Chiral symmetry restoration in linear and nonlinear **O**(N) models within the auxiliary field method

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Introduction

- The global $\mathbf{SU(N}_f\mathbf{)}_L \times \mathbf{SU(N}_f\mathbf{)}_R$ symmetry of QCD with $\mathbf{N}_f$ massless quark flavours is spontaneously broken at the ground state to $\mathbf{SU(N}_f\mathbf{)}_V$
- Lattice simulations indicate a restoration of chiral symmetry at temperatures of ~150 MeV
- At nonzero temperature it is possible to investigate the thermodynamics of QCD applying low-energy effective theories like the O(4) linear and nonlinear model in 3+1 dimensions.
- For $\mathbf{N}_f = \mathbf{2} : \mathbf{O(4)} \sim \mathbf{SU(2)}_L \times \mathbf{SU(2)}_R,\ \mathbf{O(3)} \sim \mathbf{SU(2)}_V$
- The O(4) linear and nonlinear models in 3+1 dimensions are used to describe the dynamics of three pions and one σ-field

Introduction

- At very low temperatures only the pions are excited

  → in the nonlinear O(N) model: the *σ*-field is eliminated as a dynamical degree of freedom by sending its mass to infinity

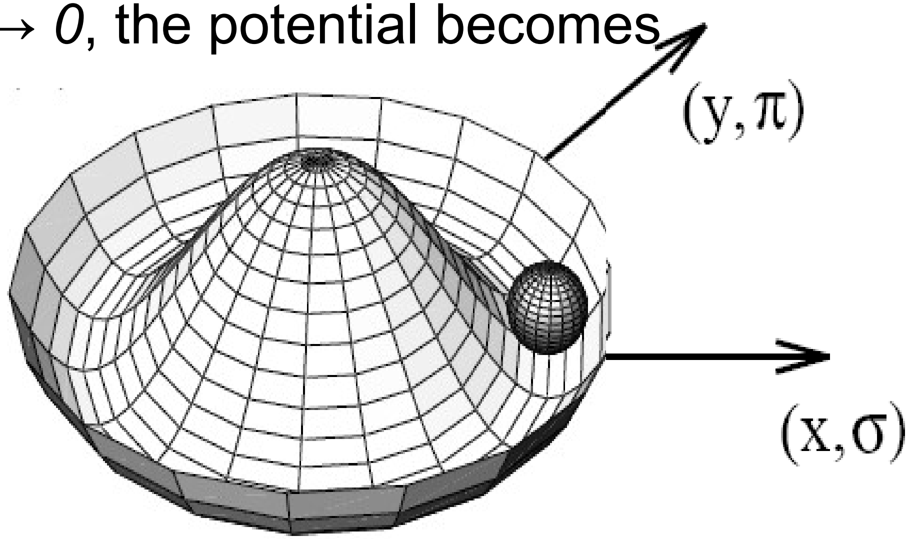
- This can be realised with an infinitely large coupling constant in the potential of the linear O(N) model:

$$U(\Phi) = \frac{1}{\varepsilon}\left(\Phi^2 - f_\pi^2\right)^2 \quad \text{where} \quad \Phi = \begin{pmatrix} \sigma \\ \vec{\pi} \end{pmatrix}$$

- Sending the coupling to infinity, ε → *0*, the potential becomes infinitely steep in the σ-direction

- The dynamics of the fields is constrained on the " chiral circle"

$$\boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2 = \mathbf{f}_\pi^2$$



The O(N) Model

- The generating functional of the linear O(N) model is given by

$$Z_L(\varepsilon, h) = N \int \mathcal{D}\alpha\, \mathcal{D}\Phi\, \exp\left(\int_0^{1/T} d\tau \int_V d^3x\, \mathcal{L}_{\sigma\text{-}\alpha}\right)\ ,\quad \beta = 1/T$$

  with the Lagrangian:

$$\mathcal{L}_{\sigma\text{-}\alpha} = \frac{1}{2}\partial_\mu \Phi^t \partial^\mu \Phi - U(\Phi, \alpha)\ ,\quad U(\Phi, \alpha) = \frac{i}{2}\alpha(\Phi^2 - v_0^2) + \frac{N\varepsilon}{8}\alpha^2 - h\sigma\ ,$$

  where $\Phi^t = (\sigma, \pi_1, \ldots \pi_{N-1})$ and α is an auxiliary field

- Integrating out α the generating functional reads

$$Z_L(\varepsilon, h) = \int \mathcal{D}\Phi\, \exp\left(\int_0^{1/T} d\tau \int_V d^3x\, \mathcal{L}_\sigma\right)\ ,$$

  with

$$\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu \Phi^t \partial^\mu \Phi - \frac{1}{2N\varepsilon}\left(\Phi^2 - v_0^2\right)^2 + h\sigma\ ,$$

  where $1/\varepsilon$ is the coupling constant

The O(N) Model

- In the *nonlinear* version of the O(N) model the fields are constrained by the condition $\Phi^2 = \upsilon_0^2$
- The nonlinear O(N) model is obtained by studying the limit ε → *0* :

$$Z_{NL}(h) = \lim_{\varepsilon \to 0+} Z_L(\varepsilon, h) = \lim_{\varepsilon \to 0+} N \int \mathcal{D}\alpha\, \mathcal{D}\Phi \, \exp\left[\int_0^{1/T} d\tau \int_V d^3x\, \mathcal{L}_{\sigma\text{-}\alpha}\right]$$

$$= \int \mathcal{D}\Phi\, \delta(\Phi^2 - \upsilon_0^2) \exp\left[\int_0^{1/T} d\tau \int_V d^3x \left(\frac{1}{2}\partial_\mu \Phi^t \partial^\mu \Phi + h\sigma\right)\right] ,$$

where $\delta(\Phi^2 - \upsilon_0^2)$ is identified with the mathematically well-defined, (i.e. convergent) form of the representation of the *δ*-function:

$$\delta(\Phi^2 - \upsilon_0^2) = \lim_{\varepsilon \to 0+} N \int \mathcal{D}\alpha \exp\left\{- \int_0^\beta d\tau \int_V d^3x \left[\frac{i}{2}\alpha(\Phi^2 - \upsilon_0^2) + \frac{N\varepsilon}{8}\alpha^2\right]\right\}$$

The Model

- The effective potential is computed using the CJT-Formalism

$$V = U(\phi) + \frac{1}{2}\int_k [\ln G^{-1}(k) + D^{-1}(k;\phi)G(k) - 1] \; + V_2(\phi, G)$$

