Conjecture concerning the fractional Hall hierarchy

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We present numerical evidence in support of a conjecture concerning the hierarchy of incompressible states that are responsible for the fractional quantum Hall effect (FQHE). We propose that for filling factors in the range $\frac{1}{3} \leq \nu \leq \frac{2}{3}$, the FQHE occurs *only* when $\nu = \nu_n = n/(2n+1)$ (or when $\nu = 1-\nu_n$) and at *no* other fractional filling factors with odd denominators. If correct, this conjecture *would imply that* important qualitative features of the hierarchy physics of the FQHE are not understood.

I. INTRODUCTION

The quantum Hall effect occurs in a two-dimensional electron gas (2D EG) whenever a discontinuity occurs in the chemical potential (i.e., whenever the system becomes incompressible) at a magnetic-field-dependent density.[1] Quantization of the kinetic energy of a cyclotron orbit leads to incompressibilities at integer values of the Landau-level filling factor, $\nu \equiv 2\pi l^2 n$, and hence to the integer quantum Hall effect.[2] [$l \equiv (\hbar c/eB)^{1/2}$, and n is the electron areal density.] The fractional[3] quantum Hall effect (FQHE) occurs at fractional values of ν and is due to the interactions between electrons that share the same quantized value of the kinetic energy, i.e., are in the same Landau level.

Our understanding of the physics responsible for the incompressibilities which lead to the FQHE grew from the observation by Laughlin[4] that simple Jastrow-type many-body wave functions could be constructed for electrons in the lowest Landau level only when $\nu = 1/q$, where q is an odd integer. Laughlin's wave functions are nondegenerate exact zero-energy eigenstates of the many-electron Hamiltonian for short-ranged repulsive interactions in which electrons interact only if their relative angular momentum[5] is less than $q-1$. For $\nu < 1/q$, there are many zero-energy eigenstates of such a Hamiltonian, and electrons can be added to the system without any pair of electrons ever occupying a state of relative angular momentum less than $q-1$. For $\nu > 1/q$, this is no longer possible,[1,6] so that the chemical potential jumps when the filling factor crosses $1/q$. The realistic[7] effective interaction between electrons in a 2D EG is sufficiently similar to the hard-core models, for which the Laughlin wave functions become exact, that the chemical potential jumps, and hence the FQHE, still occurs[8] when $\nu = 1/q$.

The physical origin of the FQHE at $\nu = 1/q$, and (by invoking the exact particle-hole symmetry which exists within a single Landau level) at $\nu = 1-1/q$, is thus clearly understood. However, the FQHE occurs at a large number of additional fractional filling factors.[9] To explain these observations, an intuitive hierarchy picture[10,11] was developed in which the additional filling factors were associated with Laughlin states formed by the fractionally charged quasiparticles[4] of the primary Laughlin states. This picture can accommodate a FQHE at any rational filling factor with an odd denominator. While it has not been possible to provide a convincing mathematical justification of the hierarchy picture, there is considerable evidence[12] that it captures much of the physics responsible for the occurrence of the FQHE at additional filling factors. It explains the *odd-denominator rule.* (The FQHE *does* occur only at fractional filling factors with odd denominators, provided that there is complete spin polarization.[13]) The composite nature of the hierarchy's incompressible ground states is shared with various proposed[14] microscopic incompressible many-body wave functions. In addition, many aspects of the hierarchy picture are supported by small-system exact-diagonalization studies.[15] However, the hierarchy picture has not been extremely successful in predicting the filling factors at which strong FQHE's will occur. For example, it has long[16,17] been recognized that the hierarchy picture does not capture the fact that excited states that do not occupy pair states of unit relative angular momentum are available for the quasiholes of the $\nu = \frac{1}{3}$ state but not for the quasiparticles. Moreover, for $\frac{1}{3} < \nu < \frac{2}{3}$, the FQHE has been observed only[18] when $\nu = \nu_n \equiv n/(2n+1) = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots$, or when $\nu = 1-\nu_n$. The FQHE does not occur at many other values of ν, where the qualitative hierarchy picture would suggest equally strong or stronger effects. In this paper we report on an exact-diagonalization study of the FQHE for electrons on the surface of a sphere.[10] Most of our calculations are based on a hard-core model in which electrons repel each other only when they have unit relative angular momentum. As explained above, this is the *ideal* model for the $\nu = \frac{1}{3}$ and $\frac{2}{3}$ FQHE's, which are well understood. We find convincing numerical evidence in support of the conjecture that for this model the FQHE occurs only at the filling factors where it is observed experimentally, namely, at $\nu = \nu_n$, and at $\nu = 1-\nu_n$. Further calculations with more realistic interaction models give similar results and suggest that the FQHE will occur only at these filling fractions for any physically realistic interaction.

