Exercise Sheet #3

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Problem 1 (*Largest Eigenvalue of Bipartite Graphs*) 8 Pts

It was shown in the lecture that the largest eigenvalue of the normalized Laplacian of a complete bipartite graph is 2 and it was stated that $\lambda_{N-1} = 2$ is true if and only if the graph is bipartite. However, this was not shown generally. One way to prove this statement is to use the fact that the *Rayleigh quotient* of a Matrix, in this case L, can be used to define the maximum eigenvalue by

$$\lambda_{\max} = \max_v \left\{ \frac{v^T L v}{||v||^2} \right\} \tag{1}$$

where v is the corresponding eigenvector.

1. Show that in the case of the normalized Laplacian, this can be written as
$$\max_v \left\{ \frac{\sum_{i \leftrightarrow j} (v_i - v_j)^2}{\sum_i v_i^2 k_i} \right\} \tag{2}$$
where $i \leftrightarrow j$ denotes the sum over all connected pairs of vertices (not counted twice).

2. Using the general relation $(x-y)^2 \leq 2\,(x^2 + y^2) \quad \forall x, y \in \mathbb{R}$, show that one finds $\lambda_{\max} \leq 2$.

3. When do we find equality and why does this correspond only to cases of bipartite graphs (thus completing the proof)?

Problem 2 (*Network Robustness*) 8 Pts

Consider an electricity network, where the probability of a node of degree k failing is a_k. As a first approximation, consider the network to have a random degree distribution (Erdös–Rényi). Then, suppose the power supply of a node can fail with a probability a_k, where k is the degree of the node. Consider the two simple cases of failure:

- $a_k = const.$ (i.e. all nodes have the same probability of failing)
- $a_k \propto \Theta(k^* - k)$ (i.e. only the heavy load nodes, the one with a degree bigger than k^*, can fail)

Find:

1. Make a plot of the critical threshold $k^* \to k_c$ at which the network stops being functional at the thermodynamic limit $N \to \infty$, as a function of z.

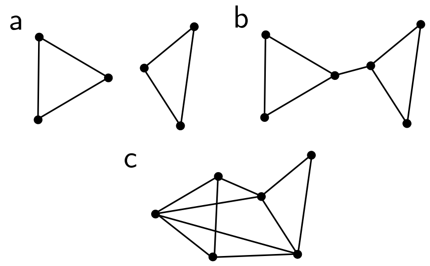
2. Calculate numerically the critical fraction f_c as a function of the average degree z. What fraction of all nodes can fail if, $z = 1.5$, 5 or 10?

3. Compare the results with the ones for a (more realistic) scale-free network with exponent α. For one value of z compare the critical fraction for random and scale-free networks.

4. Which kind of network would you say is more robust to which kind of failures and why?

Problem 3 (*Cluster Connectivity*) 4 Pts

Consider the following three graphs:



1. Calculate the Laplacian matrices of these three graphs.

2. Calculate the eigenvalues (you may do this numerically).

3. If $\lambda_0, ..., \lambda_{N-1}$ is the list of eigenvalues of the Laplacian, sorted from small to large (including possible degeneracies), does λ_1 tell you something about the connectivity within the graph? You may search for *algebraic connectivity* to get further information.