Once and twice subtracted dispersion relations in analysis of $\pi\pi$ amplitudes

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Outline

1. **Introduction**
   - Motivation
2. **Dispersion relations with imposed crossing symmetry condition**
   - main idea and short historical review
   - Twice subtracted dispersion relations
   - example of application
   - Once subtracted dispersion relations
   - Threshold behavior of output amplitudes
3. **Numerical results in theory and in practice**
   - Numerical results in theory
   - Numerical results for S, P, D G and F $\pi\pi$ amplitudes
   - σ pole (resonance)
4. **Conclusions**

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Why dispersive approach?

- it is model independent, only analyticity and crossing symmetry,
- it can well determine amplitudes even where is no data,
- allows to test the data on $\pi\pi$ scattering,
- relates different $\pi\pi$ partial waves,
- for each $m_{\pi\pi}$ various $\pi\pi$ amplitudes are combined and integrated,
- increases precision of output amplitudes

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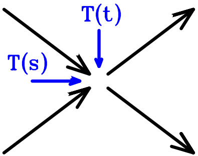
main idea

- Crossing symmetry:
  $\pi\pi$ amplitudes should be invariant under change of channel



- So $T(s) = C_{st} T(t)$ where
  C_{st} is crossing matrix.
  - General form of twice subtracted dispersion relations:

$$\begin{aligned} Re f_\ell^I(s) &= Const_1 + Const_2(s-4) + \\ & \sum_{I'} \sum_{\ell'} \int_4^\infty ds' K_{\ell\ell'}^{II'}(s,s') \text{Im}\, f_{\ell'}^{I'}(s') \end{aligned}$$

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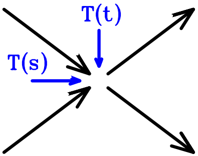

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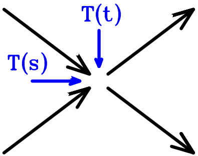

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Twice subtracted dispersion relations (Roy's equations)

- $\text{Re } f_\ell^I(s) = ST(s) + KT(s) + DT(s)$ where
- "subtracting term" $ST(s) = a_0^0 \delta_{I0}\delta_{\ell 0} + a_0^2 \delta_{I2}\delta_{\ell 0} + \frac{s-4}{12}(2a_0^0 - 5a_0^2)(\delta_{I0}\delta_{\ell 0} + \frac{1}{6}\delta_{I1}\delta_{\ell 1} - \frac{1}{2}\delta_{I2}\delta_{\ell 0})$ with a_0^0 and a_0^2 - the $\pi\pi$ scattering lengths in the $S0$- and $S2$-wave,
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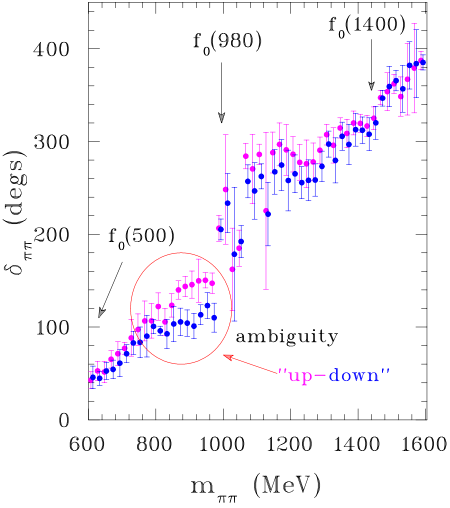
up-down ambiguity



- well known “up-down” ambiguity in the $\pi\pi$ $S0$ wave below 1 GeV,
- caused by ambiguity in sign of $\theta_S - \theta_P$ in PWA (e.g. works of (CERN-Cracow-Munich Coll. 70’),
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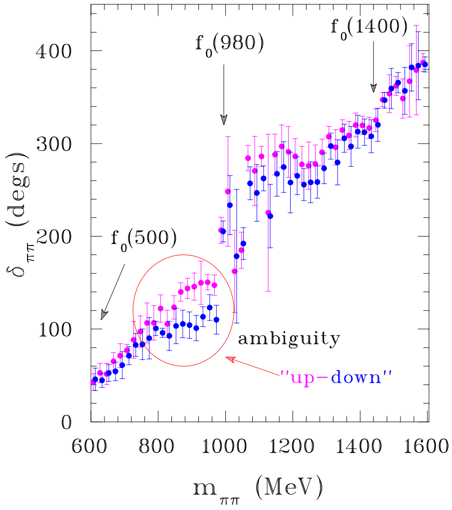