Our paper is organized as follows. In Sec. II we discuss the construction of the usual hierarchy picture for

electrons on the surface of a sphere. We emphasize the finite-size corrections to the filling factor, at which a hierarchy state is expected to occur, and point out the aspect of this construction whose validity is most uncertain. In Secs. III and IV we report on our results for the ground-state energy and the chemical-potential discontinuities, respectively, for the hard-core model. We expect that this model will exhibit the FQHE physics most clearly for $\frac{1}{3} \leq \nu \leq \frac{2}{3}$ as explained above. In addition, it has the advantage that the thermodynamic limit of both these quantities can be evaluated analytically for this model for ν outside the range $\frac{1}{3} \leq \nu \leq \frac{2}{3}$. This fact is helpful in accessing the importance of finite-size corrections to our results. The filling-factor dependence of the chemical potential itself is discussed in Sec. V.

By using particle-hole symmetry, we are able to derive an exact expression for $\lim_{\nu \to 1} \mu(\nu)$ which is valid for any effective electron-electron interaction. This result is used to place a limit of the size of the chemical potential jumps, which is consistent with the system having thermodynamic stability at all filling factors. As discussed in Sec. VI, our numerical results for the hard-core model show that the chemical-potential jump at $\nu = \nu_n$ decreases slowly with n, and that the chemical potential must decrease with increasing density between incompressible filling factors. We conclude that phase separation into regions with filling factors ν_{n+1} and ν_n will occur for $\nu_n < \nu < \nu_{n+1}$ in the hard-core model. In Sec. VII we compare our numerical results for the hard-core model with the results for the Coulomb model. We show that, when expressed in appropriate units, the finite-size estimates of chemical-potential jumps are very similar for the two cases, and argue that the qualitative conclusions that we draw from our study of the hard-core model apply for any physically realistic interaction. In Sec. VIII we conclude by suggesting a new direction for attempts in understanding the physics of the fractional Hall hierarchy.

II. HALDANE'S MAGIC TABLE

Haldane's[10] hierarchical classification of states for electrons on the surface of a sphere predicts incompressible ground states when the number of single-particle states in the lowest Landau level, N_L, is related to the number of electrons N_e and the filling factor ν by

$$N_L \equiv \nu^{-1} N_e + K(\nu) , \tag{1}$$

where $\nu = p/q$ is a fraction with an odd denominator, N_e is a multiple of p, and $K(\nu)$ is not necessarily an integer independent of N_e, which can be viewed as a finite-size correction [$k(\nu=1)=0$]. Table I shows the values of N_L at which incompressible states are expected for various values of N_e. To construct[15] this "magic" table, we need three rules.

(i) Starting from any family of finite-size states obeying Eq. (1), another family exists, where for each N_e we get

$$\begin{aligned} N_L' &= N_L + 2(N_e - 1) = (\nu^{-1} + 2) N_e + K(\nu) - 2 \\ &\equiv (\nu')^{-1} N_e + K(\nu') . \end{aligned} \tag{2}$$

This rule corresponds to excluding an angular momentum channel and is the process which generates the Laughlin[4] states, starting from a full Landau level. In terms of wave functions, this process is equivalent to multiplication by a Jastrow factor $\prod_{i<j} (z_i - z_j)^2$.

(ii) Particle-hole symmetry guarantees that, for any family of states obeying Eq. (1), another family exists with N_e replaced by $N_L - N_e'$, so that

TABLE I. (Magic Table). This table lists the values N_L of Landau-level degeneracy, at which an incompressible state at filling ν is predicted by the hierarchy scheme for a given number of particles N_e. Only a selected range of N_e and fillings ν are listed. The best way to read this table is from the top down. For a fixed $\nu = p/q$, an entry occurs at e very p'th value of N_e, and the increment in N_L is given by q. Note the "twins" of filling factors, e.g., the state at $N_e = 4$ and $N_L = 10$ can be viewed either as a $\nu = \frac{1}{3}$ or as a $\nu = \frac{4}{7}$ state. Details are discussed in the text.

N_e	$\frac{1}{3}$	$\frac{4}{11}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{7}{11}$	$\frac{2}{3}$	N_e
1	1										1
2	4		2							4	2
3	7			3				7			3
4	10	8	7		4		10		8	7	4
5	13					13					5
6	16		12	10				12		10	6
7	19										7
8	22	19	17		13		17			13	8
9	25			17				17			9
10	28		22			22				16	10
11	31								19		11
12	34	30	27	24	22		24	22		19	12
13	37										13
14	49		32							22	14
15	49			31		31		27			15

$$N_L=(1-\nu)^{-1}N_e'-\nu K(\nu)/(1-\nu)$$
$$\equiv(1-\nu)^{-1}N_e'+K(1-\nu)\ . \tag{3}$$

