Observing scale-invariance in non-critical dynamical systems

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overview

critical vs. criticality

- criticality in statistical physics

critical boolean networks

- undersampling of phase space
- observational criticality

vertex routing models

- exactly solvable
- scaling vs. log-corrections

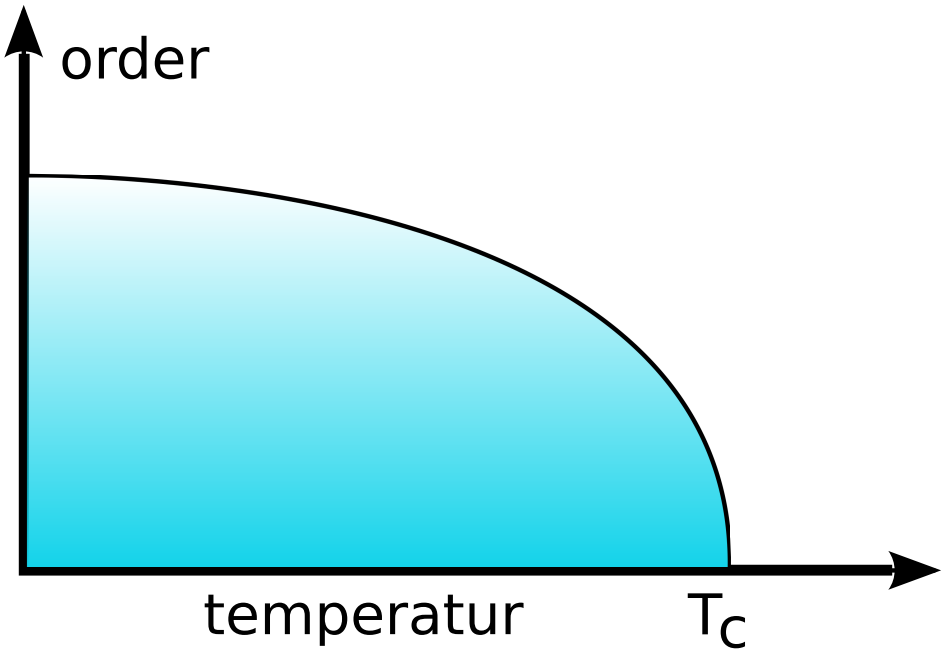
what does ’criticality’ mean?

thermodynamic phase transition

- critical point T_c

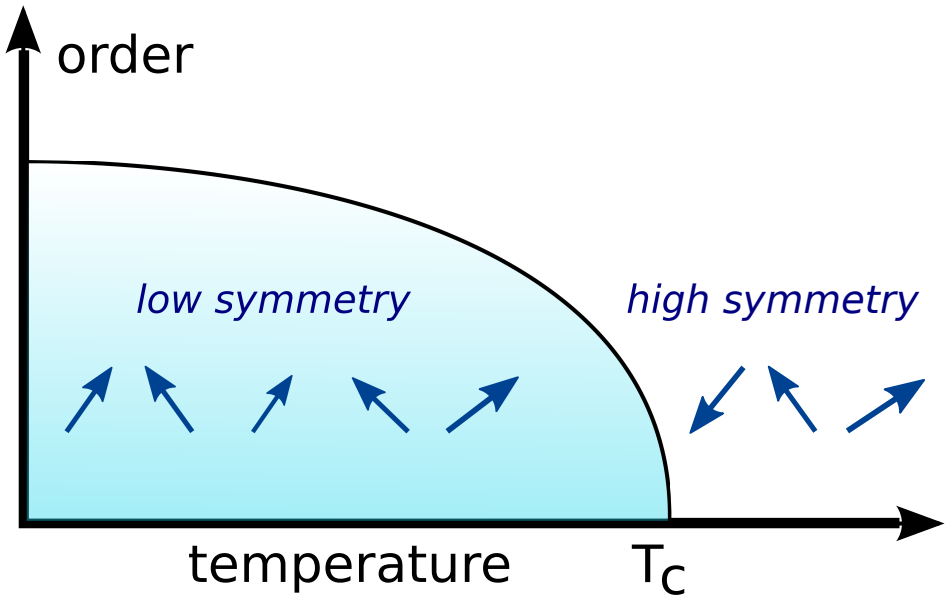
- order
  magnetization
  superconductivity
  . . .

- control parameter
  temperature
  pressure
  . . .



symmetry breaking

continuous phase transition $\hat{=}$ second order



disordered phase $T > T_c$

- spins point in all directions
- high rotational symmetry

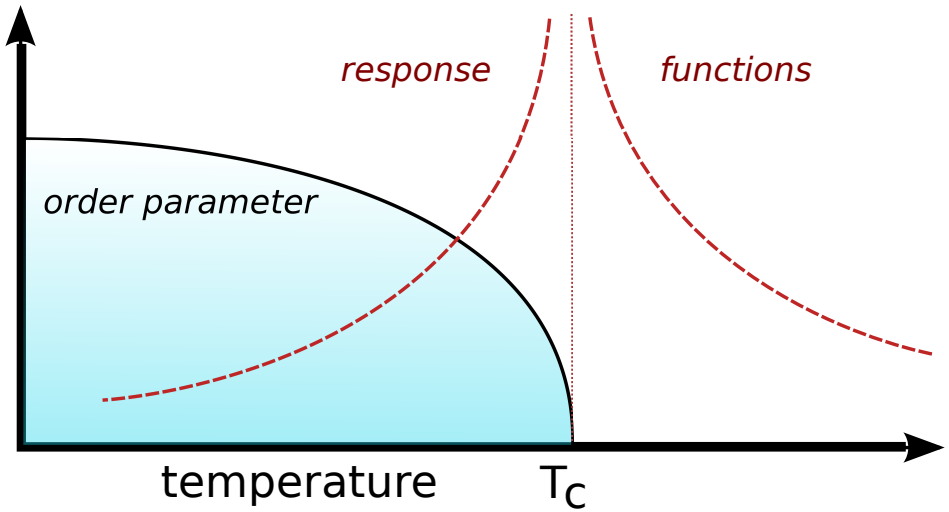
ordered phase $T < T_c$

- spins point mostly in one direction
- rotational symmetry broken

scaling towards criticality

response to external perturbations

- temperature gradient $c_V(T)$
- magnetic field $\chi(T)$
- ...