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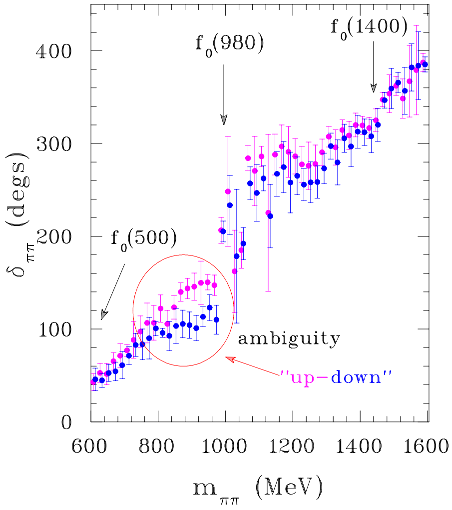

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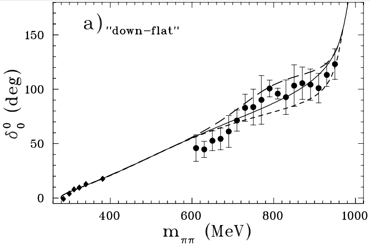
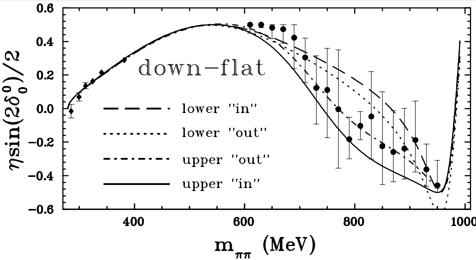
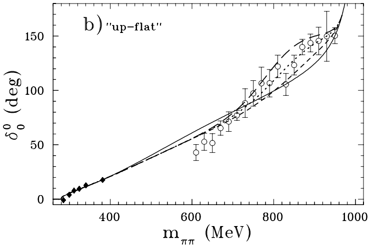
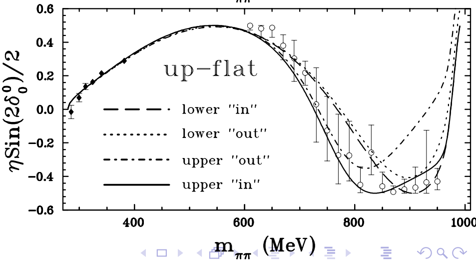
Elimination of “up-down” ambiguity









Once subtracted dispersion relations (GKPY equations)

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Threshold behavior of output amplitudes

- Threshold expansion:
  $Ref^I_\ell(s \approx 4) = (s-4)^\ell \left[a^I_\ell + b^I_\ell(s-4) + ...\right]$
- Let's compare the Roy's and GKPY equations:

Wave	Thr. exp	ST_{Roy}	$KT\&DT_{Roy}$	ST_{GKPY}	$KT\&DT_{GKPY}$
$S0$	a^0_0	$a^0_0 + C_{S0}(s-4)$	$\beta_{S0}(s-4)$	$a^0_0 + 5a^2_0$	$\alpha_{S0} + \beta_{S0}(s-4)$
P	0	$C_P(s-4)$	$\beta_{P1}(s-4)$	$a^0_0 - \frac{5}{2}a^2_0$	$\alpha_{P1} + \beta_{P1}(s-4)$
$S2$	a^2_0	$a^2_0 + C_{S2}(s-4)$	$\beta_{S2}(s-4)$	$a^0_0 + \frac{1}{2}a^2_0$	$\alpha_{S2} + \beta_{S2}(s-4)$

- so, in GKPY equations necessary are mutual cancellations of constant terms in the P-wave and partial cancellations in the S-waves.

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$S0$	a^0_0	$a^0_0 + C_{S0}(s-4)$	$\beta_{S0}(s-4)$	$a^0_0 + 5a^2_0$	$\alpha_{S0} + \beta_{S0}(s-4)$
P	0	$C_P(s-4)$	$\beta_{P1}(s-4)$	$a^0_0 - \frac{5}{2}a^2_0$	$\alpha_{P1} + \beta_{P1}(s-4)$
$S2$	a^2_0	$a^2_0 + C_{S2}(s-4)$	$\beta_{S2}(s-4)$	$a^0_0 + \frac{1}{2}a^2_0$	$\alpha_{S2} + \beta_{S2}(s-4)$

- so, in GKPY equations necessary are mutual cancellations of constant terms in the P-wave and partial cancellations in the S-waves.