(iii) The last rule is the magic one, which allows the iterative construction of the full hierarchy. It can be motivated in several superficially different ways.[10,11,16] For example, we can see from Eq. (3) that the incompressible state at filling factor $1-\nu'$ can be formed by adding $\nu' N_e/(1-\nu')+K(1-\nu')$ holes to the *full* Landau level, where $K(1-\nu')$ is given by Eq. (3): $N_L=(1-\nu')^{-1}N_e+K(1-\nu')=N_e+\nu' N_e/(1-\nu')+K(1-\nu')$. This state can be viewed as the ν' incompressible state of N_L-N_e holes in the full Landau level. This suggests[19] that the same number of quasiholes added to a family at ν would form a ν' incompressible state in these quasiholes. More generally, we may define, for a given family of states at ν and ν', another family at

$$N_L''=\left[\frac{1}{\nu}N_e+K(\nu)\right]\pm\left[\frac{\nu'}{1-\nu'}N_e+K(1-\nu')\right]$$
$$=\left[\frac{1}{\nu}\pm\frac{\nu'}{1-\nu'}\right]N_e+[K(\nu)\pm K(1-\nu')]$$
$$\equiv(\nu'')^{-1}N_e+K(\nu'')\ . \tag{4}$$

In the hierarchy picture,[11] these states are ν' incompressible states formed by the quasiholes (+) or quasiparticles (−) of the ν incompressible state. Rule (ii) is exact, and compelling arguments can be advanced in favor of rule (i). But rule (iii), up to the present, has a more intuitive justification. The full hierarchy can be generated by iterating these rules.

The strong fractions, $\nu_n=n/(2n+1)$ defined above, are obtained by our starting from a Laughlin state at $\nu=\frac{1}{3}$, using (ii) and only quasiparticle states in (iii). Note that these states correspond to deeper and deeper hierarchy levels. Naively, one might expect that the energy scales for the low-energy excitations would decrease quickly with increasing n. Instead, as we will show in this paper, we find that the size of the chemical-potential jumps decreases slowly with n.

It is interesting to consider what happens if one mixes different "hierarchy trees." For example, let us consider what is, in the hierarchy language, a $\frac{1}{5}$ state of quasiparticles in a $\frac{1}{3}$ state. The hierarchy rules give a family with $N_L=(11/4)N_e-3$. This is the $\frac{4}{11}$ state. In the hierarchy scheme, it is at the same level depth as the $\frac{2}{5}$ state. Naively, one might have expected that it should then be seen experimentally more or less as strongly as the $\frac{2}{5}$ state. Instead, it has not yet been observed with certainty.[18] We will show that this family is not stabilized by short-range interactions, and that it is probably not stabilized by any realistic interaction.

Table I is best read from the top down. For $\nu=p/q$, an entry occurs at every pth N_e, and N_L incremented by q. At a given N_e, each N_L would appear twice in the table if all fractions were listed. For example, the N_L values associated with $\nu=\nu_{n'}$ and with $\nu=1-\nu_n$ are the same at $N_e=n'(n+1)$. Similarly, the values associated with $\nu=1-\nu_{n'}$ and $\nu=\nu_n$ are the same at $N_e=(n'+1)n$. Thus, as N_e increases, finite-size approximations to $\nu_{n'}<\frac{1}{2}$ are twinned with finite-size approximations to states at ever-deeper levels of the hierarchy. This sequence of "twin"-filling factors approaches $\frac{1}{2}$ from above.

III. GROUND-STATE ENERGIES FOR THE HARD-CORE MODEL

We have used a Lanczos method to examine the hard-core model on the sphere. In this model, two particles repel each other only if they are in a state of relative angular momentum 1. The strength of this *hard-core* repulsion is V_1. We have computed the ground-state energy as a function of N_L for all even values of N_e from $N_e=4$ to 12, and in each case up to the largest values of N_L for which we were able to perform the calculation.[20] The ground-state energy is zero whenever N_L exceeds $3N_e-2$; just as in the thermodynamic limit, the ground-state energy per electron, ϵ_0, is zero for $\nu\le\frac{1}{3}$. Using particle-hole symmetry, we are readily shown that

$$\epsilon_0=2V_1(2\nu-1)/\nu \tag{5}$$

for $\frac{2}{3}\le\nu\le 1$. Thus the hard-core model can only yield chemical-potential jumps for $\frac{1}{3}\le\nu\le\frac{2}{3}$ (the chemical potential is zero for $\nu<\frac{1}{3}$ and $4V_1$ for $\nu>\frac{2}{3}$.)

A physical state is said to be incompressible if a jump in the chemical potential occurs at the density for which it is the ground state. (For a fixed magnetic field, the density can be parametrized by the Landau-level filling factor.) For the energy, as a function of density, a chemical-potential jump produces a kink. In Fig. (1) we have plotted the ground-state energy, E_0, in units of V_1, as a function of the Landau-level degeneracy, N_L, for a fixed-particle number, $N_e=4$, 6, 8, 10, and 12. A striking feature of Fig. (1) is the apparent regularity of the curves.