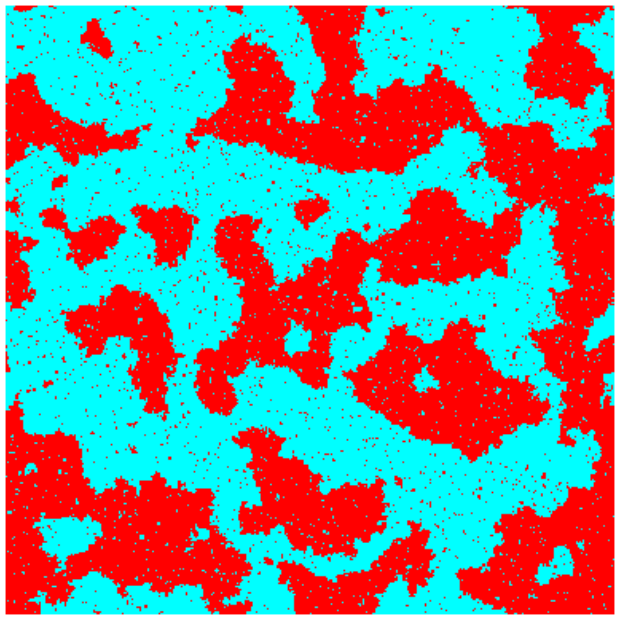
power-law scaling

$$\sim \frac{1}{|T - T_c|^{\alpha}}$$

- critical exponent α
- only when coupling to order parameter

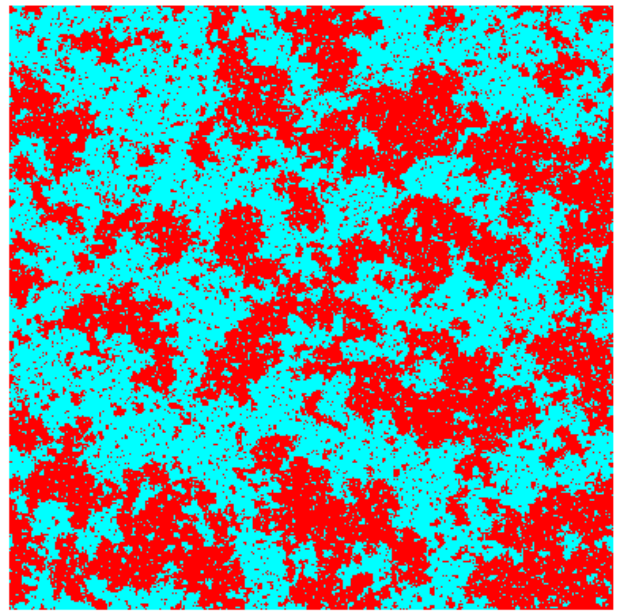
criticality – $T = T_c$

spatial correlations: 2D Ising



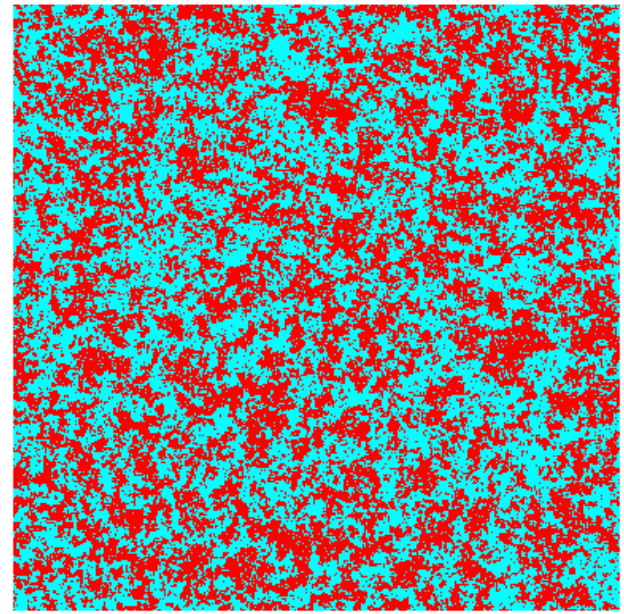
$T < T_c$

typical
domain size



$T = T_c$

no typical
length scale



$T > T_c$

microscopic
length scale

correlations

spatial (or temporal) correlation functions

$$D(\mathbf{r}) \equiv D(\mathbf{x}-\mathbf{y}) = \left\langle S(\mathbf{x})S(\mathbf{y})\right\rangle - \langle S\rangle^2$$

- correlation length ξ

$$D(\mathbf{r}) \propto \begin{cases} \mathrm{e}^{-r/\xi} & T \neq T_c \\ r^{-\gamma} & T = T_c \end{cases}$$

- typical length scale $\xi \propto 1/|T-T_c|^z$

scale invariance

$$\boxed{\frac{1}{r^\alpha} \sim \frac{1}{\tilde{r}^\alpha}} \qquad \tilde{r} = ar$$

microscopic scales

example: Schrödinger equation

$$i\hbar\frac{\partial\Psi(t,\mathbf{r})}{\partial t} = H\Psi(t,\mathbf{r}), \qquad H = -\frac{\hbar^2\Delta}{2m} - \frac{e^2}{|\mathbf{r}|}$$

- Hamilton Operator

$$H = -E_R\left(a_0^2\Delta + \frac{2a_0}{|\mathbf{r}|}\right), \qquad E_R = \frac{me^4}{2\hbar^2}, \qquad a_0 = \frac{\hbar^2}{me^2}$$

length scale

Bohr radius $a_0 = 0.53$

energy scale

Rydberg energy $E_R = 13.6\,\mathrm{eV}$

temporal scale

$E_R/\hbar = 3.39\cdot 10^{15}\,\mathrm{Hz}$

universality

scale invariance at criticality

at criticality all microscopic scales (length, energy, time, . . .) become irrelevant

universality at criticality

at criticality only the symmetry of the order parameter is relevant

magnetism

rotational symmetry: O(3), SU(2), . . .

superconductivity

U(1) gauge symmetry: phase $e^{i\varphi}$

thermodynamics vs. dynamical systems

classical theory of phase transition

- thermal equilibrium $e^{\beta E}$
- quantum phase transition $T_c = 0$

dynamical system

$$\frac{d}{dt} x_i(t) \;=\; f_i(x_1, .., x_N; \gamma), \qquad\qquad i = 1, .., N$$

- control parameter γ
- classical: real-valued variables
- the brain is a classical dynamical system

  $\Rightarrow$ phase transitions in dynamical systems

thermodynamic limit

only in the thermodynamic limit

$$\boxed{\lim_{N\to\infty}}$$

- phase transitions, criticality

$$\frac{d}{dt}x_i(t) \;=\; f_i(x_1,..,x_N;\gamma), \qquad f_i \equiv f$$

- identical units

finite size scaling

$$\text{system property} \;\propto\; \begin{cases} \mathrm{e}^{-N/\xi} & T \neq T_c \\ N^{-\gamma} & T = T_c \end{cases}$$

Kauffman nets

random boolean network

$$\sigma_i(t+1) \;=\; f_i(\sigma_1(t),..,\sigma_K(t)), \qquad i = 1,..,N$$

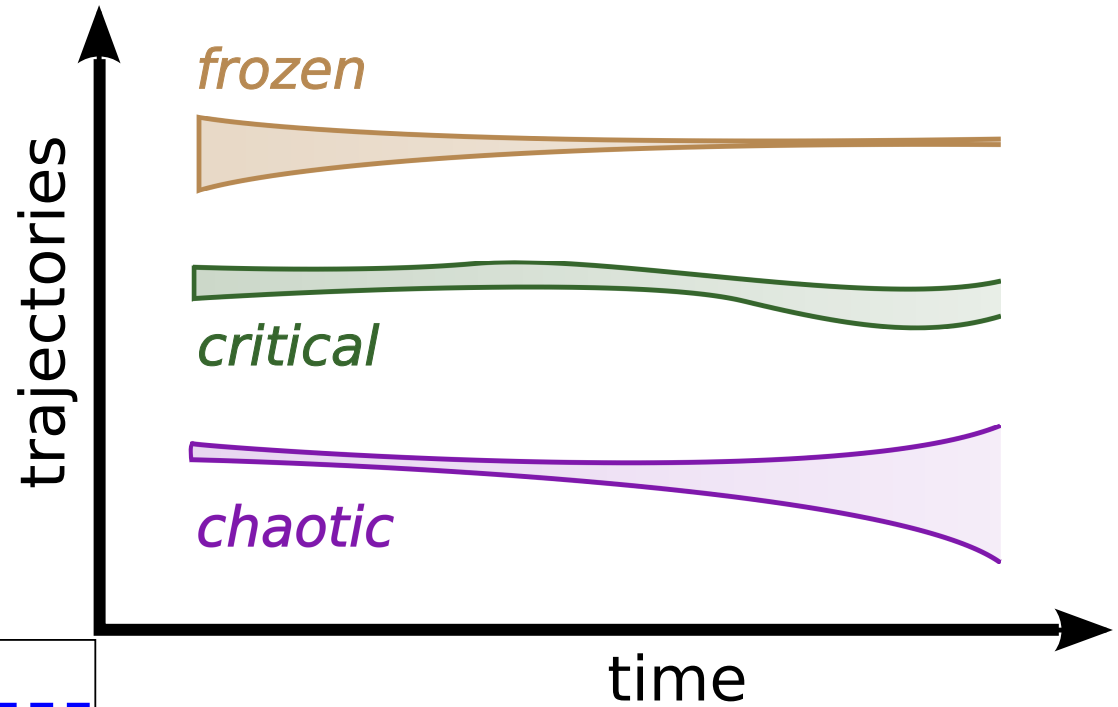
- boolean: $\sigma_i = 0,1$
- random boolean function: $f_i(\cdot) = 0,1$
- connectivity: K
  $\Rightarrow$ control parameter