Threshold behavior of output amplitudes

- Threshold expansion:
  $Re f^{I}_{\ell}(s \approx 4) = (s-4)^{\ell}\left[a^{I}_{\ell} + b^{I}_{\ell}(s-4) + \ldots\right]$
- Let's compare the Roy's and GKPY equations:

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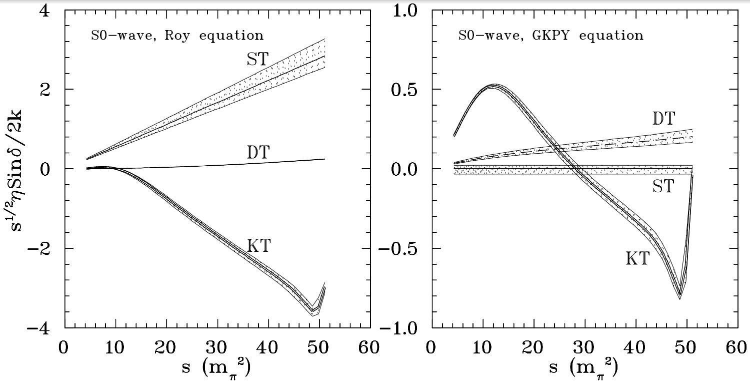
- so, in GKPY equations necessary are mutual cancellations of constant terms in the *P*-wave and partial cancellations in the *S*-waves.

Decomposition of Roy's and GKPY eqs: $S0$-wave



- $f_\ell^I(s) = \frac{\sqrt{s}}{2i\sqrt{s-4}}\left[\eta_\ell^I(s)e^{2i\delta_\ell^I(s)} - 1\right] \longrightarrow Ref_\ell^I(s)$ should be smaller than ≈ 0.6
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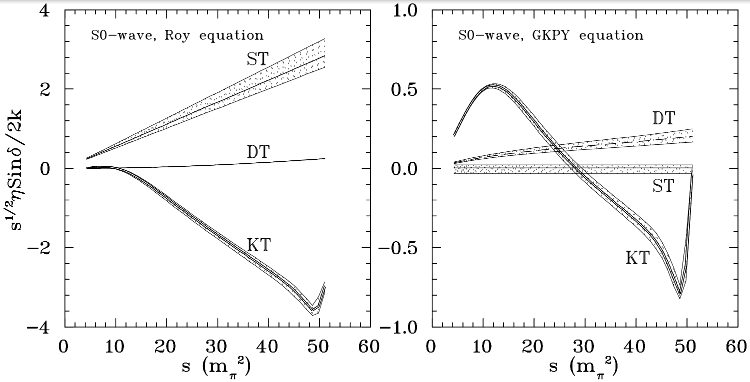

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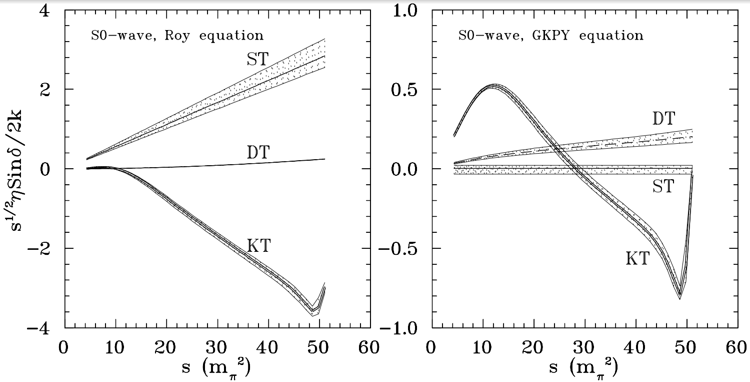

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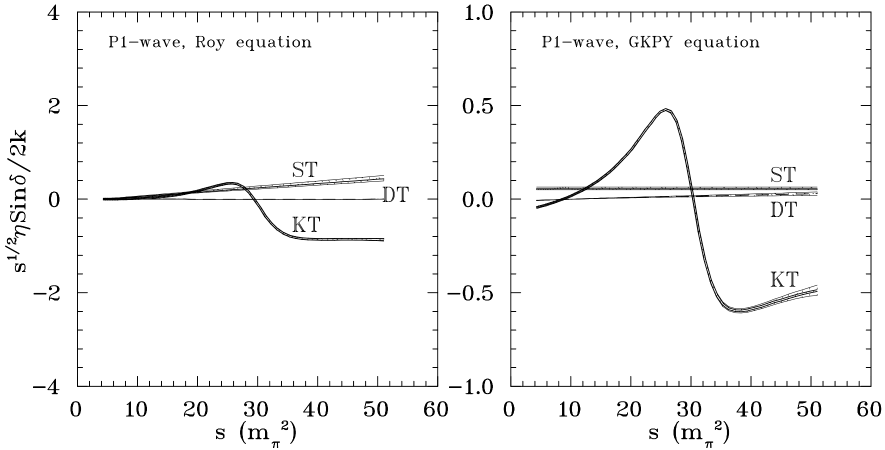
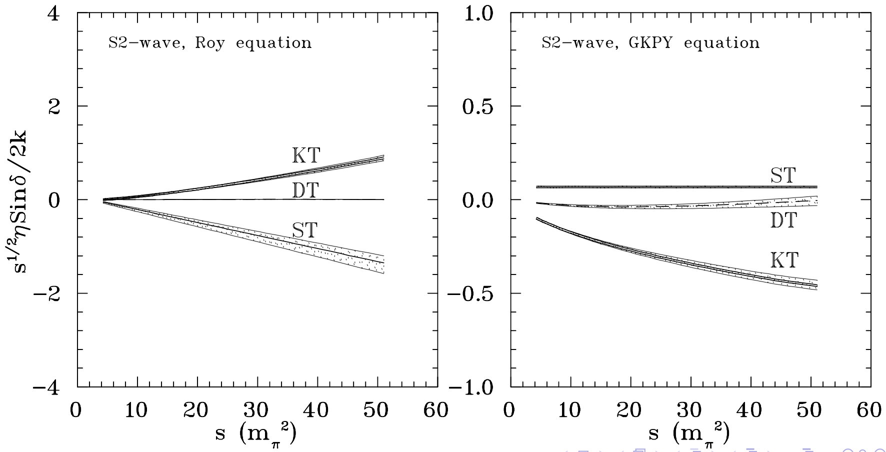
Decomposition of Roy's and GKPY equations: *P* wave