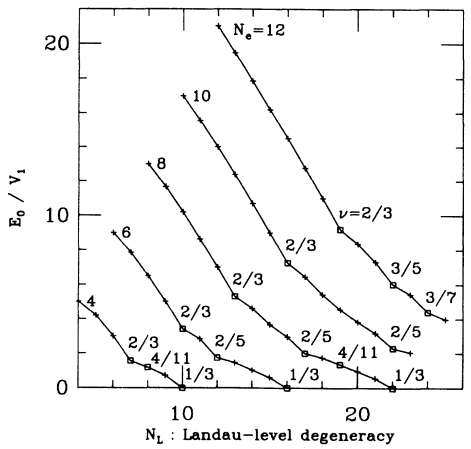

FIG. 1. The ground-state energy, in units of the strength V_1 of the hard-core model on the sphere for various number of particles, N_e, the Landau-level degeneracy, N_L. The number of flux quanta through the sphere is N_L-1. The squares denote the fillings for which an incompressible state is expected according to the magic table. There is clear evidence for a kink in the slope of energy versus N_L for $\nu=\frac{2}{5}$ and $\frac{3}{7}$, but not for $\nu=\frac{4}{11}$.

(Physically, this might be a consequence of having only one energy scale, V_1.) This regularity gives us some confidence in the interpretation of Fig. (1). We find that for all but one fraction listed in Table I, a well-pronounced kink shows up in E_0 (note that the $\frac{3}{7}$ state is very pronounced, too). All the kinks are associated with fractions of the form $\nu_n = n/(2n+1)$, or $\nu = 1-\nu_n$, as discussed above. In particular, no kink appears for the $\nu = \frac{4}{11}$ family. According to Table I, finite-size realizations of this family occur within the range of N_L values we studied for $N_e = 4$ and 8.

IV. CHEMICAL-POTENTIAL DISCONTINUITIES

The chemical potential is defined as the derivative of the total ground-state energy, with respect to the particle number:

$$\mu = \frac{d}{dN_e} E_0 = \frac{d}{dN_e}(N_e \epsilon_0) = \epsilon_0 + \frac{\partial(\nu\epsilon_0)}{\partial \nu} \ . \tag{6}$$

To estimate μ for a finite system, the derivatives in the above formula are replaced by finite differences. The jump in the chemical potential is given by[21]

$$\begin{aligned} \mu_+ - \mu_- &= \left.\frac{\partial(\nu\epsilon_0)}{\partial \nu}\right|_+ - \left.\frac{\partial(\nu\epsilon_0)}{\partial \nu}\right|_- \\ &= (N_L - 1)\epsilon_0(N_L) - N_L \epsilon_0(N_L - 1) \\ &\quad - N_L \epsilon_0(N_L + 1) + (N_L + 1)\epsilon_0(N_L) \\ &= -N_L[\epsilon_0(N_L + 1) - 2\epsilon_0(N_L) + \epsilon_0(N_L - 1)] \ . \end{aligned} \tag{7}$$

We have used Eq. (7) to estimate the jumps in the chemical potential. The results are shown in Fig. 2 as a function of inverse particle number, and the numerical values are listed in Table II. The greatest number of data points are available for $\nu = \frac{2}{3}$. The results behave smoothly, as a function of system size; an extrapolation to the thermodynamic limit can be done with an accuracy of about 10%. This smooth behavior for $\nu = \frac{2}{3}$ gives us some confidence in the extrapolation of the data for the other fractions, where fewer data points are available. The chemical-potential jump for the $\frac{1}{3}$ state differs from that

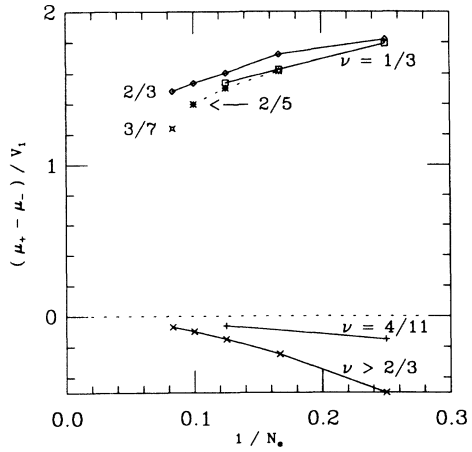

FIG. 2. The chemical-potential discontinuity, $\mu_+ - \mu_-$, Eq. (7), as a function of inverse particle number N_e^{-1}. For $\nu = \frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{5}$, and $\frac{3}{7}$, the chemical-potential discontinuity estimates are similar and extrapolate to a finite value $\sim V_1$ as N_e^{-1} goes to zero. For $\frac{2}{3} < \nu < 1$, $\mu_+ - \mu_-$ extrapolates quadratically to zero, indicating a state with infinite compressibility. For $\nu = \frac{4}{11}$, $\mu_+ - \mu_-$ is negative and small. The plot indicates a linear extrapolation to zero as a function of N_e^{-1}. This would be indicative of state unstable against phase separation.

of the $\frac{2}{3}$ state because of a finite-size correction, but closes in with increasing N_e. In the thermodynamic limit, they must be identical, because of particle-hole symmetry.