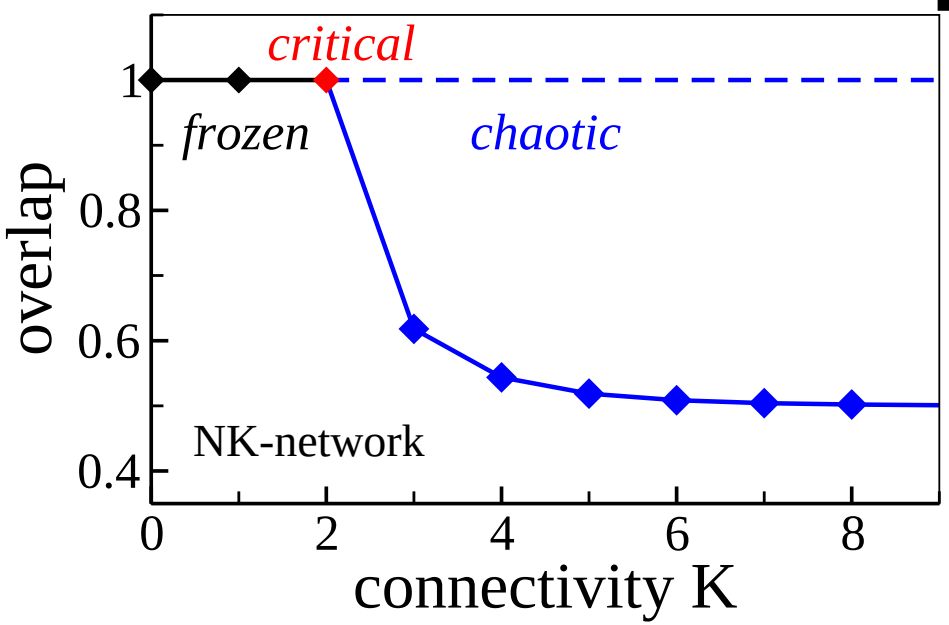
$K < 2$	$K = 2$	$K > 2$
frozen	critical	chaotic

