Observing scale-invariance in non-critical dynamical systems

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overview

critical vs. criticality
- criticality in statistical physics

critical boolean networks
- undersampling of phase space
- observational criticality

vertex routing models
- exactly solvable
- scaling vs. log-corrections
what does ’criticality’ mean? ________________

thermodynamic phase transition

- critical point $T_c$
- order
  - magnetization
  - superconductivity
    - ...
- control parameter
  - temperature
  - pressure
    - ...

![Diagram](image-url)
symmetry breaking

continuous phase transition $\hat{=} \text{ second order}$

disordered phase $T > T_c$
- spins point in all directions
- high rotational symmetry

ordered phase $T < T_c$
- spins point mostly in one direction
- rotational symmetry broken
scaling towards criticality

response to external perturbations

- temperature gradient $c_V(T)$
- magnetic field $\chi(T)$
- ...

power-law scaling

\[
\sim \frac{1}{|T - T_c|^\alpha}
\]

- critical exponent $\alpha$
- only when coupling to order parameter
criticality – $T = T_c$

spatial correlations: 2D Ising

$T < T_c$

$T = T_c$

$T > T_c$

typical domain size

no typical length scale

microscopic length scale
correlations

spatial (or temporal) correlation functions

\[ D(r) \equiv D(x - y) = \langle S(x)S(y) \rangle - \langle S \rangle^2 \]

- correlation length \( \xi \)
  \[ D(r) \propto \begin{cases} 
  e^{-r/\xi} & T \neq T_c \\
  r^{-\gamma} & T = T_c 
  \end{cases} \]

- typical length scale \( \xi \propto 1/|T - T_c|^z \)

scale invariance

\[ \frac{1}{r^{\alpha}} \sim \frac{1}{\tilde{r}^{\alpha}} \]
\[ \tilde{r} = ar \]
microscopic scales

example: Schrödinger equation

\[ i\hbar \frac{\partial \Psi(t, \mathbf{r})}{\partial t} = H \Psi(t, \mathbf{r}), \quad H = -\frac{\hbar^2 \Delta}{2m} - \frac{e^2}{|\mathbf{r}|} \]

- Hamilton Operator

\[ H = -E_R \left( a_0^2 \Delta + \frac{2a_0}{|\mathbf{r}|} \right), \quad E_R = \frac{me^4}{2\hbar^2}, \quad a_0 = \frac{\hbar^2}{me^2} \]

length scale
- Bohr radius \( a_0 = 0.53 \)

energy scale
- Rydberg energy \( E_R = 13.6 \text{ eV} \)

temporal scale
- \( E_R/\hbar = 3.39 \cdot 10^{15} \text{ Hz} \)
universality

scale invariance at criticality

at criticality all microscopic scales (length, energy, time, ...) become irrelevant

universality at criticality

at criticality only the symmetry of the order parameter is relevant

magnetism

rotational symmetry: O(3), SU(2), ...

superconductivity

U(1) gauge symmetry: phase $e^{i\phi}$
thermodynamics vs. dynamical systems

classical theory of phase transition

- thermal equilibrium $e^{\beta E}$
- quantum phase transition $T_c = 0$

dynamical system

$$\frac{d}{dt} x_i(t) = f_i(x_1,..,x_N;\gamma), \quad i = 1,..,N$$

- control parameter $\gamma$
- classical: real-valued variables
- the brain is a classical dynamical system
  $$\Rightarrow$$ phase transitions in dynamical systems
only in the thermodynamic limit

\[ \lim_{N \to \infty} \]

- phase transitions, criticality

\[ \frac{d}{dt} x_i(t) = f_i(x_1, \ldots, x_N; \gamma), \quad f_i \equiv f \]

- identical units

finite size scaling

system property \( \propto \begin{cases} e^{-N/\xi} & T \neq T_c \\ N^{-\gamma} & T = T_c \end{cases} \)
Kauffman nets

random boolean network

\[ \sigma_i(t+1) = f_i(\sigma_1(t),..,\sigma_K(t)), \quad i = 1,..,N \]

- boolean: \( \sigma_i = 0,1 \)
- random boolean function: \( f_i(\cdot) = 0,1 \)
- connectivity: \( K \)
  \Rightarrow control parameter

\[
\begin{array}{ccc}
K < 2 & K = 2 & K > 2 \\
\text{frozen} & \text{critical} & \text{chaotic}
\end{array}
\]
$N - K$ networks

**Order parameter**

- Activity overlap

![Diagram showing NK-networks in different states: frozen, critical, and chaotic.](image)

- In the limit $t \to \infty$, the trajectories evolve through these states.
cycles and attractors

* gene regulation networks
* brain ⇒ transient attractors
critical Kauffman net: $K = 2$

number of attractors - numerics
- $\propto \sqrt{N}$
- power law – scale invariant

number of attractors - exact
- grows faster than $N^\alpha$, $\forall \alpha$
- no power law – not scale invariant

numerics is wrong and right
- wrong measurement of intrinsic property
- corresponds to experimental observation
basins of attraction
big attractors can dominate

(fictitious) example: isolated point attractors & big attractors

- phase space volume: \( \Omega = 2^N \)
- number of point attractors, f.i.: \( 2^{\sqrt{N}} \)
- relative phase space of point attractors, including basins of attraction
  \[
  \lim_{N \to \infty} \frac{2^{\sqrt{N}}}{2^N} \to 0
  \]

big attractors may dominate phase space

- number of big attractors, f.i.: \( \sqrt{N} \)
what does ’critical’ mean?