Decomposition of Roy's and GKPY equations: $S2$-wave



Parameterization of amplitudes

We START by parametrizing the data

To avoid model dependences we only require analyticity and unitarity

For the integrals any data parametrization could do.
We use something SIMPLE at low energies (usually <932 MeV)

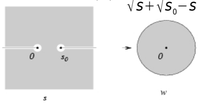
We use an
effective range formalism:
+a conformal expansion
(just two or three terms enough)

$$f_L(s) = \frac{2\sqrt{s}}{\pi k} \frac{1}{2\sqrt{s}\,k^{-2L-1}\Phi_L(s) - i}$$

$$\Phi_L(s) = \sum B_n\, \omega(s)^n$$

We use for input in the REAL axis

$$\omega(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}$$



s_0=1450

If needed we explicitly factorize a value where f(s) is imaginary or has a zero:

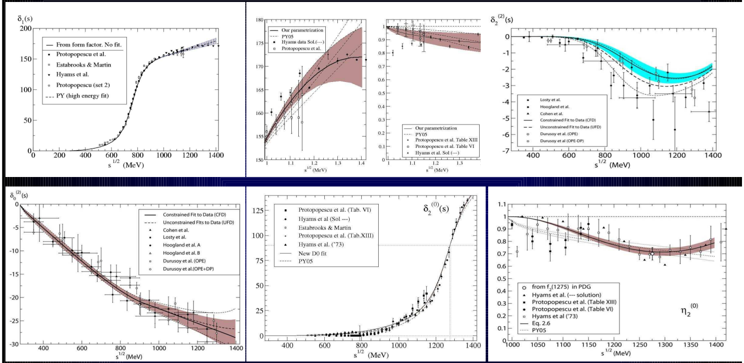
$$\Phi_L(s) = \frac{s - M^2}{s - z^2} \sum B_n\, \omega(s)^n$$

Sometimes another coefficient added
to remove spurious poles near left cuts

t higher energies phenomenological fits (polynomials of relevant momenta)
pecial care for continuous matching between low-high energy fits

Fit to partial waves amplitudes

Similar Initial UNconstrained FIts for all other waves and High energies



R. Kaminski, JRP, F.J. Ynduráin. Phys. Rev. D77:054015,2008.
Eur.Phys.J.A31:479-484,2007,
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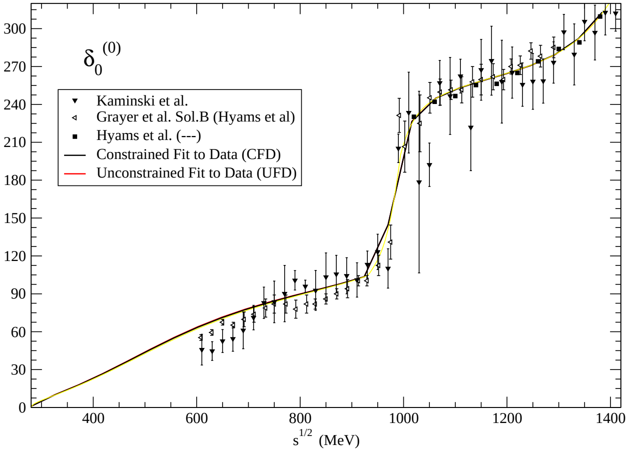
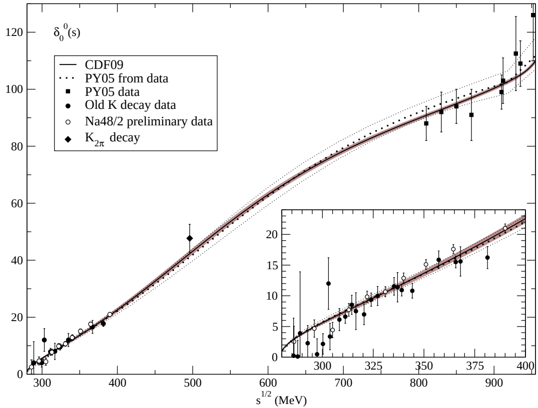
phase shifts for the $S0$-wave