$N - K$ networks

order parameter

- activity overlap

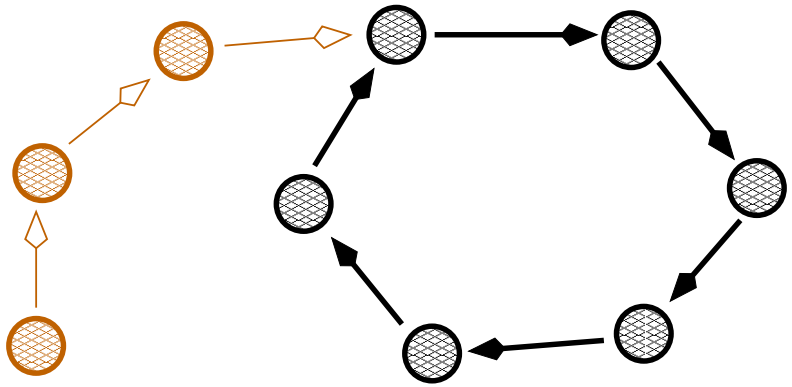


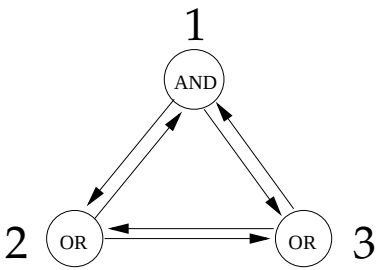


in the limit $t \to \infty$

cycles and attractors

- gene regulation networks
- brain $\Rightarrow$ transient attractors





OR

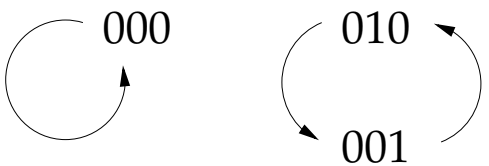
1	2	3
0	0	0
0	1	1
1	0	1
1	1	1

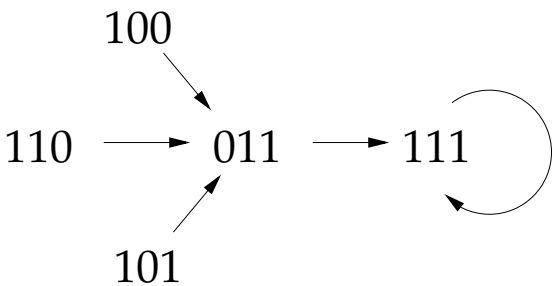
OR

1	3	2
0	0	0
0	1	1
1	0	1
1	1	1

AND

2	3	1
0	0	0
0	1	0
1	0	0
1	1	1





critical Kauffman net: $K = 2$

number of attractors - numerics

- $\propto \sqrt{N}$ [Kauffman, '69]
- power law – scale invariant

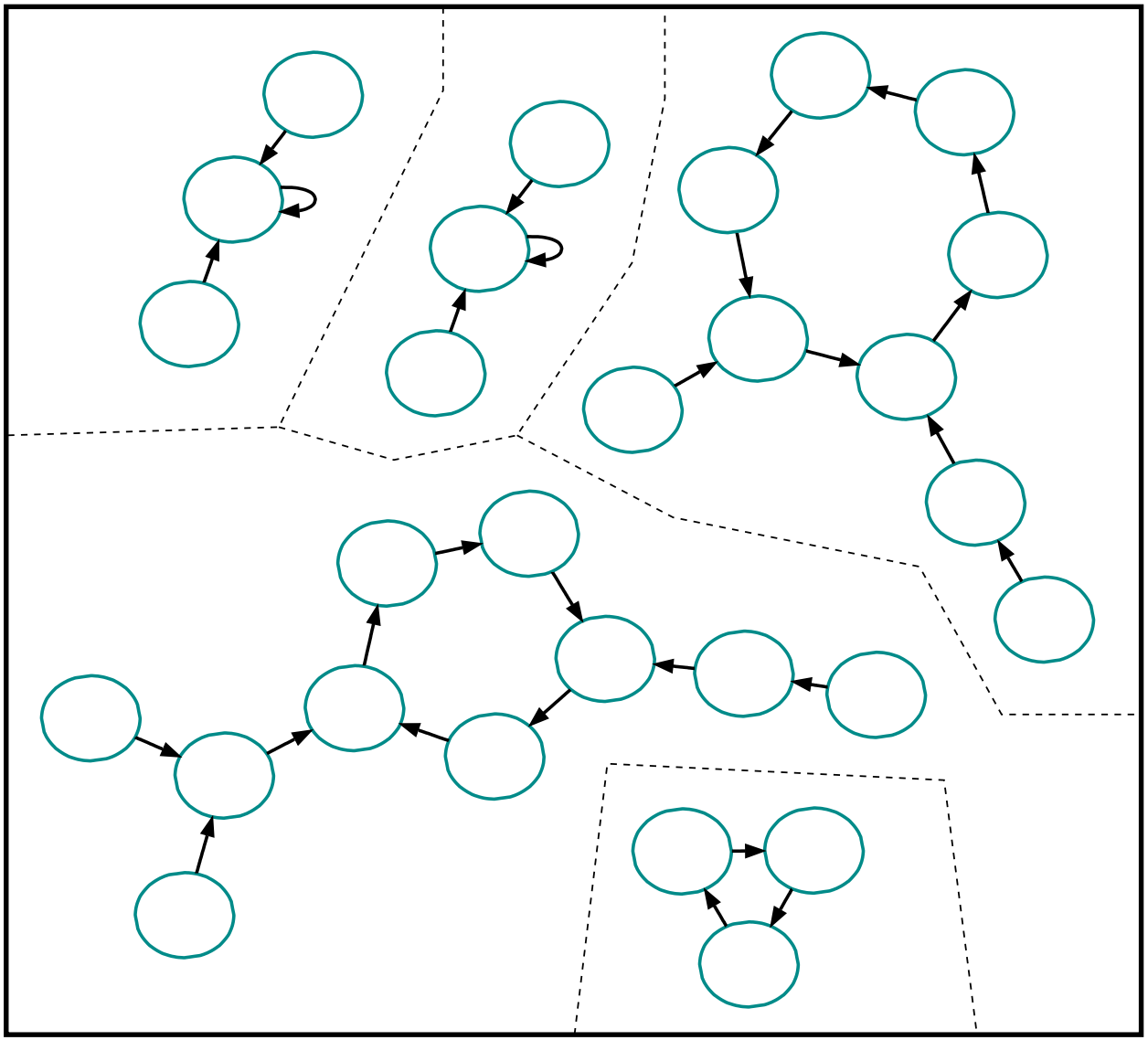
number of attractors - exact

- growths faster than $N^{\alpha}, \quad \forall \alpha$ [Samuelsson & Troein, '03]
- no power law – not scale invariant

numerics is wrong and right

- wrong measurement of intrinsic property
- corresponds to experimental observation

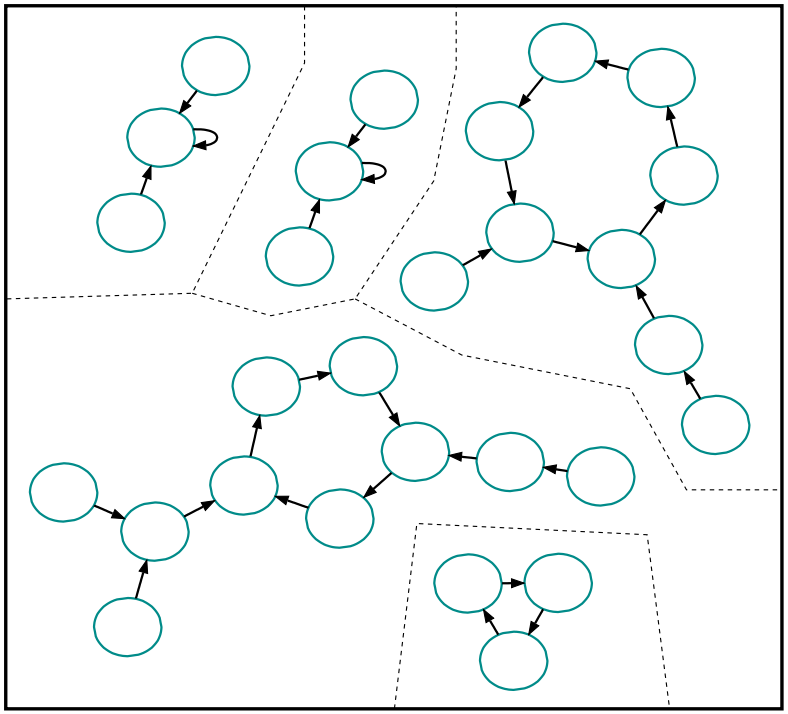
basins of attraction



big attractors can dominate

(fictitious) example: isolated point attractors &
big attractors

- phase space volume: $\Omega = 2^N$
- number of point attractors, f.i.: $2^{\sqrt{N}}$
- relative phase space of point attractors, including basins of attraction



$$\lim_{N\to\infty} \frac{2^{\sqrt{N}}}{2^N} \to 0$$

big attractors may dominate phase space

- number of big attractors, f.i.: $\sqrt{N}$

what does 'critical' mean?

experimental observations

- stochastic sampling of phase space
- $K = 2$ Kaufman net
  - $\Rightarrow$ experimental sampling $\hat{=}$ statistics of big attractors
  - $\Rightarrow$ scale invariance

is the $K = 2$ Kaufman net critical?

- yes: boundary between two phases
- no: state is not critical (?)
  - $\Rightarrow$ no universality

observing scale invariance in a dynamical system does not necessarily imply a critical dynamical state

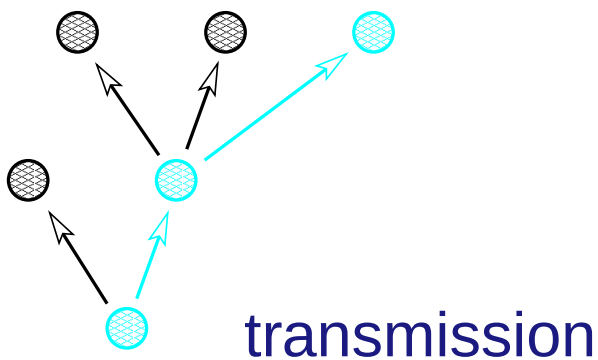
information transmission & routing

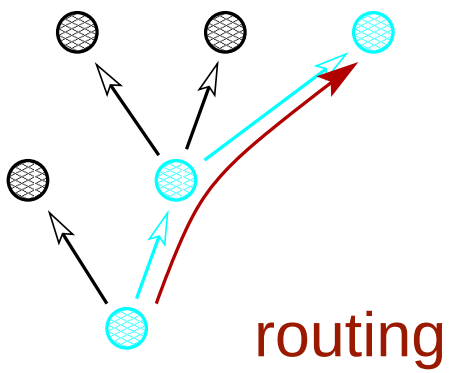
information transmission

- vertex $\Rightarrow$ vertex
  neuron $\Rightarrow$ neuron
- phase space: number of vertices

information routing

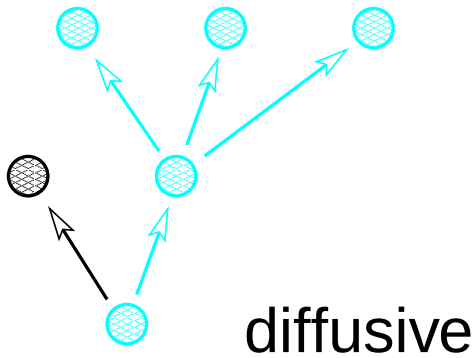
- link (incoming) $\Rightarrow$ link (outgoing)
- phase space: number of directed links

