Exercise Sheet #1

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Problem 1 (*Graph Spectrum*) 7 Pts

1. For the graphs given in Fig.1 calculate:

   (a) The Adjecancy matrix and the Laplacian matrix of the graphs.

   (b) Find the spectra of these matrices.

   (c) Calculate the moments of these adjacency spectra to third order and interpret the result (What do these moments tell you about the topology?).

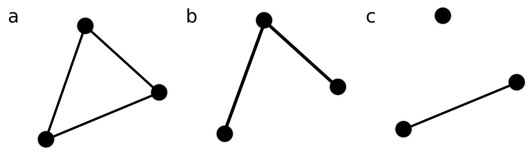


Figure 1: Random graphs with N=3 vertices

Problem 2 (*Node Centrality*) 7 Pts

1. There are many possible measures for the relevance of a node in the network, i.e. the *centrality* of a node.

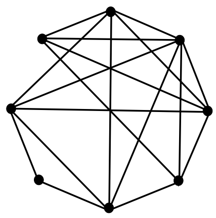
   Let us define x_i as the "eigenvalue centrality" of node i as a value directly proportional to the sum of the centrality of its first neighbors, that is:

$$x_i \propto \sum_{j \in \mathcal{N}_i^{(1)}} x_j \;, \tag{1}$$

   where $\mathcal{N}_j^{(1)}$ is the group of first-neighbors of i. Using the adjacency matrix A, reduce this expression to an eigenvalue problem:

$$A\mathbf{x} = \lambda \mathbf{x} \;, \tag{2}$$

   where $\mathbf{x}$ is a vector with components x_i. Apply this measure to identify the three most "relevant" nodes in the following network:



Hint: Only the eigenvector with the largest eigenvalue results in the desired centrality properties. The eigenvectors are to be calculated numerically.

Problem 3 (*Degree Distribution*) 6 Pts

In a random network with connection probability p, the probability of a node to have k edges is

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k} \ . \tag{3}$$

1. Use Stirling's approximation $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ to show that for large N, p_k approaches a Poisson distribution with mean pN:

$$p_k \approx e^{-pN} \frac{(pN)^k}{k!} \tag{4}$$

2. Let $\mathrm{P}\left(pN(1-\delta) \leq k \leq pN(1+\delta)\right)$ be the probability of finding an edge number within a δ-environment proportional to the mean degree pN. Show that, for a *fixed* p and any $\delta > 0$, we can find a sufficiently large N, such that $\mathrm{P}\left(pN(1-\delta) \leq k \leq pN(1+\delta)\right) \to 1$. In other words fluctuations of k relative to its mean become arbitrarily small for large N.

   You should first analyze the behavior of the Poisson distribution for large N and show that it approaches a Gaussian distribution with moments depending on pN. You can then proceed to express $\mathrm{P}\left(pN(1-\delta) \leq k \leq pN(1+\delta)\right)$ as a Gaussian integral and describe its behavior for $N \to \infty$.