We can observe the $\frac{2}{5}$ state for three system sizes. The corresponding chemical-potential jumps are so close to those of the $\frac{1}{3}$ state, that they are practically indistinguishable on the scale we have used. The same would have been true for the $\frac{3}{5}$ state, which we did not include in Fig. 2 in order to avoid overcrowding (See Table II). Most remarkably, $\mu_+ - \mu_-$ is also very large for the $\frac{3}{7}$ state; indeed, it is of the same order as the chemical-potential jumps for the $\frac{1}{3}$ and $\frac{2}{5}$ states. We will explore further the consequences of this result below.

Before we discuss our results for $\nu = \frac{4}{11}$, we comment on our results for $\nu > \frac{2}{3}$. As mentioned previously, it follows from Eq. (5) that, in the thermodynamic limit,

TABLE II. The jump in the chemical potential, $\mu_+ - \mu_-$ in units of V_1, calculated according to Eq. (7) for all the fractions predicted by Table I. Note that all the positive entries are roughly of the same magnitude and only weakly dependent on the particle number N_e. This is in contrast to the data for the $\frac{4}{11}$ state, which are negative and strongly decreasing in magnitude for the larger system. Note, also, that for $N_e = 12$, the jump in chemical potential is actually slightly larger for the $\frac{3}{7}$ state than for the $\frac{3}{5}$ state, although the latter occurs at an earlier level in the hierarchy scheme.

N_e	$\frac{1}{3}$	$\frac{4}{11}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{3}{5}$	$\frac{2}{3}$	N_e
4	+1.792	−0.151				+1.818	4
6	+1.623		+1.614			+1.722	6
8	+1.538	−0.064	+1.504			+1.602	8
10			+1.397			+1.536	10
12				+1.238	+1.260	+1.482	12

$\mu=4V_1$ for any $\nu>\frac{2}{3}$. If the chemical potential is well defined (does not have a jump) at some filling factor in the thermodynamic limit, then, from Eq. (7), the finite-size estimate of $\mu_+-\mu_-$ reduces at large N_e to

$$\mu_+-\mu_-=\nu\kappa^{-1}/nN_e \ , \tag{8}$$

where n is the areal density, and the inverse compressibility is

$$\kappa^{-1}=n^2\frac{d\mu}{dn} \ . \tag{9}$$

For $\nu>\frac{2}{3}$, $\mu=4V_1$ is independent of density, and $\kappa^{-1}=0$. The infinite compressibility for $\nu>\frac{2}{3}$ in this model is expected and is associated with the macroscopic ground-state degeneracy. For $\nu>\frac{2}{3}$, we see in Fig. 2 that our finite-size estimate for $\mu_+-\mu_-$ decreases as N_e^{-2}, rather than N_e^{-1}, because $\kappa^{-1}=0$. For $\nu=\frac{4}{11}$, on the other hand, we see in Fig. 2 that the finite-size estimate for $\mu_+-\mu_-$ appears to decrease linearly towards zero with N_e^{-1}, as we would expect from Eq. (8) whenever κ^{-1} is finite. The apparent negative compressibility at $\nu=\frac{4}{11}$ implies that the hard-core-model system is thermodynamically unstable at this filling factor.

V. EXCITATION GAPS AND A SUM RULE

In Fig. 3 we have plotted finite-size estimates ($N_e=8$ and 12) for the chemical potential itself, calculated according to Eq. (6). In the thermodynamic limit, for large N_e, it will go from $4V_1$, for $\frac{2}{3}<\nu<1$, to zero, for $0<\nu<\frac{1}{3}$. [This follows from Eq. (5) and (6).] Figures 3 shows that this limiting behavior is being approached outside the filling-factor range of primary interest, $\frac{1}{3}<\nu<\frac{2}{3}$. In Fig. 3 the chemical-potential estimates have been plotted at $N_L+0.5$, if $\epsilon_0(N_L)$ and $\epsilon_0(N_L+1)$ have been used to make the estimates. The dashed lines join two estimates, at $N_L+0.5$ and -0.5, whenever an incompressible state is expected at N_L, according to the magic table. In fact, we easily recognize the large jumps associated with $\nu=\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{3}{5}$, and $\frac{2}{3}$. (Note, that $\mu_\pm$ is at $N_L\mp0.5$, since the filling factor decreases with increasing N_L.)