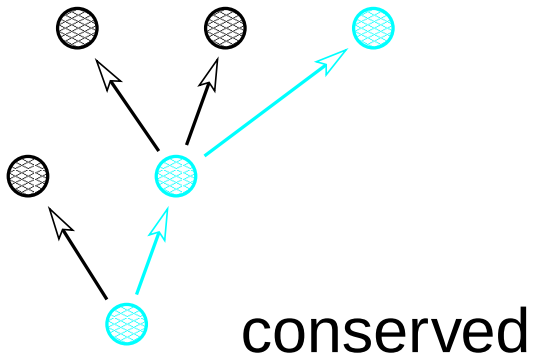




vertex routing models

critical information routing





- (on the average) conserved quantities
  $\Rightarrow$ critical dynamics
- information packages

vertex routing models

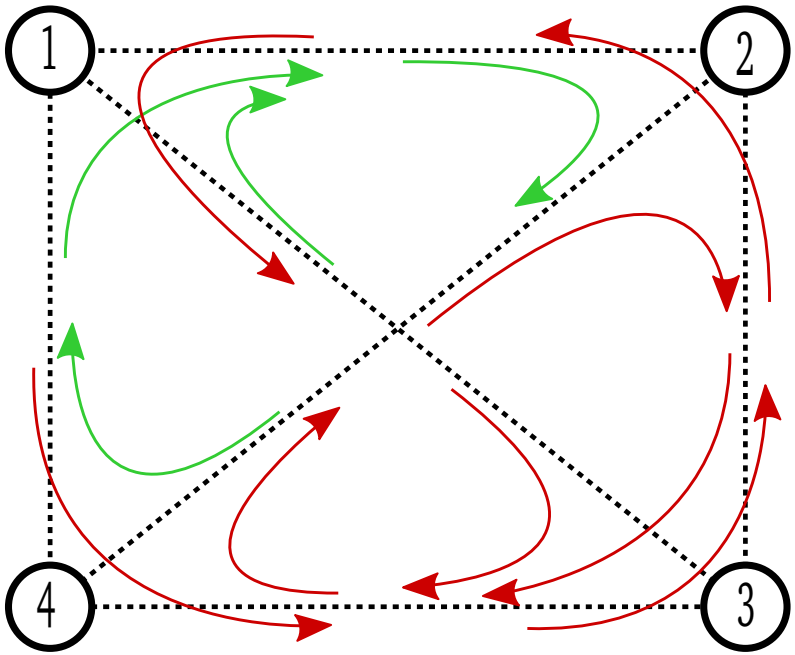
- routing of conserved information packages

routing dynamics

- random routing tables at every vertex
- quenched dynamics (fixed routing tables)

$$(1) \to (2) \to (4) \to \dots$$

$$(2) \to (3) \to (4) \to \dots$$



information centrality

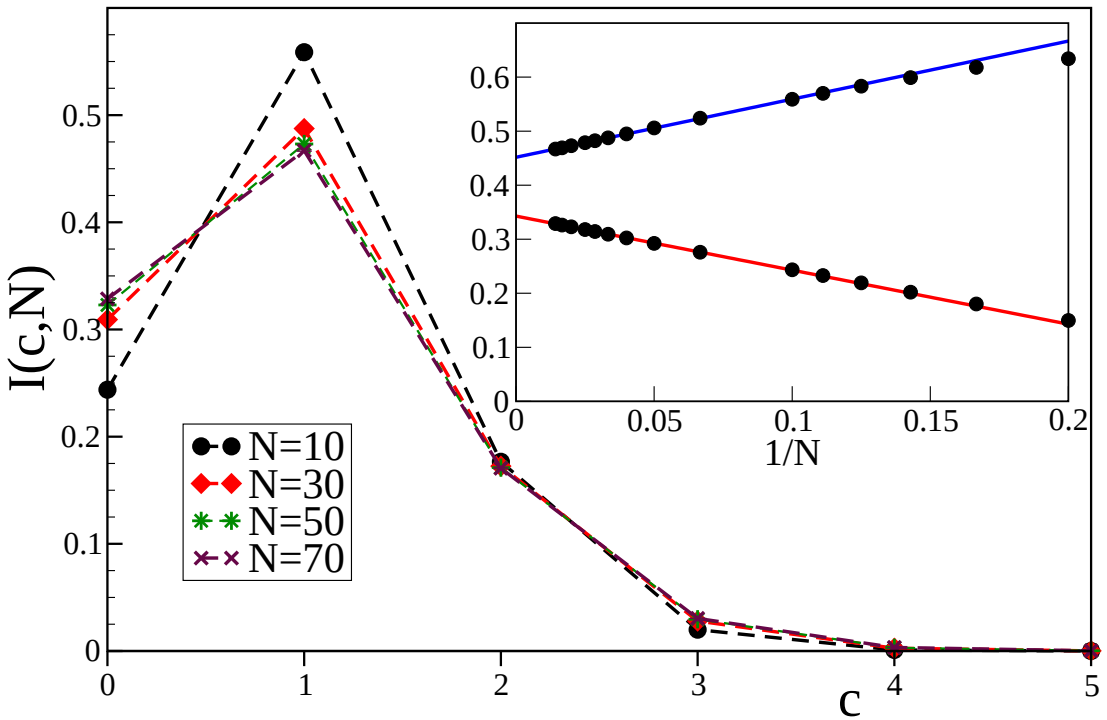
- overlapping cyclic attractors

number of attractors passing through a given vertex

non-trivial information centrality

numerical simulations

fully connected graphs



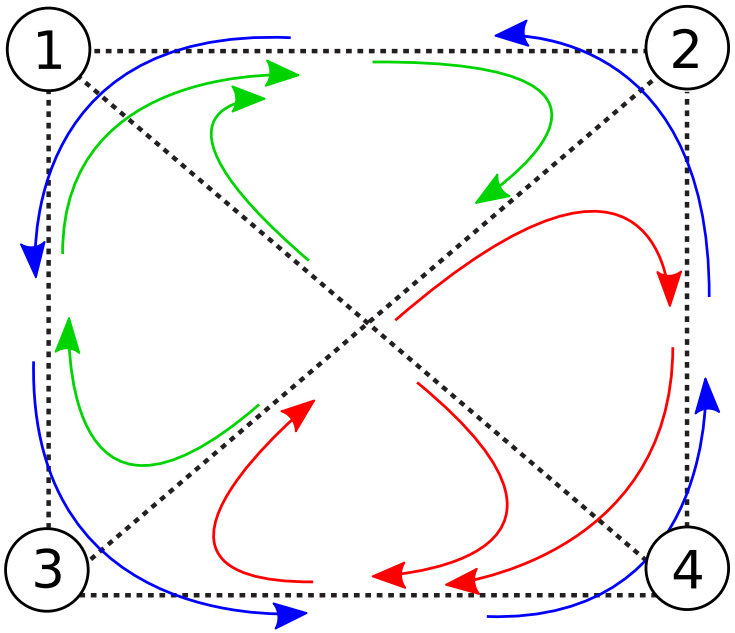
[Markovic & Gros, NJP '09]

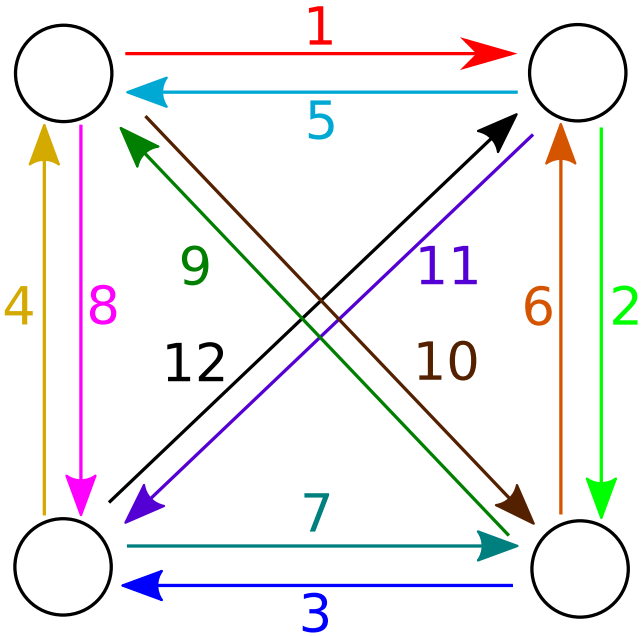
democratic distribution of the information centrality

random walk in phase space

phase space of directed links

- polynomial: $\Omega = N(N-1)$
- mapping to random walk in Ω
  for: number of cycles
  mean cycle length / cycle length distribution
  but not for information centrality