- Shifting σ and *α* around their vacuum expectation values

$$\alpha \longrightarrow \; \alpha_0 + \alpha\,(x,\tau)\,,\; \sigma \longrightarrow \phi + \sigma\,(x,\tau)$$

produces a mixing term $-i\alpha\sigma\phi$ in the tree-level potential:

$$U = \frac{i}{2}(\alpha_0 + \alpha)(\sigma^2 + \pi_i^2 + 2\sigma\phi \; + \phi^2 - v_0^2)$$

$$+ \frac{N\varepsilon}{8}(\alpha_0 + \alpha)^2 - h(\phi + \sigma)$$

➡ This mixing term renders the mass matrix non-diagonal in the fields σ and *α*

The Lagrangian

- Performing a further shift of *α* the mixing term can be eliminated

$$\alpha \longrightarrow \alpha - 4\frac{i\phi}{N\varepsilon}\,\sigma$$

- The resulting Lagrangian reads

$$\mathcal{L}_{\sigma\text{-}\alpha} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi_i\partial^\mu\pi_i - \frac{\sigma^2}{2}\left(i\alpha_0 + 4\frac{\phi^2}{N\varepsilon}\right) - \frac{\pi_i^2}{2}\left(i\alpha_0\right)$$
$$- \frac{\alpha^2}{8}N\varepsilon - \frac{i}{2}\alpha(\sigma^2 + \pi_i^2) - \frac{2\ \phi\sigma}{N\varepsilon}(\sigma^2 + \pi_i^2)$$
$$- \frac{i}{2}\alpha_0(\phi^2 - v_0^2) - N\frac{\varepsilon}{8}\alpha_0^2 + h\phi$$

The Lagrangian

- The inverse tree-level propagators

$$D_i^{-1}(k;\phi,\alpha_0) = -k^2 + m_i^2 \; ; \qquad i = \sigma \; , \; \vec{\pi}$$

$$D_\alpha^{-1}(k;\phi,\alpha_0) = m_\alpha^2 = \frac{N\varepsilon}{4}$$

- The tree-level masses

$$m_\sigma^2 = \; i\alpha_0 + 4\frac{\phi^2}{N\varepsilon} \; , \qquad m_\pi^2 = i\alpha_0$$

- Features of the shift:

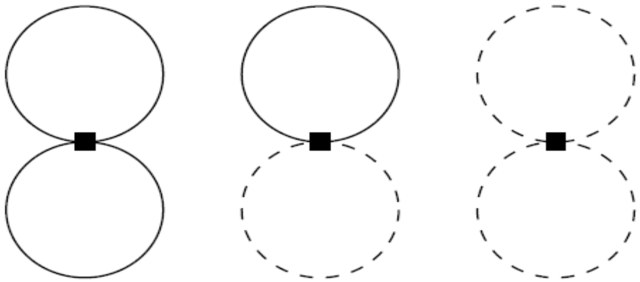
1) The Jacobian is unity

2) The σ-mass becomes infinitely heavy for ε→ *0*

⟹ the σ -field is not dynamical in the nonlinear limit

The effective potential

- Restricting to double-bubble type diagrams the contribution of $V_2 = 0$ to the effective potential vanishes



- The CJT-effective potential only contains tree-level and one-loop terms

$$V_{eff}(\phi, \alpha_0, G_\sigma, G_\pi, G_\alpha) = \frac{i}{2}\alpha_0(\phi^2 - f_\pi^2) + \frac{\varepsilon}{2}\alpha_0^2 - h\phi$$

$$+\frac{1}{2}\sum_{i=\sigma,\vec{\pi},\alpha}\int_k [\ln G_i^{-1}(k) + D_i^{-1}(k;\phi,\alpha_0)G_i(k) - 1]$$

- We use the imaginary-time formalism to compute quantities at nonzero *T*, our notation is:

$$\int_k G(k) = \int_k d^4k \frac{1}{-k^2 + M^2} = T\sum_{n=-\infty}^{\infty}\int \frac{d^3\vec{k}}{(2\pi)^3} G(i\omega_n, \vec{k})$$

The effective potential

- From the stationary conditions for the effective potential

$$\frac{\delta V_{eff}}{\delta \phi} = 0, \quad \frac{\delta V_{eff}}{\delta \alpha_0} = 0, \quad \frac{\delta V_{eff}}{\delta G_i(k)} = 0; \quad i = \sigma, \pi, \alpha$$

one derives two condensate equations

$$h = i\alpha_0 \phi + \frac{4\phi}{N\varepsilon} \int_k G_\sigma(k) \ ,$$
$$i\alpha_0 = \frac{2}{N\varepsilon} \left[\phi^2 - v_0^2 + \int_k G_\sigma(k) + (N-1) \int_k G_\pi(k) \right]$$

and the so-called Dyson-Schwinger equations for the full propagators

$$G_i(k) = -k^2 + M_i^2 \ = D_i^{-1}(k; \phi, \alpha_0)$$
$$M_i^2 \ = m_i^2 + \Sigma_i = m_i^2 \ .$$

The effective potential

- α is a Lagrange-multiplier and not an independent dynamical degree of freedom
- Substituting α by

$$i\alpha_0 = \frac{2}{N\varepsilon}\left[\phi^2 - v_0^2 + \int_k G_\sigma(k) + (N-1)\int_k G_\pi(k)\right]$$

we obtain the usual “Mexican hat” shape for the effective potential:

$$V_{eff}(\phi; G_\sigma, G_\pi) = \frac{1}{2N\varepsilon}(\phi^2 - v_0^2)^2 - h\phi$$

$$+ \frac{1}{2}\sum_{i=\sigma,\vec{\pi}}\int_k \left[\ln G_i^{-1}(k) + D_i^{-1}(k;\phi)G_i(k) - 1\right]$$

$$+ \frac{1}{2N\varepsilon}\left(\int_k G_\sigma(k) + (N-1)\int_k G_\pi(k)\right)^2 .$$

The gap equations

- The corresponding gap equations read:

$$h = \phi \left[M_\pi^2(\varepsilon, h) + \frac{4}{N\varepsilon} \int_k G_\sigma(k) \right] \ ,$$

$$M_\sigma^2(\varepsilon, h) = M_\pi^2(\varepsilon, h) + \frac{4\phi^2}{N\varepsilon} \ ,$$

$$M_\pi^2(\varepsilon, h) = \frac{2}{N\varepsilon} \left[\phi^2 - v_0^2 + \int_k G_\sigma(k) + (N-1) \int_k G_\pi(k) \right]$$

- In the large-N limit the gap equations simplify:

$$h = \phi M_\pi^2(\varepsilon, h) \ ,$$

$$M_\sigma^2(\varepsilon, h) = M_\pi^2(\varepsilon, h) + \frac{4\phi^2}{\varepsilon N} \ ,$$

$$M_\pi^2(\varepsilon, h) = \frac{2}{N\varepsilon} \left[\phi^2 - v_0^2 + N \int_k G_\pi(k) \right]$$

