## Contents

1 Network Theory ......................................................... 1
   1.1 Properties of Real-World Networks .......................... 1
      1.1.1 The Small World Effect ............................... 1
      1.1.2 Basic Graph-Theoretical Concepts ..................... 4
      1.1.3 Network Degree Distribution ......................... 9
   1.2 Spectral Properties ........................................... 11
      1.2.1 Graph Laplacian ........................................ 14
   1.3 Percolation in Generalized Random Graphs .................. 17
      1.3.1 Graphs with Arbitrary Degree Distributions ........... 17
      1.3.2 Probability Generating Function Formalism ............ 23
      1.3.3 Distribution of Component Sizes ....................... 25
   1.4 Robustness of Random Networks ............................. 29
   1.5 Small-World Models .......................................... 34
   1.6 Scale-Free Graphs .............................................. 36
   Exercises ............................................................ 41
   Further Reading ..................................................... 43
   References .......................................................... 43

2 Bifurcations and Chaos in Dynamical Systems .................... 45
   2.1 Basic Concepts of Dynamical Systems Theory ................ 45
   2.2 Fixpoints, Bifurcations and Stability ....................... 52
      2.2.1 Fixpoints Classification and Jacobian ................. 53
      2.2.2 Bifurcations and Normal Forms ....................... 56
      2.2.3 Hopf Bifurcations and Limit Cycles ................... 59
   2.3 Global Bifurcations ........................................... 62
      2.3.1 Infinite Period Bifurcation ............................ 64
      2.3.2 Catastrophe Theory ..................................... 65
      2.3.3 Rate Induced Tipping ................................... 69
   2.4 Logistic Map and Deterministic Chaos ....................... 71
      2.4.1 Colliding Attractors ................................... 77
   2.5 Dynamical Systems with Time Delays ........................ 78
      2.5.1 Distributed Time Delays ............................... 81
### 3 Dissipation, Noise and Adaptive Systems

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Chaos in Dissipative Systems</td>
<td>87</td>
</tr>
<tr>
<td>3.1.1 Phase Space Contraction and Expansion</td>
<td>87</td>
</tr>
<tr>
<td>3.1.2 Strange Attractors and Dissipative Chaos</td>
<td>91</td>
</tr>
<tr>
<td>3.1.3 Partially Predictable Chaos</td>
<td>94</td>
</tr>
<tr>
<td>3.2 Adaptive Systems</td>
<td>97</td>
</tr>
<tr>
<td>3.2.1 Conserving Adaptive Systems</td>
<td>102</td>
</tr>
<tr>
<td>3.3 Diffusion and Transport</td>
<td>106</td>
</tr>
<tr>
<td>3.3.1 Random Walks, Diffusion and Lévy Flights</td>
<td>106</td>
</tr>
<tr>
<td>3.3.2 Markov Chains</td>
<td>110</td>
</tr>
<tr>
<td>3.4 Stochastic Systems</td>
<td>113</td>
</tr>
<tr>
<td>3.4.1 Langevin Equation</td>
<td>113</td>
</tr>
<tr>
<td>3.4.2 Stochastic Calculus</td>
<td>116</td>
</tr>
<tr>
<td>3.5 Noise-Controlled Dynamics</td>
<td>117</td>
</tr>
<tr>
<td>3.5.1 Fokker–Planck Equation</td>
<td>117</td>
</tr>
<tr>
<td>3.5.2 Stochastic Escape</td>
<td>119</td>
</tr>
<tr>
<td>3.5.3 Stochastic Resonance</td>
<td>121</td>
</tr>
<tr>
<td>Exercises</td>
<td>125</td>
</tr>
</tbody>
</table>