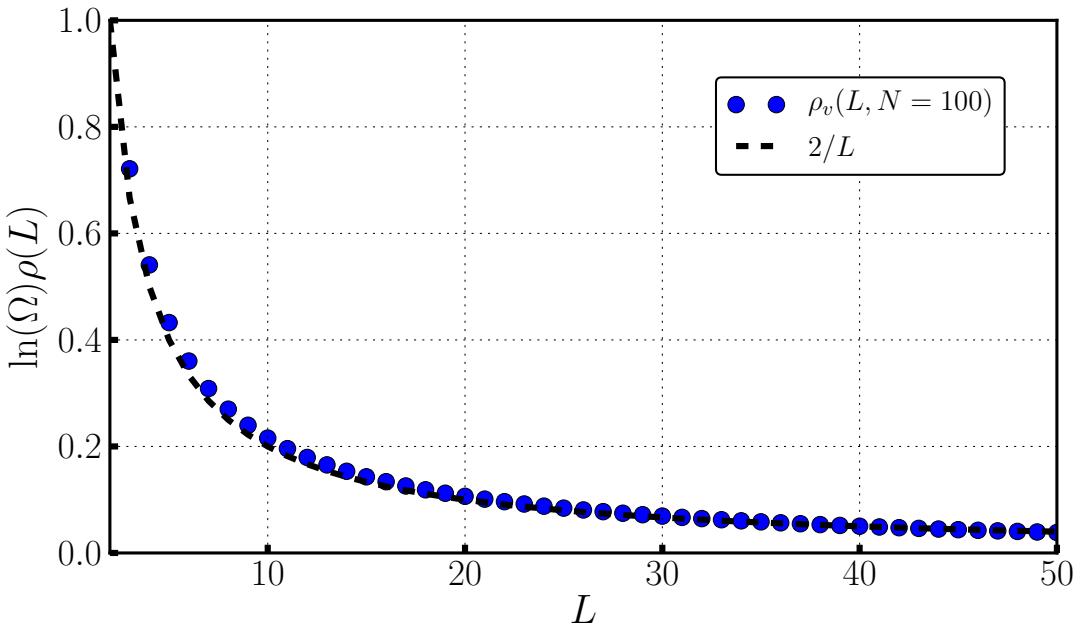
solution of vertex routing model

cycle length distribution

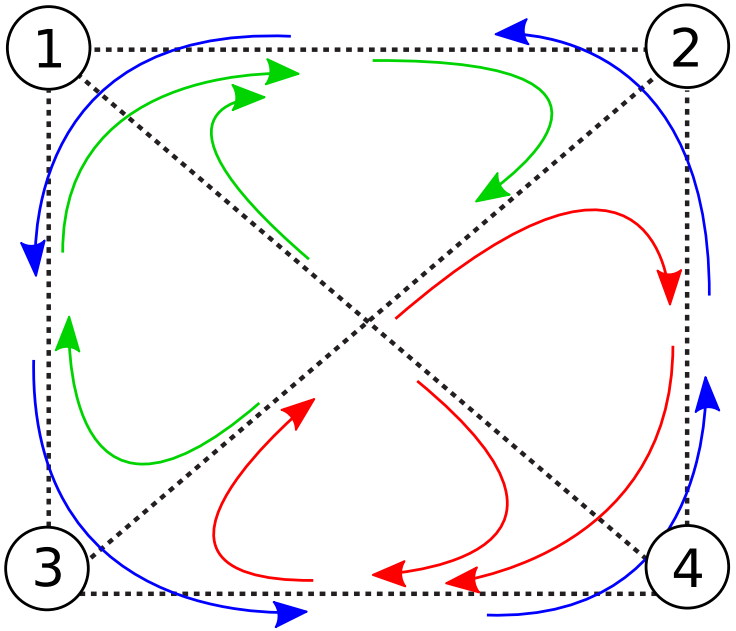
- number of cycles of length L

$$\langle C_L \rangle (N) = \frac{N((N-1)^2)!}{L(N-1)^{2L-1}((N-1)^2+1-L)!}$$

- fully connected graphs
- numerical
  $N \leq 10^4$



experimental observation



quenched dynamics

- model (routing tables) fixed
- intrinsic properties

on-the-fly dynamics

- model (routing tables) generated when needed (on-the-fly)

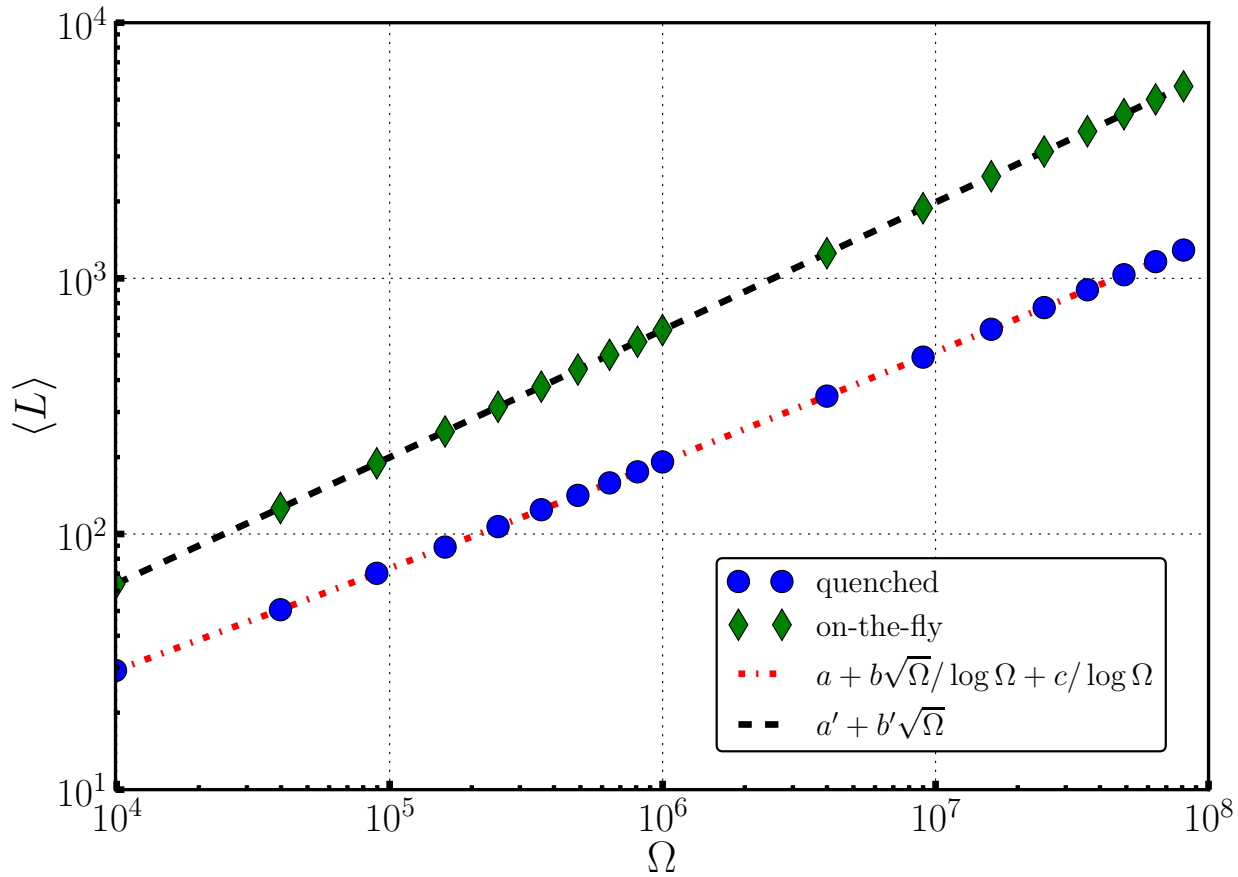
  $\Rightarrow$ random sampling of phase space, experimental sampling

- solvable

$$\langle C_L \rangle (N) \propto \sum_{t=L}^{L_{max}} \frac{((N-1)^2)!}{(N-1)^{2t}((N-1)^2+1-t)!}$$

scaling of mean cycle length

[Markovic, Schuelein & Gros, '12]