Counterterm Regularisation

- Matsubara summation of the thermal tadpole integral gives

$$\int_k G_i(k) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + M_i^2}} \left[1 + \frac{2}{\exp\left(\sqrt{\vec{k}^2 + M_i^2}/T\right) - 1} \right]$$

- Using the residue theorem the vacuum contribution can be rewritten as

$$Q_\mu = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + M^2}} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + M^2} \; .$$

- This term exhibits logarithmic and quadratic divergences and has to be regularized accordingly

$$Q_\mu = \frac{1}{(4\pi)^2} \left[M^2 \ln(\frac{M^2}{\mu^2}) - M^2 + \mu^2 \right]$$

Results

- The numerical results are presented for

  1) N = 4: $\Phi^t = (\sigma, \pi_1, \pi_2, \pi_3)$

  2) The O(N) linear and nonlinear model

  3) Explicit symmetry breaking and the chiral limit

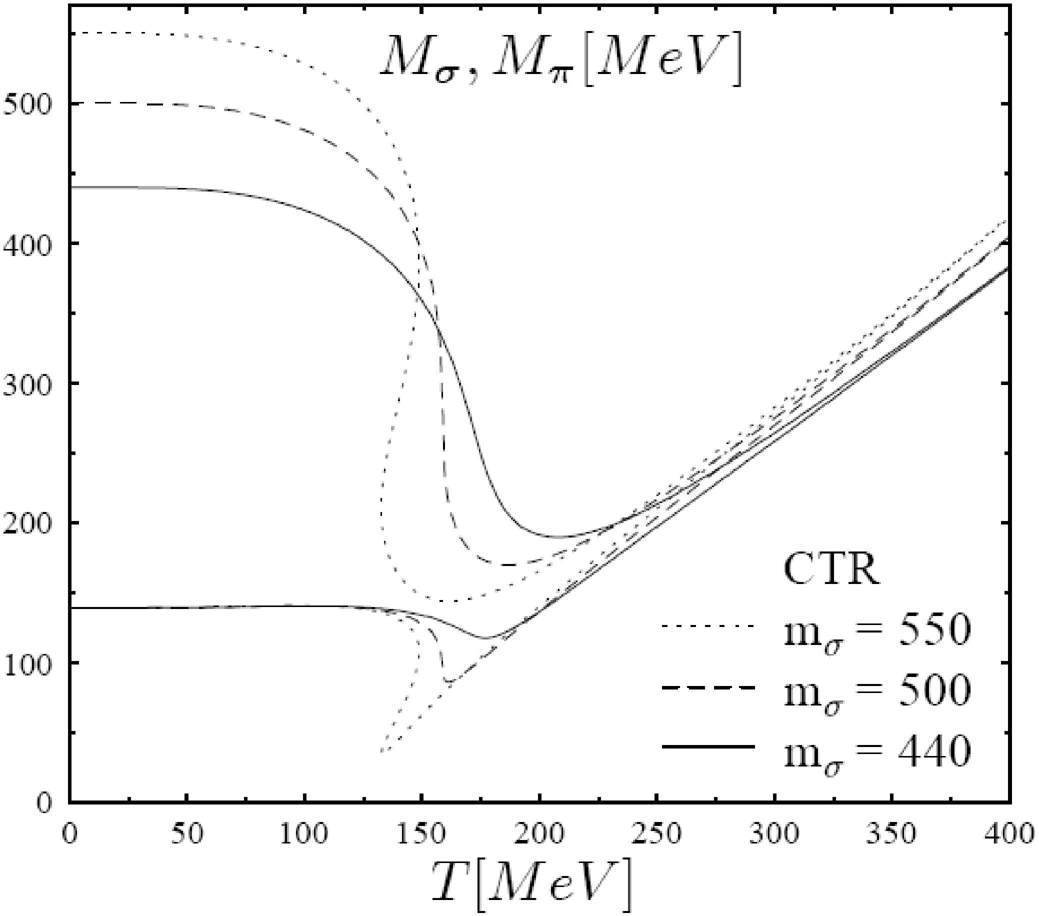
  4) Counter-term regularization method (CTR) and trivial regularization (TR)

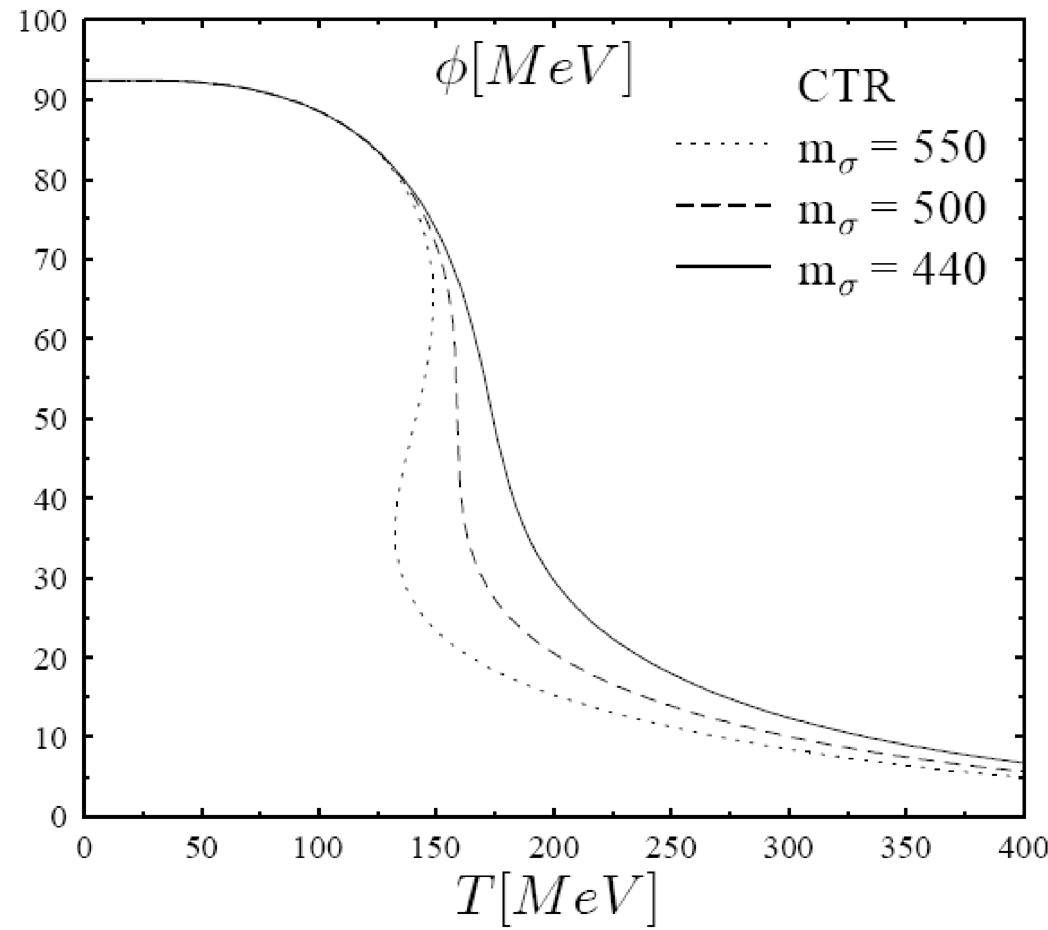
  5) Large-N limit in the counter-term regularization method (LN-CTR) and trivial regularization (LN-TR)