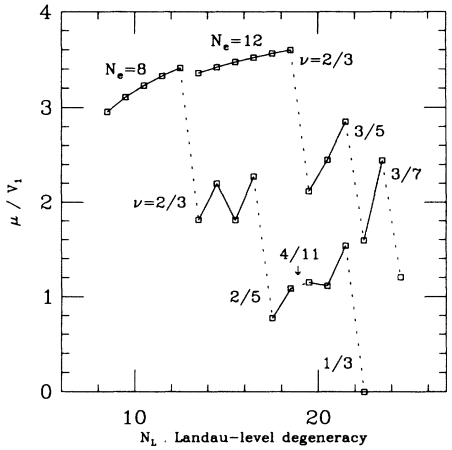

FIG. 3. The finite-size estimates for the chemical potential, calculated according to Eq. (6). For systems with 8 and 12 particles, μ is given in units of V_1 as a function of N_L. Note the large drops in μ across an N_L, where an incompressible fraction is predicted by Table I. The only notable exception occurs for eight particles and $N_L=19$, where the $\nu=\frac{4}{11}$ would have been predicted to occur.

We see in Fig. 3 that (for the hard-core model) the large chemical-potential jumps at incompressible filling factors require the chemical potential to fall with increasing filling factor between jumps to avoid exceeding $4V_1$ for $\nu>\frac{2}{3}$. The large gaps require regions of filling factor that are thermodynamically unstable and would lead to phase separation. This result is a specific example of a general relationship between gap sizes and thermodynamic stability, which has some profound consequences.

It follows, from making a particle-hole transformation within the lowest Landau level,[21] that

$$(1-\nu)\epsilon_0(1-\nu)=\nu\epsilon_0(\nu)+(1-2\nu)\epsilon_0(\nu=1) \ , \tag{10}$$

and hence, whenever the chemical potential is defined, that

$$\mu(1-\nu)+\mu(\nu)=2\epsilon_0(\nu=1)$$
$$\equiv\frac{1}{\pi l^2}\int d^2\mathbf{r}\, V_{ee}(\mathbf{r})(1-e^{-r^2/2l^2}) \ . \tag{11}$$

In particular, for any reasonable electron-electron interaction, $V_{ee}(\mathbf{r})$, $\mu(\nu)$ must vanish in the low-density limit ($\nu\to0$), so that $\mu(\nu=1)=2\epsilon_0(\nu=1)$. It follows from integrating Eq. (9) that (at fixed field)

$$\sum_{i=1}^{N}\Delta_i=2\epsilon_0(\nu=1)-\sum_{i=1}^{N}\int_{\nu_i-1}^{\nu_i}d\nu\frac{2\pi l^2}{\nu^2}\kappa^{-1}(\nu) \ , \tag{12}$$

where $\nu_0=0$, and $\nu_{N+1}=1$. The chemical potential is allowed to have jump discontinuity at ν_i for $i=1,2,\ldots,N$ and $\Delta_i=\mu(\nu_i^+)-\mu(\nu_i^-)$. If the system is thermodynamically stable at all densities ($\Delta_i\geq0$, $\kappa^{-1}\geq0$), then

$$\sum_{i=1}^{N}|\Delta_i|\leq2\epsilon_0(\nu) \ . \tag{13}$$

In particular, the hard-core model is conjectured to have a denumerably infinite set of incompressible filling factors (in the above notation, $N\to\infty$). In order to avoid instability, $\sum_n\Delta(\nu_n)$ must converge, and therefore $\Delta(\nu_n)$ must decrease faster than n^{-1}. We find no indication of such a rapid decrease in our numerical results, and, in fact, $\Delta(\nu_3)\sim\Delta(\nu_1)$. We expect that thermodynamic instabilities occur in the hard-core model.

In the light of the above, it is interesting to consider the possibility that uniform fluid states may be unstable even with long-ranged Coulomb interaction between the particles (this point was discussed some time ago, by Trugman and Kivelson[22]). If we follow the argument[4,10,11] that the quasiparticles of the hierarchy picture carry charge $1/q$ in a $\nu=p/q$ state, and that they interact like charged particles, their interaction energy should scale like q^{-2}. This quasiparticle interaction energy provides the scale for the quasiparticle excitation gap of the incompressible state. The latter should equal

$[(\mu_+ - \mu_-)|_{\nu=p/q}]/q$, since we need q quasiparticle to add one real particle. This would imply that the chemical-potential discontinuities decrease approximately as $1/q$ and would not sum to a finite value. For a Coulomb system, this does not necessarily imply instability, since $\epsilon_0(\nu=1)$ has an infinite contribution from the Hartree energy. The remaining "exchange-correlation" contribution to the energy is given at $\nu=1$ by

$$2\epsilon_{\rm xc}(\nu=1)=\frac{-1}{\pi l^2}\int d^2\mathbf{r}\frac{e^2}{\epsilon r}e^{-r^2/2l^2}\ . \tag{14}$$