- average cycle length

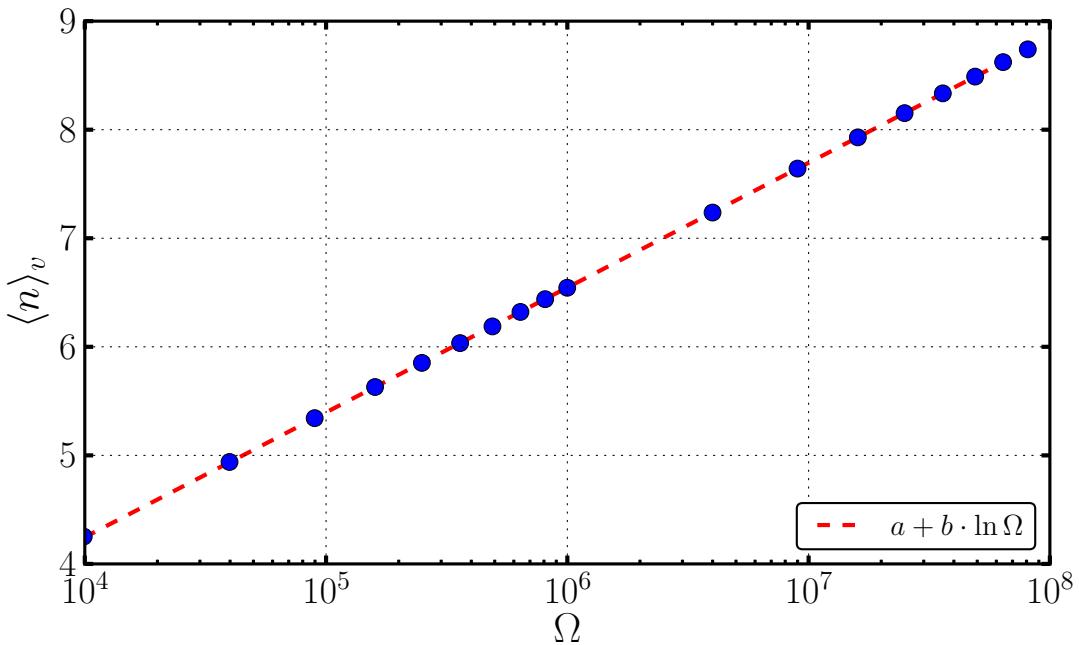
$$\langle L \rangle \sim \begin{cases} N/\log(N) & \text{quenched} \\ N & \text{on-the-fly} \end{cases}$$

scaling of cycle number

- only quenched dynamics
- on-the-fly:
  only relative quantities

mean number of attractors

$$\langle n \rangle \sim \log(N)$$



scaling for critical vertex routing

quenched dynamics	on-the-fly dynamics
intrinsic properties	experimental sampling

		quenched	on the fly
vertex routing	number of cycles mean cycle length	$\log(N)$ $N/\log(N)$	– N
markovian model	number of cycles mean cycle length	$\log(N)$ $\sqrt{N}/\log(N)$	– $\sqrt{N}$

markovian model

- variant without memory

two critical nets

vertex routing model

- number of cycles $\langle n \rangle$
  phase space Ω

$$\Omega \;=\; N(N-1) \;\simeq\; N^2, \qquad\qquad \langle n \rangle \;\propto\; \log(\Omega) \;\sim\; \log(N)$$

$K = 2$ Kauffman net

- number of cycles
  larger than any power of N

$$\Omega \;=\; 2^N, \qquad\qquad \langle n \rangle \;\not\propto\; \log(\Omega) \;\sim\; N$$

critical vs. universality

critical does not imply universality

critical model

model on the verge of a (second order) phase transition

critical state

intrinsic state is critical (scale invariant)

universality

the properties of a critical state are determined by
symmetries and not by the specific
microscopic model parameters

thermodynamics

critical models have critical states

dynamical systems

critical models do not necessarily show critical dynamics

typical vs. average

typical may differ from average

typical property

obtained by random sampling of phase space

$\hat{=}$ experimental observation

average (mean)

evaluating a property for all phase space

and taking the mean

typical and average may scale differently

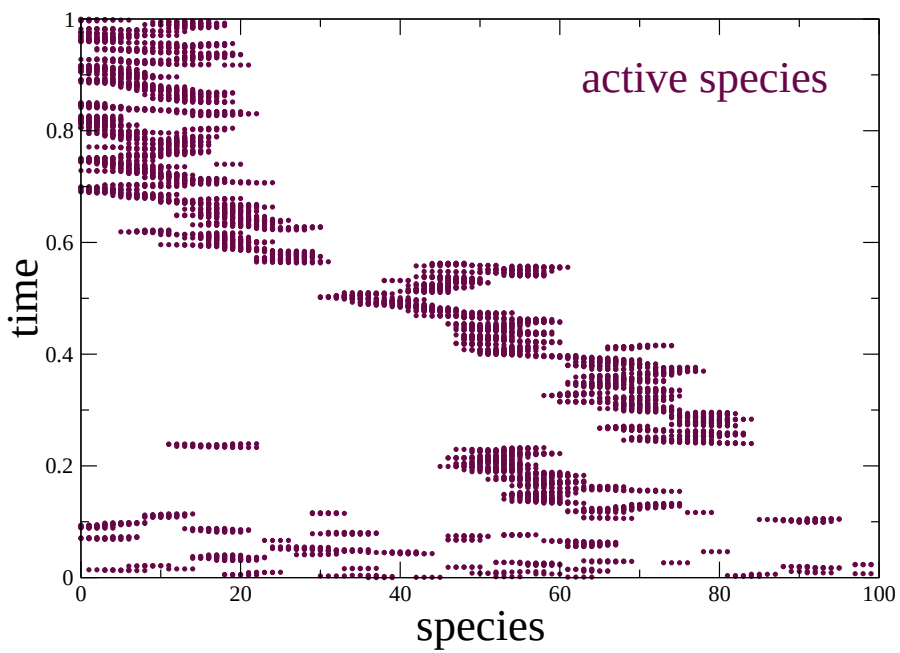
critical boolean network (Kauffman)

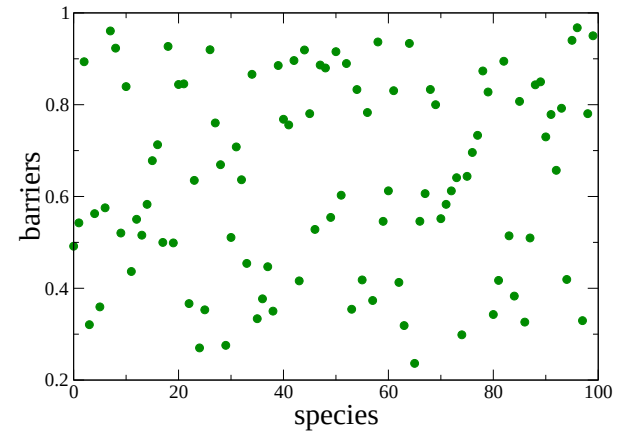
vertex routing model (information)

