Problem 1  
(Graph Laplacian)  
5 Pts

For the graphs given in Fig.1 calculate:

- The Adjacency matrix and the Laplacian matrix of the graphs.
- Find the eigenvalue spectrum of the Laplacian matrix. What is its meaning?

![Graphs](image)

Figure 1: Random graphs with N=3 vertices

Problem 2  
(Probability Generating Functions for the Matching Problem)  
10 Pts

Suppose you have a deck of $N$ cards, numbered from 1...$N$. You shuffle the cards so that they are in completely random order and put them down in a line. A *match* denotes the case where a card with some number $n$ is in the $n$th position in the line, and we shall denote this by $I_n = 1$, while $I_n = 0$ means no match. Furthermore, $p_N(k)$ is the probability of finding $k$ matches in a deck of size $N$. It is straightforward to see that the following relation holds:

$$p_N(k) = p_{N+1}(k + 1|I_{N+1} = 1)$$

(1)

where $p(x|y)$ denotes the conditional probability of $x$ given $y$.

1. Using this relation, show that the probability generating function $G_N(x)$ of $p_N(k)$ satisfies $G_N(x) = \frac{dx}{dx} G_{N+1}(x)$:

   - Use Bayes’ rule to invert the conditional probability:
     $$p(x|y) = \frac{p(y|x)p(x)}{p(y)}.$$
Next, think of expressions for \( p_{N+1} (I_{N+1} = 1) \) and \( p_{N+1} (I_{N+1} = 1|k + 1) \) to simplify the equation.

Use this equation to arrive at the given recursive relation for \( G_N(x) \).

2. The chance of a no-match for a set of size \( N \) is \( p_N (M_N = 0) = \sum_{i=0}^{N} (-1)^i /i! \).

Using this formula and the previous recursive relation, show that

\[
p_N (M_N = k) = \frac{1}{k!} \sum_{i=0}^{N-k} \frac{(-1)^i}{i!}.
\] (2)

**Problem 3** *(Percolation)* 5 Pts

We create a random graph by adding vertices with random edges connecting to other vertices. The degree of each vertex is geometrically distributed, decided by repeatedly tossing a coin until tails comes up, such that the degree is the number of heads tosses:

\[
P(k) = (1 - p)^{k-1} p
\] (3)

Assuming the coin is unfair with probability \( p \) for tails. Find the percolation threshold of the graph where a giant connected cluster forms.