Exercise Sheet #13

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Problem 1 (*Hopfield Networks*) 20 Pts

Hopfield networks are recurrent networks that can be trained to behave like an associative memory. A Hopfield network consists of N neurons. They are fully connected through symmetric, bidirectional connections with weights $w_{ij} = w_{ji}$. There are no self-connections, so $w_{ii} = 0$ for all neurons i. We will denote the activity of neuron i by x_i, where $x_i \in \{-1, 1\}$.

The states of all neurons in the network are synchronously updated via

$$a_i(t) = \sum_{j=1}^{N} w_{ij} x_j(t-1) \tag{1}$$

$$x_i(t) = \Theta(a_i(t)) \tag{2}$$

where Θ is defined by:

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \tag{3}$$

Hopfield networks are typically used for their ability to converge to previously-trained patterns. These patterns are local minima in the energy, defined by:

$$E = -\frac{1}{2} \sum_{i,j}^{N} \omega_{ij} x_i x_j \tag{4}$$

The weights are generated from the desired patterns via the sum over the outer products

$$w_{ij} = \sum_{n=1}^{n_{\mathrm{p}}} x_i^{(n)} x_j^{(n)} \tag{5}$$

where $\mathbf{x}^{(n)}$ is the nth pattern.

- Create a Hopfield network of $N = 100$ neurons, with a random weight matrix ω_{ij} generated from ten random patterns. Starting with random initial conditions, plot the energy as a function of time. repeat this for 5 different initial conditions. Interpret your results. (10 Pts)
- We would like to estimate the pattern storage capacity of the network. We do so by considering the stability of just one bit of one of the

desired patterns, assuming that the state of the network is set to that desired pattern $\mathbf{x}^{(n)}$. We will assume that the patterns to be stored are randomly selected binary patterns.

– Show that for a sufficiently large number n_p of patterns, the probability of bit i to flip on the first iteration of the Hopfield network's dynamics is

$$P(i \text{ unstable}) = \Phi\left(-\sqrt{\frac{N}{n_\mathrm{p}}}\right) \tag{6}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z \ . \tag{7}$$

(5 Pts)

– Assume that we wish all the desired patterns to be very stable – we don't want any of the bits to flip when the network is put into any desired pattern state – and the total probability of any error at all is required to be less than a small number ϵ. Using the approximation for the error function for large x,

$$\Phi(-x) \simeq \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} \ , \tag{8}$$

show that the maximum number of patterns that can be stored, n_pmax, is

$$n_\mathrm{pmax} \simeq \frac{N}{4\ln N + 2\ln(1/\epsilon)} \ . \tag{9}$$

(5 Pts)