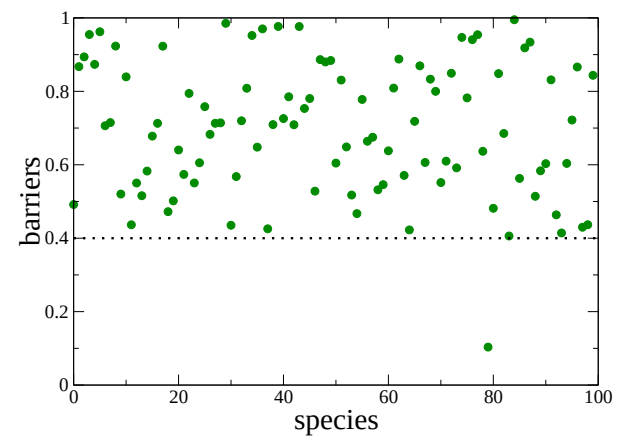
open issues – SOC

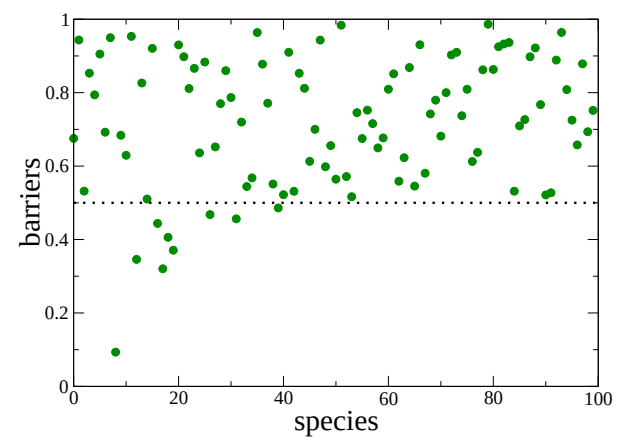
self-organized criticality

- driven systems (sandpile model)
  - ∗ beyond dynamical system theory
- autonomous systems (Bak & Sneppen model)
  - ∗ has global operations
  - ∗ beyond dynamical system theory









open issues – powerlaws in nature

heavy-tailed vs. power-law

- experimentally difficult to differentiate
- ubiquitous
  - financial markets
  - Internet - degree distribution
  - brain - activity avalanches
  - solar flares / earthquakes
- deeper significance?
  - quantitative enhancements (information processing)
  - qualitative new effects? (universality)

upcoming review

"Powerlaws and Self-Organized Criticality in Theory and Nature"

[Markovic & Gros, '13]

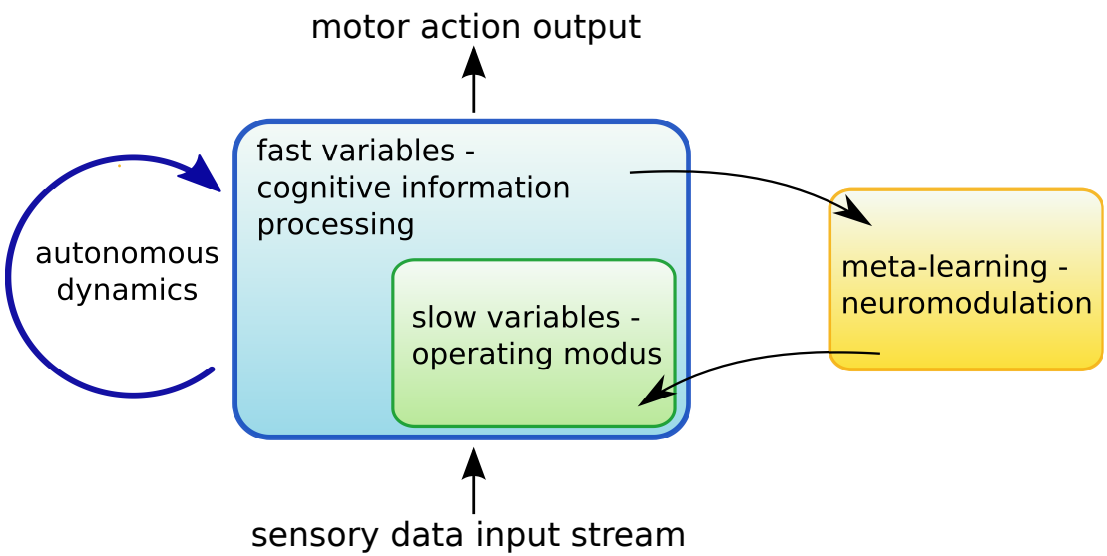
further research

transient state dynamics

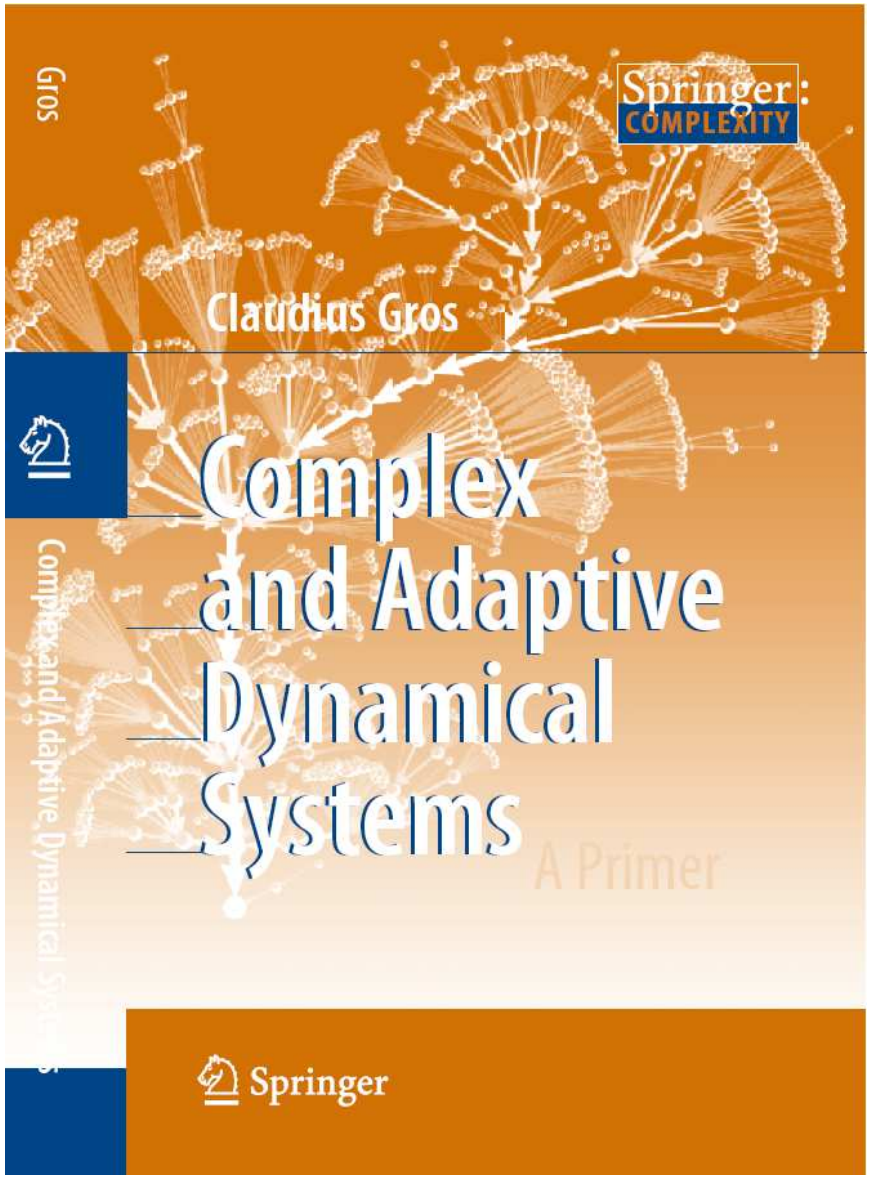
attractors not relevant for the brain
attractor ruins lead to transient firing patterns

diffusive emotional control

emotional control system works diffusively
abstracting from semantic content



graduate level textbook



Complex and Adaptive Dynamical Systems, a Primer

- The small world phenomenon in social and scale-free networks
- Phase transitions and self-organized criticality
- Life at the edge of chaos and coevolutionary avalanches
- Living dynamical systems and emotional diffusive control within cognitive system theory

(Springer, 2008, second edition 2010)