Low energy phase shifts for the S0-wave



- Adler zero at $s = m_\pi^2/2$
- the $Kl4$ and $K \longrightarrow \pi\pi$ data including the newest NA48/2 results
- average $\pi N \longrightarrow \pi\pi N$ data with enlarged errors at 870 - 970 MeV where they are consistent within $10^o - 15^o$

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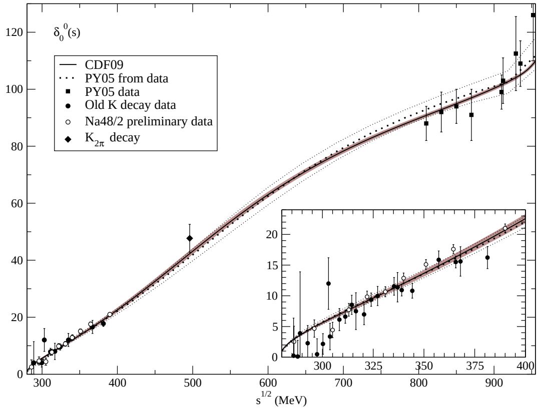

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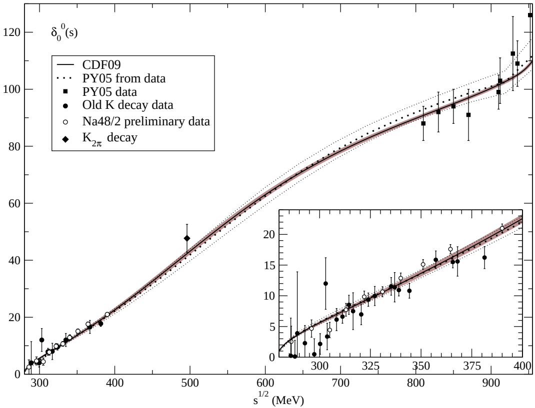

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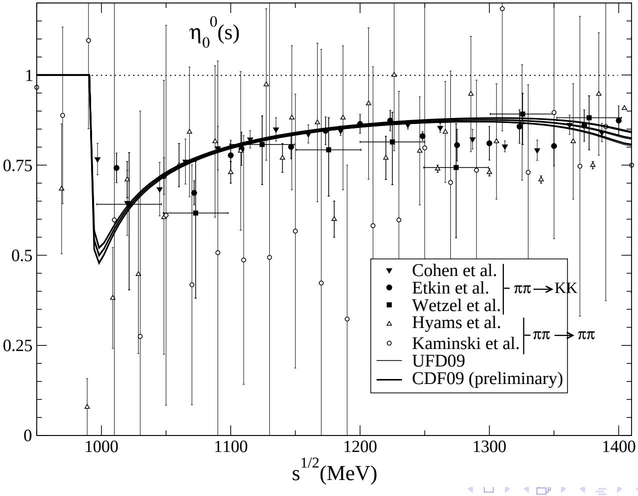
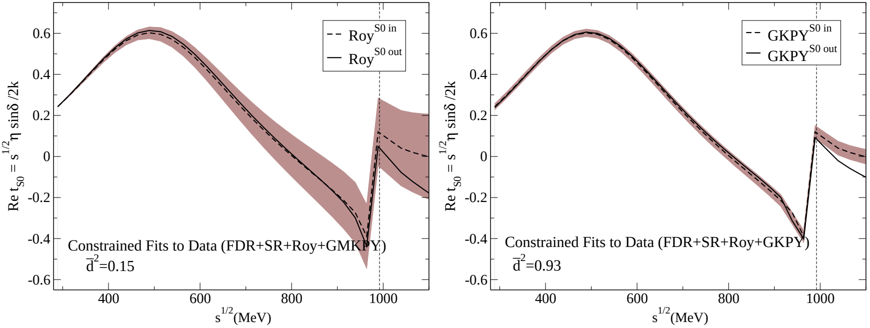


output from Roy and GKPY equations, $S0$-wave



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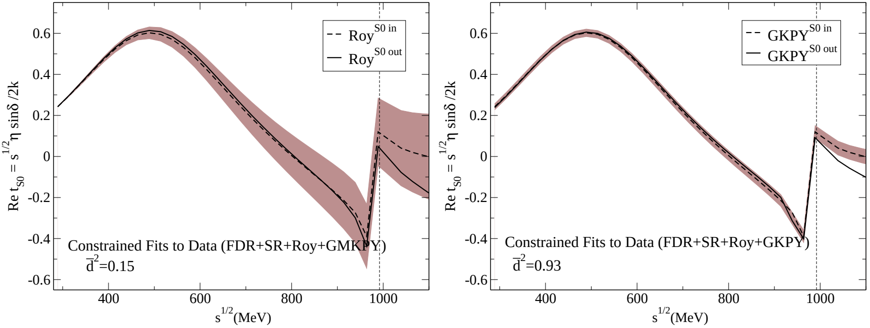