**experimental observations**
- stochastic sampling of phase space
- $K = 2$ Kaufman net
  - $\Rightarrow$ experimental sampling $\doteq$ statistics of big attractors
  - $\Rightarrow$ scale invariance

**is the $K = 2$ Kaufman net critical?**
- **yes:** boundary between two phases
- **no:** state is not critical (?)
  - $\Rightarrow$ no universality

observing scale invariance in a dynamical system does not necessarily imply a critical dynamical state
information transmission & routing

**information transmission**
- vertex $\Rightarrow$ vertex
- neuron $\Rightarrow$ neuron
- phase space: number of vertices

**information routing**
- link (incoming) $\Rightarrow$ link (outgoing)
- phase space: number of directed links
vertex routing models

critical information routing

- (on the average) conserved quantities
  ⇒ critical dynamics
- information packages

vertex routing models
- routing of conserved information packages
routing dynamics

- random routing tables at every vertex
- quenched dynamics (fixed routing tables)

\[(1) \rightarrow (2) \rightarrow (4) \rightarrow \ldots\]
\[(2) \rightarrow (3) \rightarrow (4) \rightarrow \ldots\]

information centrality
- overlapping cyclic attractors

number of attractors passing through a given vertex
non-trivial information centrality

numerical simulations  fully connected graphs

democratic distribution of the information centrality

[Markovic & Gros, NJP ‘09]
random walk in phase space

phase space of directed links

- polynomial: $\Omega = N(N - 1)$
- mapping to random walk in $\Omega$
  - for: number of cycles
  - mean cycle length / cycle length distribution
  - but not for information centrality

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![Graph of phase space with directed links and cycle counts]
solution of vertex routing model

cycle length distribution

- number of cycles of length $L$

$$\langle C_L \rangle (N) = \frac{N((N-1)^2)!}{L(N-1)^{2L-1}((N-1)^2 + 1 - L)!}$$

- fully connected graphs

- numerical $N \leq 10^4$
quenched dynamics

- model (routing tables) fixed
- intrinsic properties

on-the-fly dynamics

- model (routing tables) generated when needed (on-the-fly)
  \[ \Rightarrow \text{random sampling of phase space, experimental sampling} \]

- solvable

\[
\langle C_L \rangle(N) \propto \sum_{i=L}^{L_{max}} \frac{((N - 1)^2)!}{(N - 1)^{2t}((N - 1)^2 + 1 - t)!}
\]
scaling of mean cycle length

\[ \langle L \rangle \sim \begin{cases} \frac{N}{\log(N)} & \text{quenched} \\ N & \text{on-the-fly} \end{cases} \]

- average cycle length

[Markovic, Schuelein & Gros, '12]
scaling of cycle number

- only quenched dynamics
- on-the-fly: only relative quantities

mean number of attractors

\[ \langle n \rangle \sim \log(N) \]
## Scaling for Critical Vertex Routing

<table>
<thead>
<tr>
<th></th>
<th>Quenched Dynamics</th>
<th>On-the-Fly Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intrinsic Properties</td>
<td></td>
<td></td>
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<tr>
<td>Experimental Sampling</td>
<td></td>
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<tr>
<td>Vertex Routing</td>
<td>(\log(N))</td>
<td>(\sqrt{N})</td>
</tr>
<tr>
<td>Mean Cycle Length</td>
<td>(N / \log(N))</td>
<td>(N)</td>
</tr>
</tbody>
</table>

### Markovian Model
- Variant without memory
two critical nets

vertex routing model

- number of cycles $\langle n \rangle$
- phase space $\Omega$

$$\Omega = N(N - 1) \sim N^2,$$
$$\langle n \rangle \propto \log(\Omega) \sim \log(N)$$

$K = 2$ Kauffman net

- number of cycles larger than any power of $N$

$$\Omega = 2^N,$$
$$\langle n \rangle \not\propto \log(\Omega) \sim N$$
critical vs. universality

**critical does not imply universality**

**critical model**
- model on the verge of a (second order) phase transition

**critical state**
- intrinsic state is critical (scale invariant)

**universality**
- the properties of a critical state are determined by symmetries and not by the specific microscopic model parameters

**thermodynamics**
- critical models have critical states

**dynamical systems**
- critical models do not necessarily show critical dynamics
typical vs. average

**typical may differ from average**

**typical property**
- obtained by random sampling of phase space
- ≜ experimental observation

**average (mean)**
- evaluating a property for all phase space
- and taking the mean

**typical and average may scale differently**

- critical boolean network (Kauffman)
- vertex routing model (information)
open issues – SOC

self-organized criticality

• driven systems (sandpile model)
  * beyond dynamical system theory

• autonomous systems (Bak & Sneppen model)
  * has global operations
  * beyond dynamical system theory
open issues – powerlaws in nature

heavy-tailed vs. power-law

• experimentally difficult to differentiate

• ubiquitous
  * financial markets
  * Internet - degree distribution
  * brain - activity avalanches
  * solar flares / earthquakes

• deeper significance?
  * quantitative enhancements (information processing)
  * qualitative new effects? (universality)

upcoming review

“Powerlaws and Self-Organized Criticality in Theory and Nature”

[Markovic & Gros, ’13]
further research

**transient state dynamics**

- attractors not relevant for the brain
- attractor ruins lead to transient firing patterns

**diffusive emotional control**

- emotional control system works diffusively
- abstracting from semantic content

![Diagram with labeled nodes:]
- fast variables - cognitive information processing
- slow variables - operating modus
- sensory data input stream
- motor action output
- autonomous dynamics
- meta-learning - neuromodulation
Complex and Adaptive Dynamical Systems, a Primer

- The small world phenomenon in social and scale-free networks
- Phase transitions and self-organized criticality
- Life at the edge of chaos and coevolutionary avalanches
- Living dynamical systems and emotional diffusive control within cognitive system theory