### 4 Self Organization

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Interplay Between Diffusion and Reaction</td>
<td>129</td>
</tr>
<tr>
<td>4.1.1 Travelling Wavefronts in the Fisher Equation</td>
<td>131</td>
</tr>
<tr>
<td>4.1.2 Sum Rule for the Shape of the Wavefront</td>
<td>135</td>
</tr>
<tr>
<td>4.1.3 Self-Stabilization of Travelling Wavefronts</td>
<td>136</td>
</tr>
<tr>
<td>4.2 Interplay Between Activation and Inhibition</td>
<td>138</td>
</tr>
<tr>
<td>4.2.1 Turing Instability</td>
<td>139</td>
</tr>
<tr>
<td>4.2.2 Pattern Formation</td>
<td>140</td>
</tr>
<tr>
<td>4.2.3 Gray–Scott Reaction Diffusion System</td>
<td>142</td>
</tr>
<tr>
<td>4.3 Collective Phenomena and Swarm Intelligence</td>
<td>147</td>
</tr>
<tr>
<td>4.3.1 Phase Transitions in Social Systems</td>
<td>147</td>
</tr>
<tr>
<td>4.3.2 Collective Decision Making and Stigmergy</td>
<td>149</td>
</tr>
<tr>
<td>4.3.3 Collective Behavior and Swarms</td>
<td>152</td>
</tr>
<tr>
<td>4.3.4 Opinion Dynamics</td>
<td>154</td>
</tr>
<tr>
<td>4.4 Car Following Models</td>
<td>156</td>
</tr>
<tr>
<td>4.4.1 Linear Flow and Carrying Capacity</td>
<td>157</td>
</tr>
<tr>
<td>4.4.2 Self-Organized Traffic Congestions</td>
<td>158</td>
</tr>
<tr>
<td>Exercises</td>
<td>160</td>
</tr>
</tbody>
</table>

Further Reading: 162

References: 162
5 Information Theory of Complex Systems ........................................... 163
  5.1 Probability Distribution Functions ........................................... 163
     5.1.1 Law of Large Numbers ................................................. 167
     5.1.2 Bayesian Statistics .................................................... 169
     5.1.3 Statistical Binning .................................................... 171
     5.1.4 Time Series Characterization ....................................... 173
  5.2 Entropy and Information ...................................................... 177
     5.2.1 Maximal Entropy Distributions .................................... 182
     5.2.2 Minimal Entropy Principle ......................................... 183
     5.2.3 Mutual Information ................................................... 185
     5.2.4 Kullback-Leibler Divergence ...................................... 190
  5.3 Complexity Measures ......................................................... 193
     5.3.1 Complexity and Predictability .................................... 195
     5.3.2 Algorithmic and Generative Complexity ......................... 198
Exercises .................................................................................. 200
Further Reading ....................................................................... 202
References ............................................................................... 202

6 Self-Organized Criticality .......................................................... 203
  6.1 Landau Theory of Phase Transitions .................................... 203
  6.2 Criticality in Dynamical Systems ......................................... 209
     6.2.1 1/f Noise ..................................................................... 213
  6.3 Cellular Automata ............................................................... 214
     6.3.1 Conway’s Game of Life ................................................. 215
     6.3.2 Forest Fire Model ......................................................... 216
  6.4 Sandpile Model and Self-Organized Criticality ....................... 218
     6.4.1 Absorbing Phase Transitions ....................................... 221
  6.5 Random Branching Theory ................................................ 222
     6.5.1 Branching Theory of Self-Organized Criticality .......... 222
     6.5.2 Galton–Watson Processes ........................................... 228
  6.6 Application to Long-Term Evolution ..................................... 230
Exercises .................................................................................. 237
Further Reading ....................................................................... 238
References ............................................................................... 239