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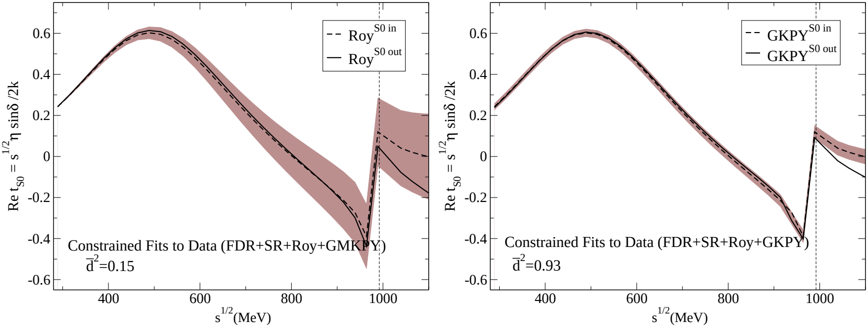

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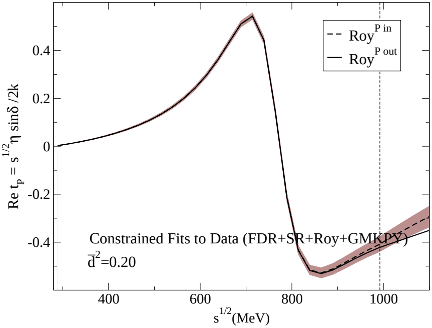
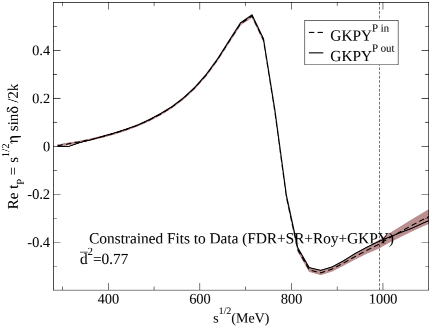
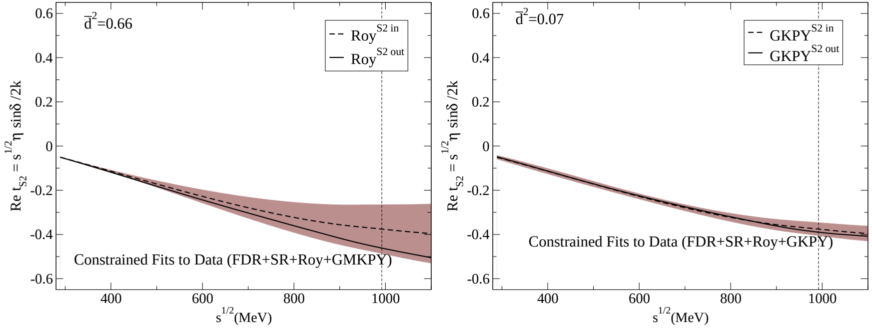




output from Roy and GKPY equations, $S2$-wave



$f_0(600)$ (σ) resonance ($I^G J^{PC} = 0^+0^{++}$)

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  - plays role in determination of chiral parameters,
  - it can be: $q\bar{q}$, $2q2\bar{q}$, glueball or mixture of these states,
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Cross sections for the $\pi\pi$ $S0$ wave

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  σ_{12} : $\pi\pi \longrightarrow K\bar{K}$

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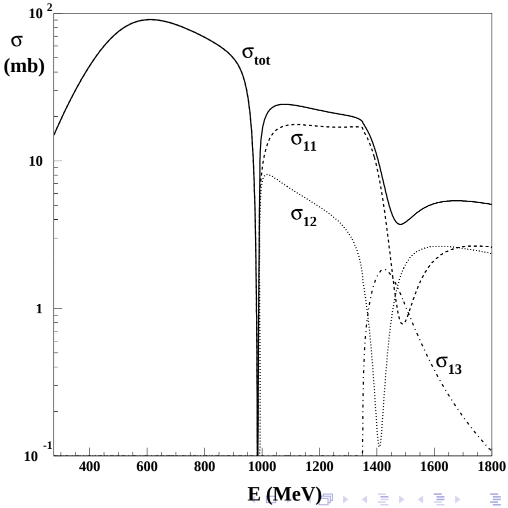