- Common observation: The smaller m_π and/or the larger the difference between m_π and m_σ the more likely the pion propagation becomes tachyonic at nonzero temperature

Results

Explicitly broken symmetry in the linear case

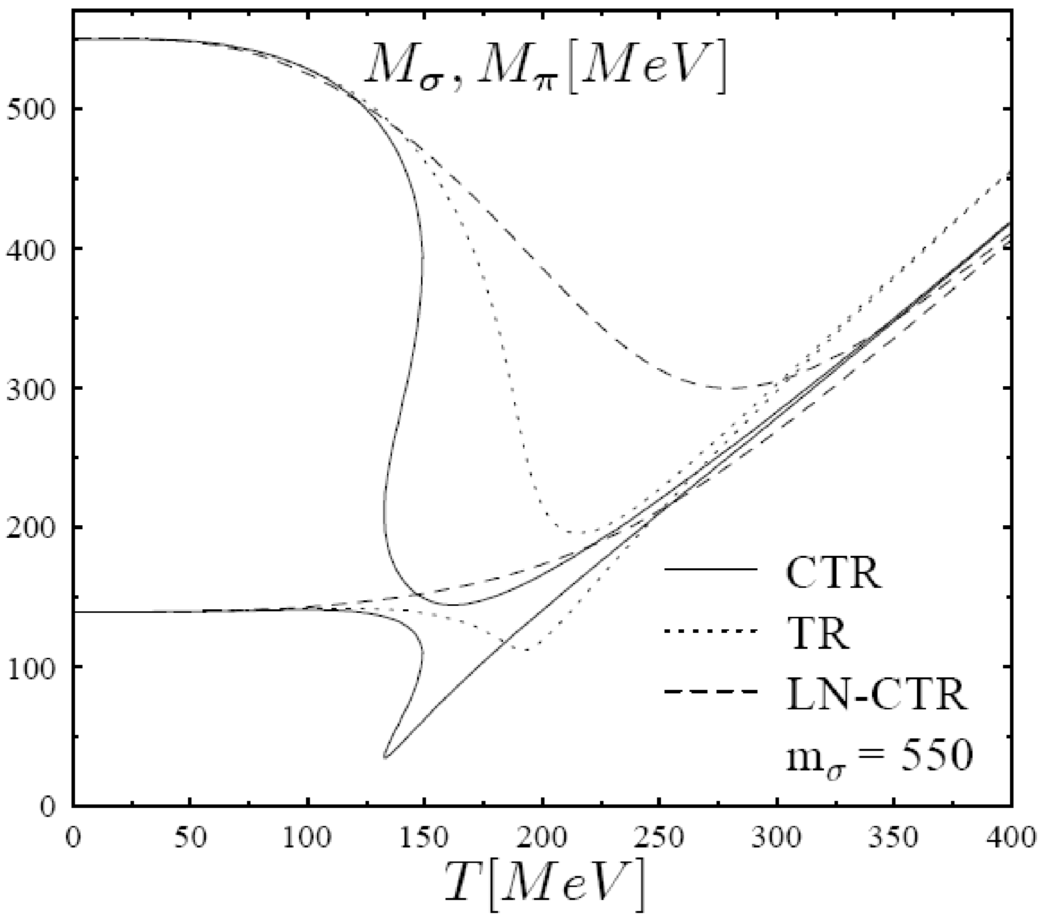


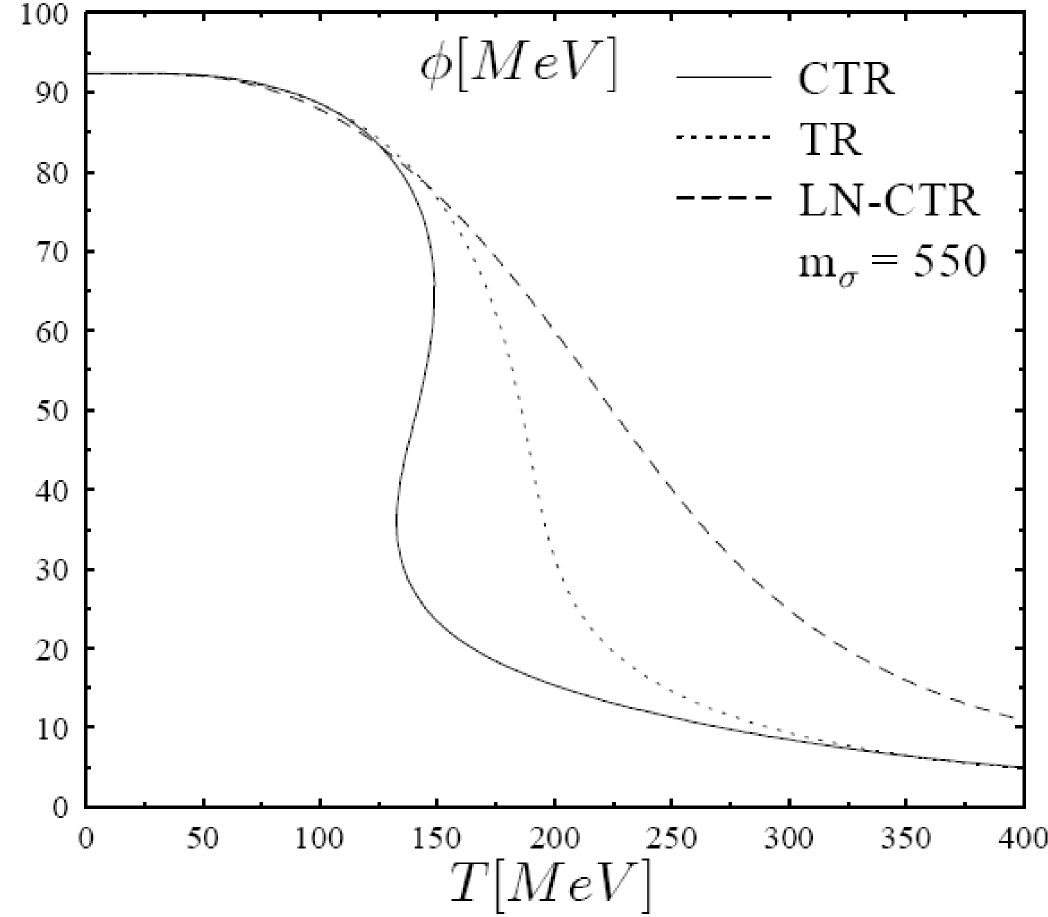


- Crossover phase transition for $m_\sigma = 440\ MeV$
- Second order phase transition for $m_\sigma = 500$ MeV
- First order phase transition for $m_\sigma > 500\ MeV$