7 Random Boolean Networks ....................................................... 241
  7.1 Introduction ......................................................................... 241
  7.2 Random Variables and Networks ......................................... 243
     7.2.1 Boolean Variables and Graph Topologies ....................... 243
     7.2.2 Coupling Functions ..................................................... 246
     7.2.3 Dynamics .................................................................... 248
  7.3 Dynamics of Boolean Networks .......................................... 249
     7.3.1 Flow of Information Through a Network ....................... 250
     7.3.2 Mean-Field Phase Diagram ........................................ 252
     7.3.3 Bifurcation Phase Diagram ......................................... 254
     7.3.4 Scale-Free Boolean Networks .................................... 258
7.4 Cycles and Attractors .................................................. 260
  7.4.1 Quenched Boolean Dynamics ................................. 261
  7.4.2 K = 1 Kauffman Network .................................... 264
  7.4.3 K = 2 Kauffman Network .................................... 266
  7.4.4 K = N Kauffman Network .................................... 267
7.5 Applications ......................................................... 270
  7.5.1 Living at the Edge of Chaos .................................. 270
  7.5.2 Yeast Cell Cycle ................................................ 271
  7.5.3 Application to Neural Networks .............................. 274
Exercises ................................................................. 276
Further Reading ......................................................... 277
References ............................................................... 278

8 Darwinian Evolution, Hypercycles and Game Theory ............... 279
  8.1 Introduction ......................................................... 279
  8.2 Mutations and Fitness in a Static Environment ................. 282
  8.3 Deterministic Evolution .......................................... 286
    8.3.1 Evolution Equations ........................................ 287
    8.3.2 Beanbag Genetics: Evolution Without Epistasis ........ 290
    8.3.3 Epistatic Interactions and the Error Catastrophe ...... 292
  8.4 Finite Populations and Stochastic Escape ...................... 297
    8.4.1 Adaptive Climbing Under Strong Selective Pressure ... 298
    8.4.2 Adaptive Climbing vs. Stochastic Escape ............... 302
  8.5 Prebiotic Evolution ............................................... 304
    8.5.1 Quasispecies Theory ........................................ 304
    8.5.2 Hypercycles and Autocatalytic Networks ................. 306
  8.6 Macroeconomy and Species Competition ....................... 310
  8.7 Coevolution and Game Theory .................................. 313
    8.7.1 Tragedy of the Commons .................................. 320
Exercises ................................................................. 322
Further Reading ......................................................... 324
References ............................................................... 324

9 Synchronization Phenomena ............................................ 327
  9.1 Frequency Locking ................................................. 327
  9.2 Coupled Oscillators and the Kuramoto Model ................... 329
  9.3 Synchronization in the Presence of Time Delays ............... 337
  9.4 Synchronization Mechanisms ..................................... 339
    9.4.1 Aggregate Averaging ....................................... 340
    9.4.2 Causal Signaling ............................................ 344
  9.5 Piecewise Linear Dynamical Systems ............................ 348
  9.6 Synchronization Phenomena in Epidemics ....................... 351
    9.6.1 Continuous Time SIRS Model .............................. 356
Exercises ................................................................. 357
Further Reading ......................................................... 359
References ............................................................... 359
10 Complexity of Machine Learning

10.1 Computation Units
   10.1.1 Structured Units
   10.1.2 The XOR Problem

10.2 Recurrent Neural Networks
   10.2.1 Random Matrix Theory
   10.2.2 Criticality in Recurrent Networks

10.3 Neural Differential Equations
   10.3.1 Residual Nets

10.4 Gaussian Processes
   10.4.1 Multivariate Gaussians
   10.4.2 Correlated Stochastic Functions
   10.4.3 Machine Learning with Neural Tangent Kernels

10.5 Attention Induced Information Routing
   10.5.1 Transformer

Exercises
Further Reading
References

11 Solutions

11.1 Solutions to the Exercises of Chap. 1
11.2 Solutions to the Exercises of Chap. 2
11.3 Solutions to the Exercises of Chap. 3
11.4 Solutions to the Exercises of Chap. 4
11.5 Solutions to the Exercises of Chap. 5
11.6 Solutions to the Exercises of Chap. 6
11.7 Solutions to the Exercises of Chap. 7
11.8 Solutions to the Exercises of Chap. 8
11.9 Solutions to the Exercises of Chap. 9
11.10 Solutions to the Exercises of Chap. 10

References

Index