If we include the Hartree contribution, the (wave-vector dependent) compressibility is given by

$$\kappa^{-1}(q)=n^2\frac{2\pi e^2}{q}+\kappa_{\rm xc}^{-1}(q)\ , \tag{15}$$

where $\kappa_{\rm xc}^{-1}(q)$, the exchange-correlation contribution is calculated neglecting the Hartree energy. For $ql\ll 1$, $\kappa_{\rm xc}^{-1}(q)$ approaches $n^2 d\mu_{\rm xc}/dn = n^2 d^2/dn^2[n\epsilon_{\rm xc}(n)]$. Equations (14) and (12) then show that $\kappa_{\rm xc}^{-1}(q\to 0)$ will tend to be negative for most values of filling factors. This negativity will not lead to phase separation, however, since the Hartree term in Eq. (15) is large and positive as $q\to 0$. If, however, $d\mu_{\rm xc}/dn$ is sufficiently large, that

$$\frac{2\pi e^2}{q^*}+\frac{d\mu_{\rm xc}}{dn}=0 \tag{16}$$

for $q^* l \ll 1$, the uniform system is unstable, and we can expect inhomogeneities on a length scale ($\nu=2\pi nl^2$) of

$$L\sim\frac{2\pi}{q^*}=2\pi l\frac{-d\mu_{\rm xc}}{d\nu}\Big/\frac{e^2}{\epsilon l}\ . \tag{17}$$

Thus, when $-d\mu_{\rm xc}/d\nu$ is much larger than $(e^2/\epsilon l)$ for filling factors that lie between two incompressible values, we may expect that the system will separate into domains of size $L\gg l$, which have one of the incompressible filling factors. Our numerical results for the Coulomb case (see below) suggest that this condition can be satisfied for filling factors near $\frac{1}{2}$. This spontaneously generated inhomogeneity may be responsible for the transport anomalies which have been found near these filling factors.[13]

VI. TWO SCENARIOS

In view of the above-stated sum rule, we want to examine in more detail the surprising fact that the estimated chemical-potential discontinuities are large and decrease slowly with n for $\nu_n=n/(2n+1)=\frac{1}{3},\frac{2}{5},\frac{3}{7},\ldots$. Two possible interpretations arise.

First, the finite-size corrections are very large, and the $\mu_+-\mu_-$ of the thermodynamic limit might have little to do with the finite-system estimates presented thus far. One might draw support for this scenario from Fig. 3. For the case of eight particles, the chemical potential is $\sim 25\%$ off the thermodynamic value of $4V_1$ for $\frac{2}{3}<\nu<1$. This difference is still $\sim 20\%$ for 12 particles. One might, therefore, argue that the finite-size corrections for $\mu_+-\mu_-$ could be even larger, and that the sum rule might be satisfied without any thermodynamic instabilities.

Second, $\mu_+-\mu_-$ might, indeed, decrease very slowly or even be constant, as one goes down the hierarchy. Note that a similar behavior has been observed for the Coulomb model.[15] (We will present some data on the Coulomb potential model in Sec. VII.) One might also draw support for this scenario from Fig. 3. If we compare the plots for the systems with eight and twelve particles, we find that the chemical potential increases more and more steeply with decreasing filling factor in between incompressible fractions, as N_e increases. This would imply that, for $\nu_n<\nu<\nu_{n+1}$, the ground state of the hard-core model becomes more and more unstable against phase separation into two phases with filling factor ν_n and ν_{n+1}, respectively, as n increases. While we favor this later interpretation, we want to emphasize that firm conclusions cannot be made on the basis of these or other finite-size calculations.

VII. RELATION TO THE COULOMB POTENTIAL MODEL

It is well established that the FQHE is a very robust effect, in that its occurrence is quite insensitive to the precise nature of the interactions.[21] Indeed, the hard-core model, which is the ideal model for which the Laughlin state becomes exact at $\nu=\frac{1}{3}$, is quite different from realistic interactions. The robustness of the FQHE, at least at $\nu=\frac{1}{3}$, results from the fact that its occurrence depends only on the repulsion strength being greater for relative angular momentum 1 (which cannot be avoided for $\nu>\frac{1}{3}$) than for relative angular momentum 3. The simplest realistic model is the Coulomb model, in which the finite thickness of the 2D EG layer is neglected and the electrons interact with a $1/r$ interaction. We have repeated some of the above calculations for this model in order to check that the conclusions drawn above are also sensitive to the detailed nature of the interactions.