Results

Explicitly broken symmetry in the linear case

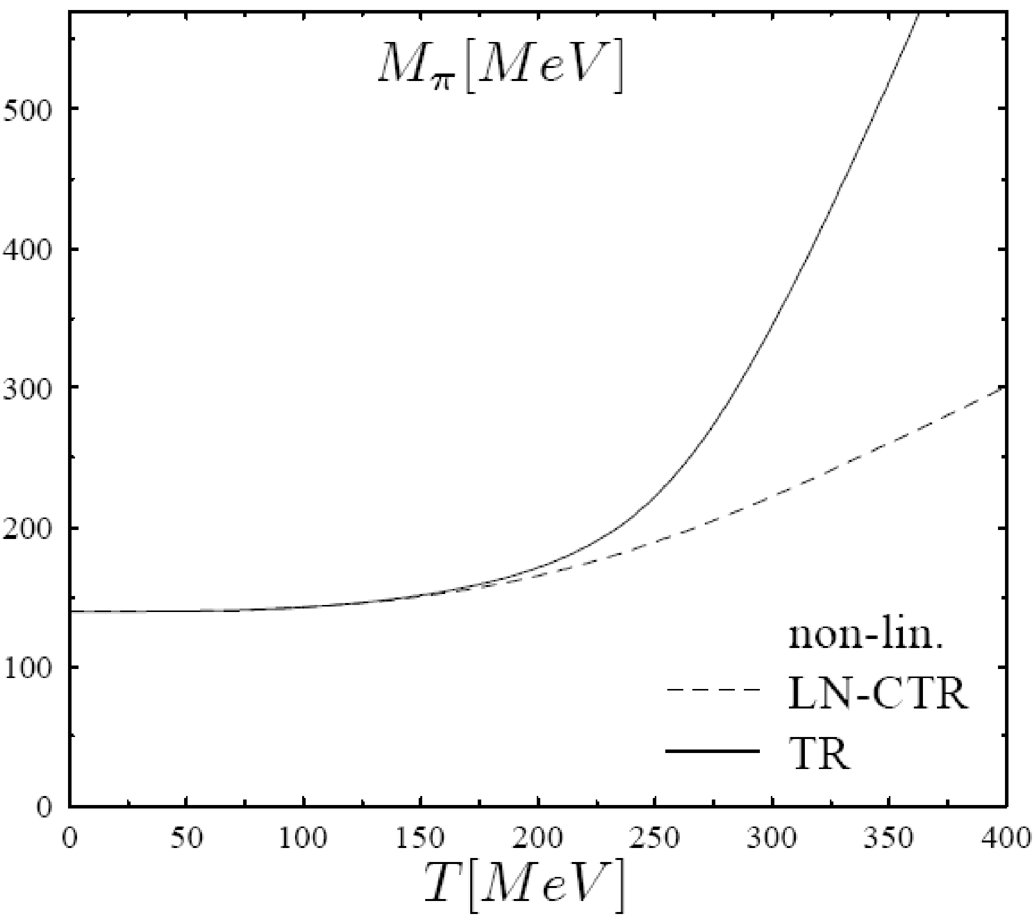


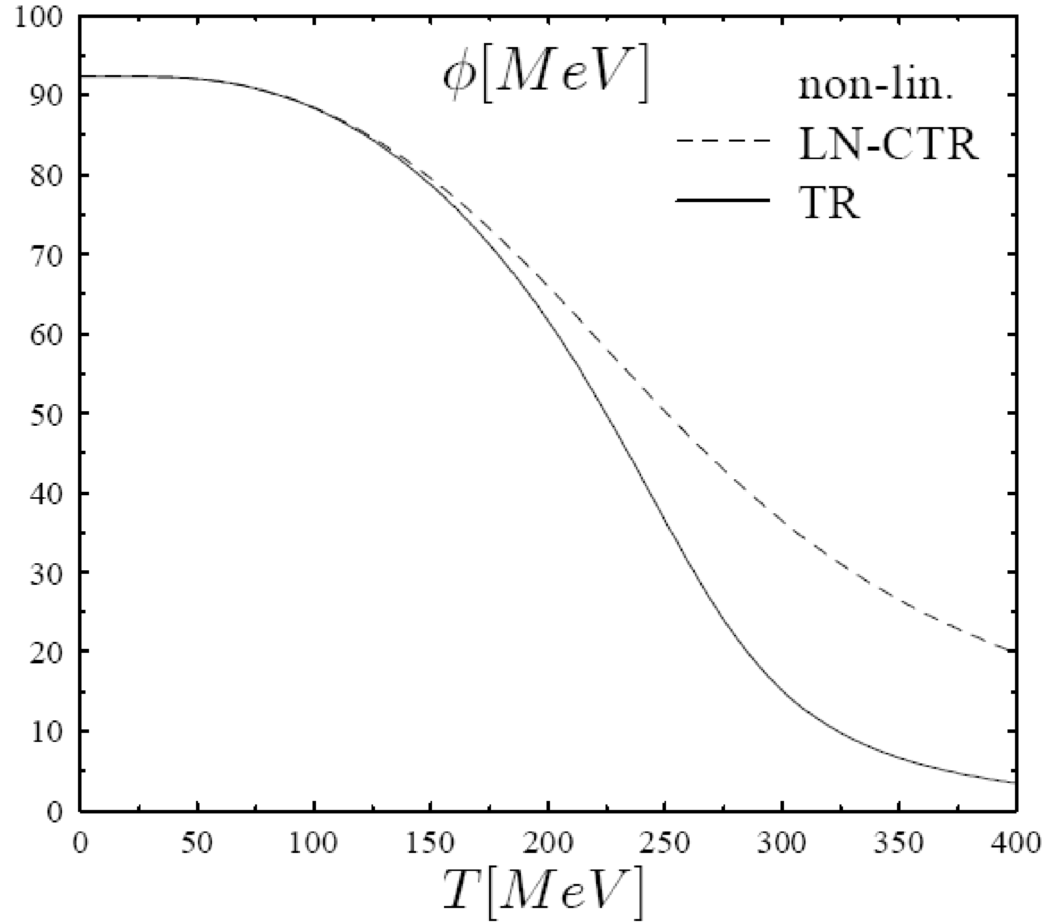


- Crossover phase transition in TR and in LN-CTR
- First order phase transition in CTR