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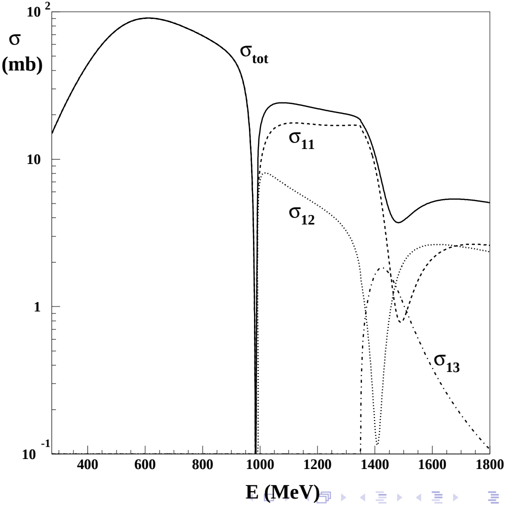

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S0 wave below 1 GeV

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Kaminski et al.
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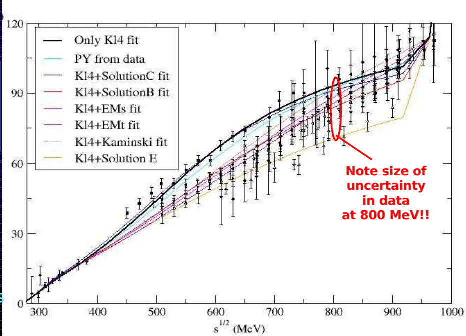

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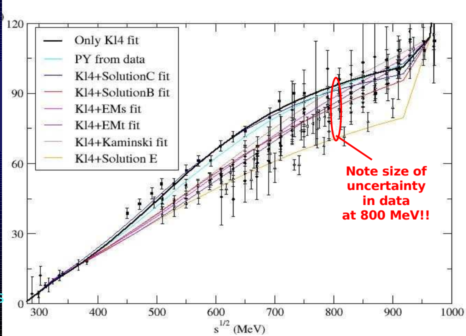

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constrains for data sets

-THE PION-PION SCATTERING AMPLITUDE-

	B_0	B_1	μ_0 (MeV)	$\frac{\chi^2}{\text{d.o.f.}}(I_t=1)$	$\frac{\chi^2}{\text{d.o.f.}}(\pi^0\pi^0)$
PY, Eq. (2.14)	21.04 [(a)]	6.62 [(a)]	782 ± 24	0.3	3.5
K decay only	18.5 ± 1.7	$\equiv 0$	766 ± 95	0.2	1.8
K decay data + Grayer, B	22.7 ± 1.6	12.3 ± 3.7	858 ± 15	1.0	2.7
K decay data + Grayer, C	16.8 ± 0.85	-0.34 ± 2.34	787 ± 9	0.4	1.0
K decay data + Grayer, E	21.5 ± 3.6	12.5 ± 7.6	1084 ± 110	2.1	0.5
K decay data + Kaminski	27.5 ± 3.0	21.5 ± 7.4	789 ± 18	0.3	5.0
K decay data + Grayer, A	28.1 ± 1.1	26.4 ± 2.8	866 ± 6	2.0	7.9
K decay data + EM, s−channel	29.8 ± 1.3	25.1 ± 3.3	811 ± 7	1.0	9.1
K decay data + EM, t−channel	29.3 ± 1.4	26.9 ± 3.4	829 ± 6	1.2	10.1
K decay data + Protopopescu, VI	27.0 ± 1.7	22.0 ± 4.1	855 ± 10	1.2	5.8
K decay data + Protopopescu, XII	25.5 ± 1.7	18.5 ± 4.1	866 ± 14	1.2	6.3
K decay data + Protopopescu, VIII	27.1 ± 2.3	23.8 ± 5.0	913 ± 18	1.8	4.2

[(a)] Errors as in Eq. (2.14b).
PY, Eq. (2.14): our *global* fit of Subsect. 2.2.2. The next rows show the fits to K decay[13] alone or combined with $\pi\pi$ scattering data. Grayer A, B, C, E: the solutions in the paper of Grayer et al.[11a] EM: the solutions of Estabrooks and Martin.[11a] Kaminski refers to the papers of Kamiński et al.[11c] Protopopescu VI, XII and VIII: the corresponding solutions in ref. 10.

