 \rangle$ of the cluster size distribution and, plugging in your previous results for $H_1'(x)$ and $H_1''(x)$, find the expressions for the mean $\langle s \rangle$ and variance $\langle s^2 \rangle - \langle s \rangle^2$ of the cluster size distribution.