Results

Explicitly broken symmetry in the nonlinear case: $\lim_{\varepsilon \to 0^+} Z(\varepsilon, h)$

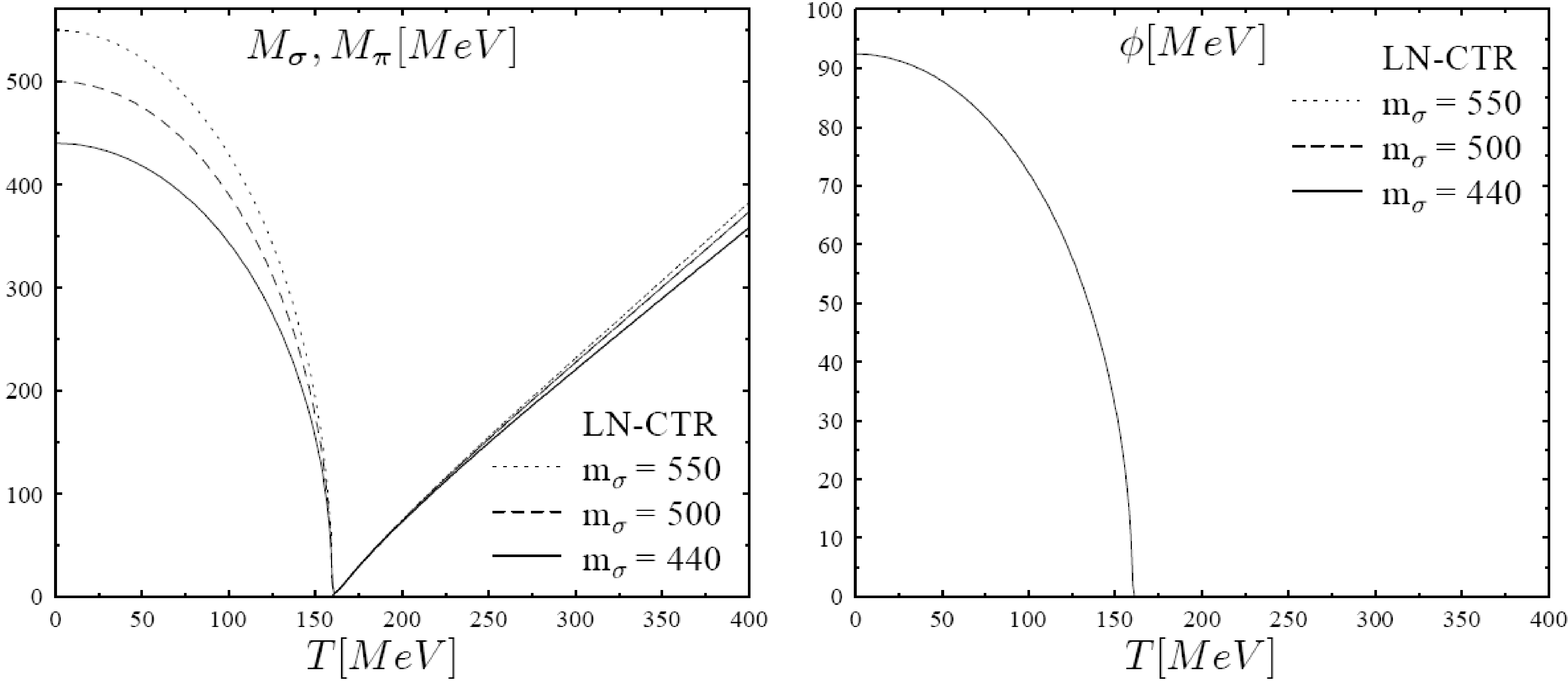




- The phase transition is crossover
- The σ-field becomes frozen due its infinitely heavy mass
  → there are only pionic excitations left

Results

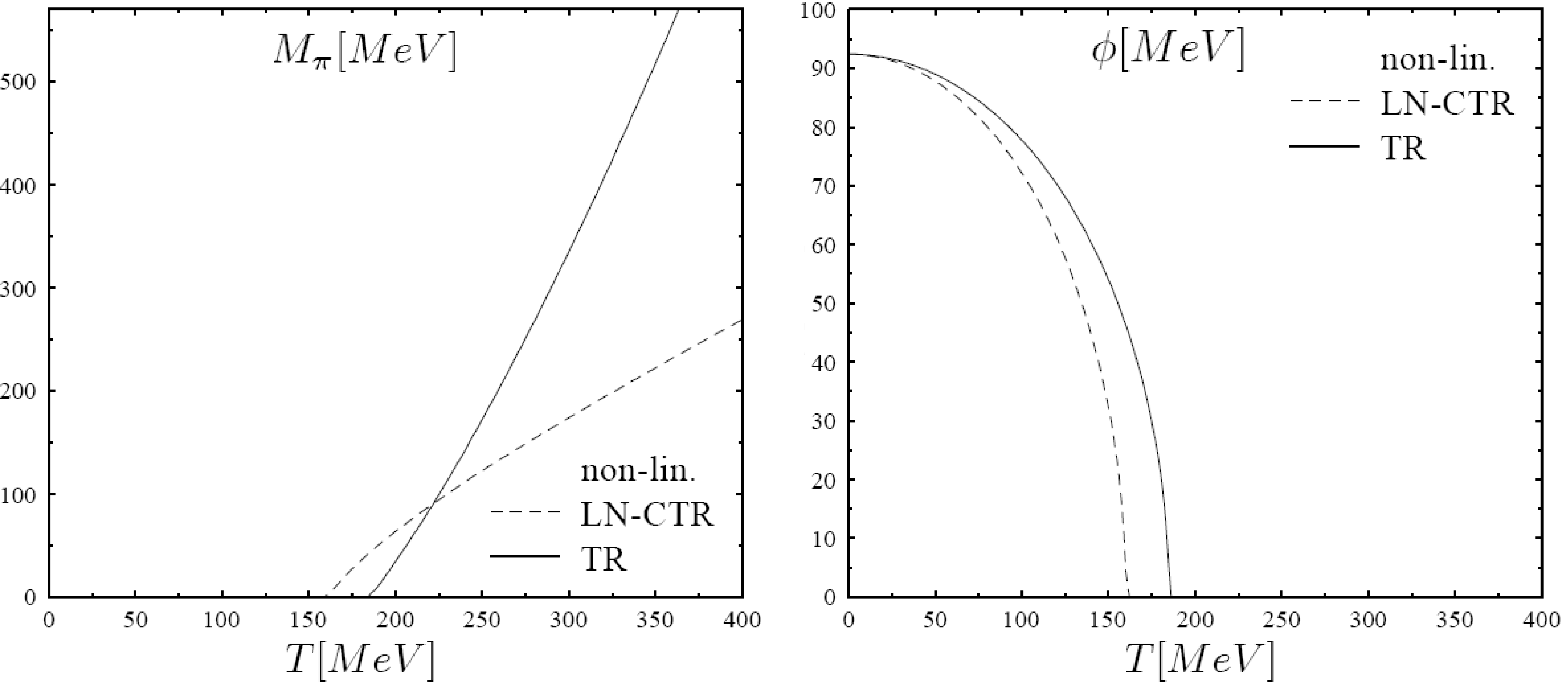
Chiral limit in the linear case: $\lim_{h \to 0^+} Z(\varepsilon, h)$