J. R. Pelaez and F. Ynduráin, Phys. Rev. D71, 074016, (2005)

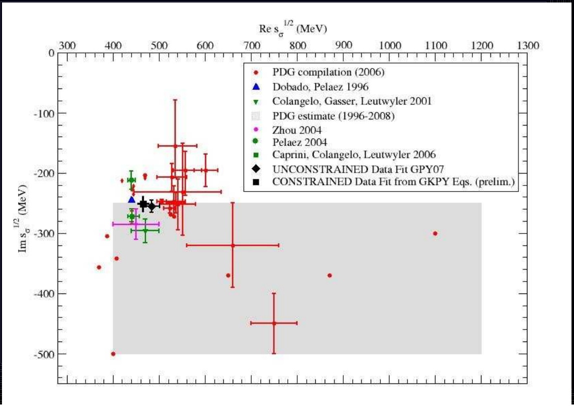
σ pole

Continuation to the complex s plane:

$Im(s_{pole})$:

- ROY: -255 ± 14 MeV
- GKPY: -251 ± 12 MeV

The results from the GKPY Eqs. with the CONSTRAINED Data Fit input



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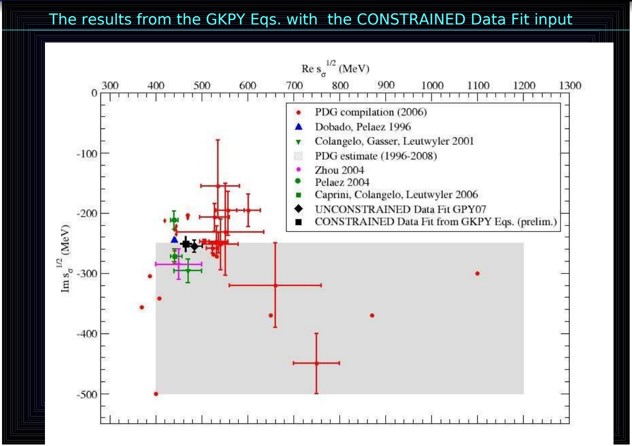

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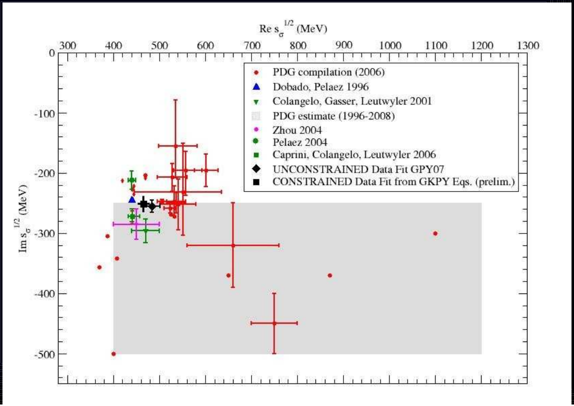

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- using dispersion relation one can constrain the data fits,
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- we do not use any ChPT predictions but
- we get from our fits $a_0^0 = 0.222 \pm 0.009$ and $a_0^2 = -0.045 \pm 0.008$
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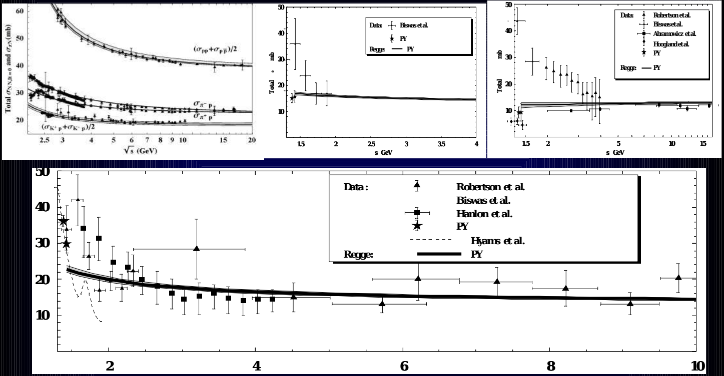
Regge

Similar Initial UNconstrained FIts for all other waves and High energies



JRP, F.J. Ynduráin. PRD69,114001 (2004)

Fit to $S0$ wave

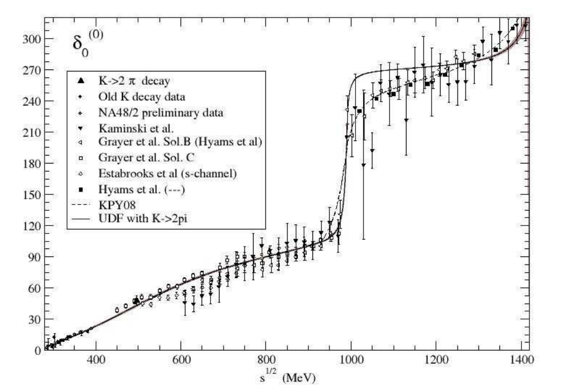
EW:S0 wave above 932 MeV with improved matcing

See R. Garcia Martín talk

CERN-Munich phases with and without polarized beams

Inelasticity from several $\pi\pi \to \pi\pi$, $\pi\pi \to KK$ experiments

K matrix fit. Tiny errors. Strong correlation of phase and inelasticity



Fit to $S0$ wave