The Coulomb-potential model has a series of closely spaced energy scales, one for each angular momentum channel. This slows down the convergence of the Lanczos method and limits the maximum system size we are able to evaluate. (For the hard-core model, the Lanczos convergence is very fast, since only one energy scale is present.) In Table III we report the chemical-potential discontinuities [$\mu_+-\mu_-$, as defined by Eq. (7)] for some of the fractions predicted by the magic table, in units of V_1-V_3. [In the thermodynamic limit, $V_1-V_3=\sqrt{\pi}(\frac{1}{4}-\frac{15}{96})$. The values for V_1 and V_3, which we have used for the estimates given in Table III, are corrected for the finite curvature of the spheres.] We believe that these units best express the physics of the incompressible states. Indeed, in these units, the difference between the estimates from the hard-core and Coulomb model is less than 10%. Again, as for the hard-core model, the magnitude of the chemical-potential discontinuity is very similar for the $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{2}{5}$ states. (We have not been able to calculate the $\frac{3}{7}$ state.)

Again, as we found for the hard-core model, the chemical-potential discontinuities for the $\nu=\frac{4}{11}$ state are negative and a factor 30–50 smaller than those for the

TABLE III. The jump in the chemical potential for the Coulomb-potential model, $\mu_+ - \mu_-$ in units of $V_1 - V_3$, calculated according to Eq. (7) for some of the fractions predicted by Table I. Note that all the positive entries are roughly of the same magnitude and only weakly dependent on the particle number N_e. This is in contrast to the data for the $\frac{4}{11}$ state, which are negative and strongly decreasing in magnitude for the larger system. These results for the Coulomb-potential model are very similar to those obtained for the hard-core model presented in Table II.

N_e	$\frac{1}{3}$	$\frac{4}{11}$	$\frac{2}{5}$	$\frac{2}{3}$	N_e
4	+1.651	−0.050		+1.523	4
6	+1.607		+1.539	+1.637	6
8	+1.607	−0.030	+1.457	+1.484	8
10				+1.621	10
12				+1.540	12

$\nu=\frac{1}{3}$, $\frac{2}{3}$, and $\frac{2}{5}$ states. Furthermore, $\mu_+ - \mu_-$ decreases strongly (for $\nu=\frac{4}{11}$) in magnitude between the cluster with four and that with eight particles. All of this is in complete agreement with the results for the hard-core model, and substantiates the conjecture that interactions in the higher angular momentum channels just renormalize quantitatively the properties of the ground state (and the energy gap) for $\frac{1}{3} \leq \nu \leq \frac{2}{3}$. The effect of the higher angular momentum channels on the ground-state properties in this range of filling factors is very weak indeed. Furthermore, the data presented in Table III support our hypotheses that the $\frac{4}{11}$ state may not be stabilized by *any* physical interaction[23] and, more generally, that chemical-potential jumps appear only at $\nu=\nu_n$, or at $\nu=1-\nu_n$.

VIII. CONCLUSIONS

We believe that these results provide convincing evidence in support of the conjecture that, for $\frac{1}{3} \leq \nu \leq \frac{2}{3}$ and complete spin polarization, the hard-core model produces a FQHE only when $\nu=\nu_n$, or when $\nu=1-\nu_n$. Since the strongest FQHE's must be associated with the lowest relative angular momenta (i.e., the channel with the strongest repulsion), this conjecture would explain why only these fractions are seen experimentally.[24] Whether or not other FQHE's occur for realistic electron-electron interactions is less certain, but we believe that none do. In particular, our results suggest that the FQHE at $\nu=\frac{4}{11}$ does *not* occur for the $1/r$ interaction. More importantly, we believe that our results suggest another point of view from which may prove it possible to obtain a clearer understanding of the "hierarchy" states. One interpretation of our results for the hard-core model is that, in many-body states representing isotropic fluids, the minimum possible probability of finding electron pairs in a state of unit ($m=1$) relative angular momentum has a cusp when ν crosses ν_n and *not* at any other filling factor. For $n=1$ ($\nu=\frac{1}{3}$), this statement can be established with some rigor as a mathematical property of the analytic wave functions for the lowest-Landau-level electrons.[1] We believe that this is also true for $n>1$, and that this *is* the essence of the FQHE at $\nu=\nu_n$, although we have attempted, without success, to construct a rigorous proof.

In this connection, it is worth noting that the analog of the simple Laughlin state for an n-component fermion system occurs at $\nu=\nu_n$, i.e., ν_n is the highest filling factor at which zero-energy states occur for hard-core models. It is this fact that motivates the microscopic trial wave functions of Ref. 14 for the ground state at $\nu=\nu_n$, which involves separating the electrons into n components (e.g., spin flavors in the proposal by Yoshioka, MacDonald, and Girvin,[14] and n orbital Landau levels in the work by Jain[14]). Our numerical results suggest that in the process of projecting from n orbital Landau levels to one (or requiring antisymmetry when the n spin flavors become indistinguishable), the Laughlin-type cusp on an n-component system is not entirely lost.

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