- The phase transition is second order with $T_c = \sqrt{3} f_\pi$ in LN-CTR
- The Goldstone's Theorem is fulfilled

Results

Chiral limit in the nonlinear case: $\lim_{h,\varepsilon \rightarrow 0^+} Z(\varepsilon, h)$



- Second order phase transition, $T_c = 2f_\pi$ in TR and $T_c = \sqrt{3}f_\pi$ in LN-CTR
- The Goldstone's Theorem is fulfilled
- $M_\sigma \rightarrow \infty$ ➡ the σ-field is excluded from the thermodynamics

Results

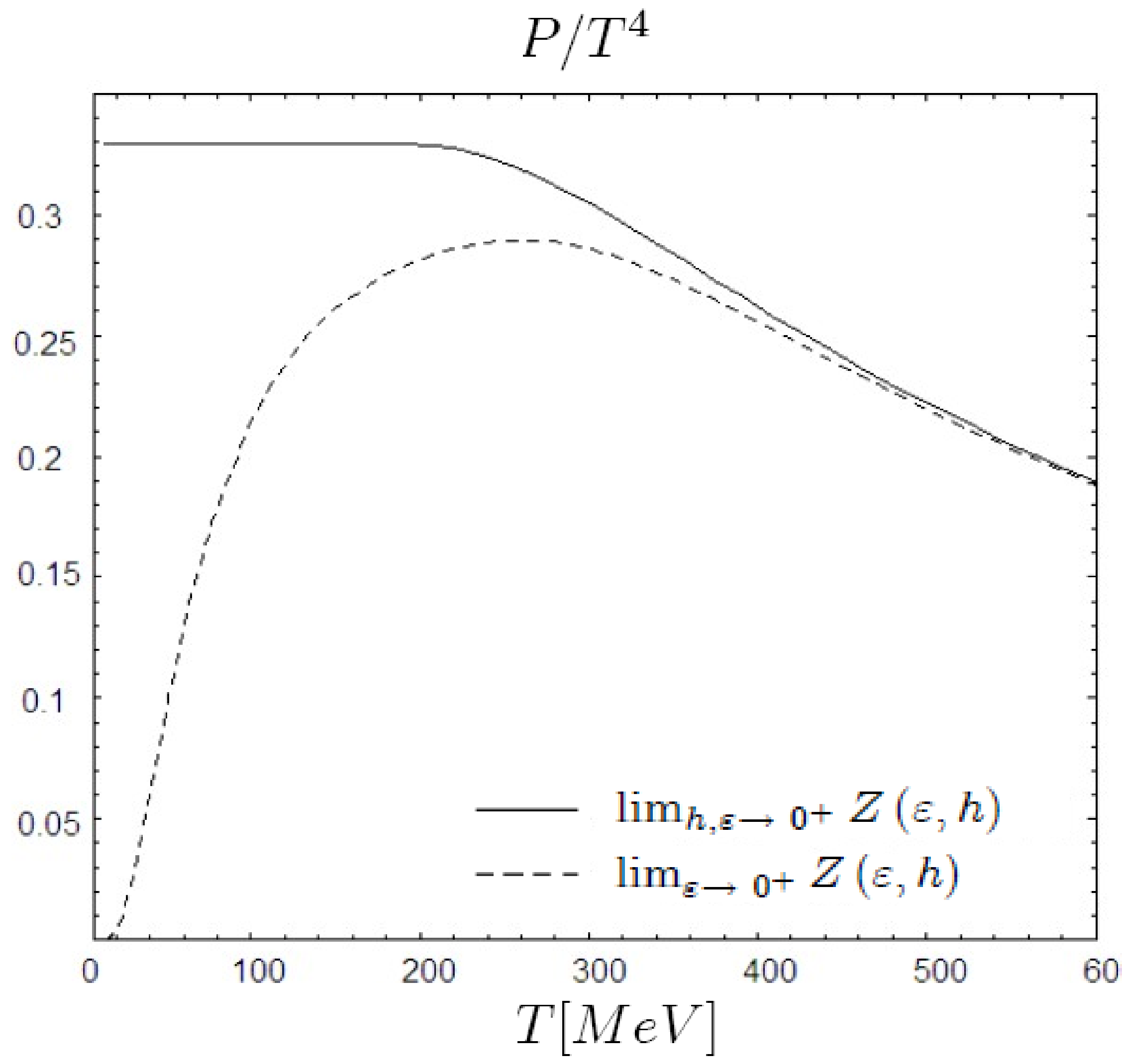
The pressure in the nonlinear case

The thermodynamic pressure is determined by the minimum of the effective potential:

$$p = -V_{eff}(\varphi; \alpha_0, \mathcal{G}_i);$$

$$i = \sigma \ , \ \vec{\pi} \ , \ \alpha \ ,$$

where $\varphi, \alpha_0, \mathcal{G}_i$ are the solutions of the gap equation



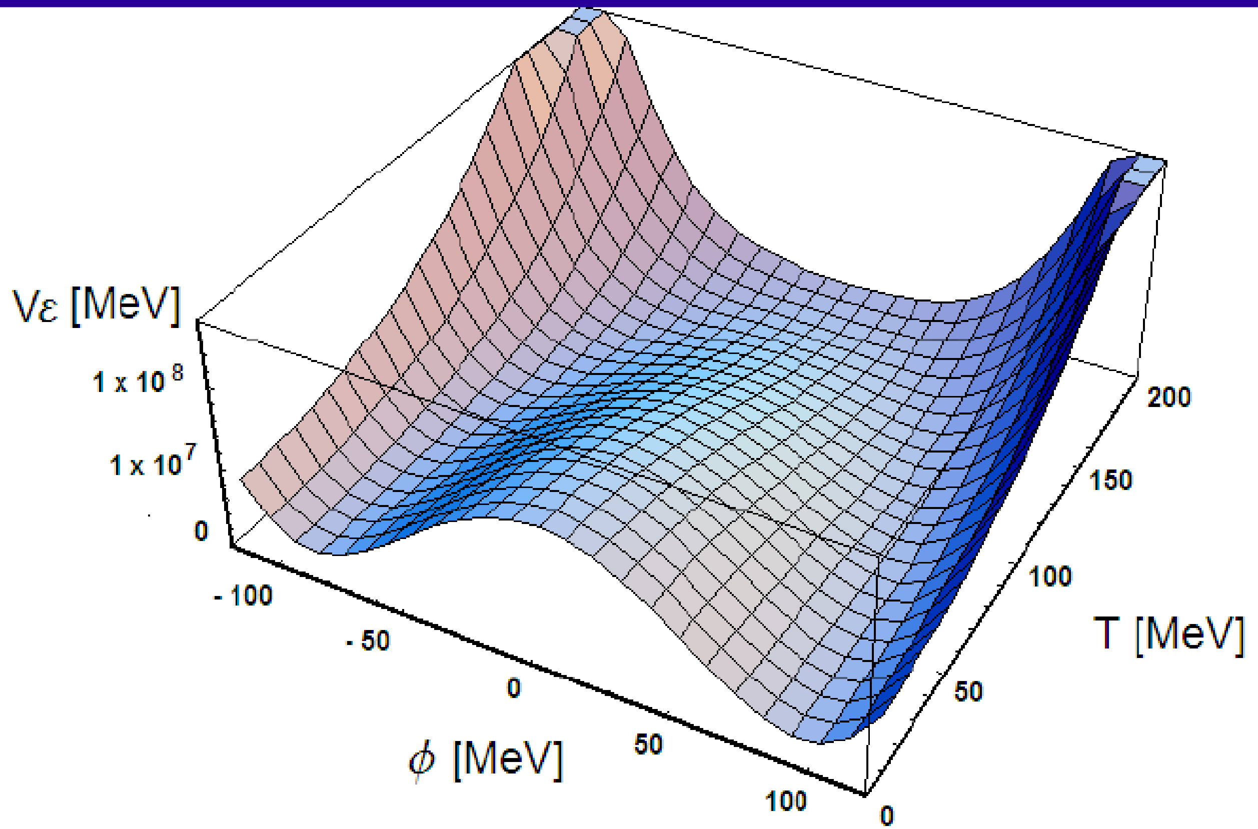
Results

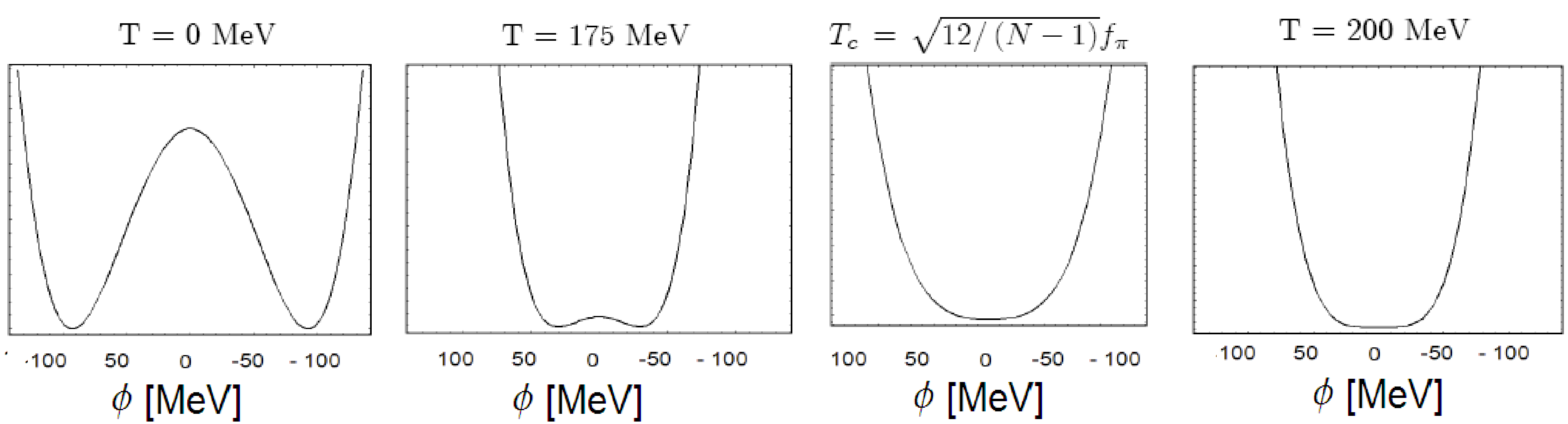
The effective potential

in the chiral limit in the

nonlinear case:

$\lim_{h,\varepsilon \to 0^+} Z(\varepsilon, h)$





Summary

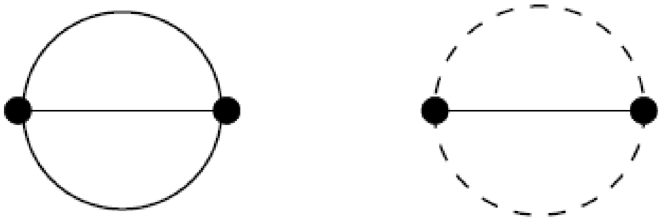
- The study of the linear and nolinear O(N) model at nonzero temperature
- The auxiliary field method allowed to properly incorporate the δ-constraint and to establish a well defined link between the linear and nonlinear model
- The CJT-effective potential and the gap equations were derived
- The regularization of divergent vacuum terms was done within the counter-term scheme
- The numerical results for the temperature dependent masses and the condensate with and without explicitly broken chiral symmetry were presented
- As required, in the nonlinear limit the thermodynamics of the system is generated by pionic excitations only
- The auxiliary field method results in a fulfilment of Goldstone's theorem and renders the order of the phase transition to be in accordance with arguments based on universality class reasoning

Outlook

- Include sunset-type diagrams

  in the 2PI effective action

  which allows the computation of the decay width



- Nonzero chemical potential
- Additional scalar singlet states
- Addition of vector and axial vector mesonic degrees of freedom

Thank you
for your attention