QCD at nonzero isospin densities

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Phase structure of lattice field theories
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• introduction: QCD with isospin

• relevant phenomena
  ▶ pion condensation
  ▶ chiral symmetry breaking

• $\lambda$-extrapolation
  ▶ naive method
  ▶ new method

• results
  ▶ phase boundary for pion condensation
  ▶ $\chi$SB transition line at low $\mu_I$
  ▶ direct check of Taylor-expansion

• outlook
Introduction

- isospin density \( n_I = n_u - n_d \)
- \( n_I < 0 \rightarrow \) excess of neutrons over protons
  \( \rightarrow \) excess of \( \pi^- \) over \( \pi^+ \)

- applications
  - neutron stars
  - heavy-ion collisions

- chemical potentials (3-flavor)
  \[ \mu_B = \frac{3(\mu_u + \mu_d)}{2} \quad \mu_I = \frac{\mu_u - \mu_d}{2} \quad \mu_S = 0 \]
Introduction

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- chemical potentials (3-flavor)
  
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- here: zero baryon number but nonzero isospin
  
  \[ \mu_u = \mu_I \quad \mu_d = -\mu_I \]
Introduction

- QCD at low energies ≈ pions
- on the level of charged pions: $\mu_\pi = 2\mu_I$
  at zero temperature

  \[
  \begin{align*}
  \mu_\pi &< m_\pi & \text{vacuum state} \\
  \mu_\pi &= m_\pi & \text{Bose-Einstein condensation} \\
  \mu_\pi &> m_\pi & \text{undefined}
  \end{align*}
  \]

- on the level of quarks: lattice simulations
  - no sign problem
  - conceptual analogies to baryon density
    (Silver Blaze, hadron creation, saturation)
  - technical similarities
    (proliferation of low eigenvalues)
Setup
Symmetry breaking

- QCD with light quark matrix
  \[ M = \not\!\Phi + m_{ud} \mathbb{1} \]
- chiral symmetry (flavor-nontrivial)
  \[ \text{SU}(2)_V \]
Symmetry breaking

- QCD with light quark matrix
  \[ M = \not{\Phi} + m_{ud} \mathbb{1} + \mu \gamma_0 \tau_3 \]

- chiral symmetry (flavor-nontrivial)
  \[ SU(2)_V \to U(1)_{\tau_3} \]
Symmetry breaking

- QCD with light quark matrix
  \[ M = \mathcal{D} + m_{ud} \mathbb{1} + \mu \gamma_0 \tau_3 \]
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  \[ SU(2)_V \to U(1)_{\tau_3} \]
- spontaneously broken by a pion condensate
  \[ \langle \bar{\psi} \gamma_5 \tau_1,2 \psi \rangle \]
- a Goldstone mode appears
Symmetry breaking

- QCD with light quark matrix
  \[ M = \bar{\psi} \gamma_5 \tau_1 \psi + m_{ud} \mathbb{1} + \mu \gamma_0 \tau_3 + i \lambda \gamma_5 \tau_2 \]

- chiral symmetry (flavor-nontrivial)
  \[ \text{SU}(2)_V \rightarrow \text{U}(1)_{\tau_3} \rightarrow \emptyset \]

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- add small explicit breaking
Symmetry breaking

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- Spontaneously broken by a pion condensate
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- Add small explicit breaking

- Extrapolate results \( \lambda \rightarrow 0 \)
Simulation details

- staggered light quark matrix with $\eta_5 = (-1)^{n_x+n_y+n_z+n_t}$

$$M = \begin{pmatrix} \mathcal{D}_\mu + m & \lambda \eta_5 \\ -\lambda \eta_5 & \mathcal{D}_{-\mu} + m \end{pmatrix}$$

- we have $\gamma_5 \tau_1$-hermiticity

$$\eta_5 \tau_1 M \tau_1 \eta_5 = M^\dagger$$

- determinant is real and positive

$$\det M = \det(|\mathcal{D}_\mu + m|^2 + \lambda^2)$$

- pioneering studies [Kogut, Sinclair '02]
  [de Forcrand, Stephanov, Wenger '07] with unimproved action

- here: $N_f = 2 + 1$ rooted stout-smeared staggered quarks + tree-level Symanzik improved gluons
Condensates: definition and renormalization

\[
\begin{align*}
\langle \bar{\psi}\psi \rangle &= \frac{T}{V} \frac{\partial \log Z}{\partial m}, \\
\langle \pi \rangle &= \frac{T}{V} \frac{\partial \log Z}{\partial \lambda}
\end{align*}
\]

- multiplicative renormalization

\[
Z_\pi = Z_\lambda^{-1} = Z_m^{-1} = Z_{\bar{\psi}\psi}
\]

- convenient normalization

\[
\Sigma_{\bar{\psi}\psi} \equiv m \cdot \langle \bar{\psi}\psi \rangle \cdot \frac{1}{m^2_f f^2_\pi}, \quad \Sigma_\pi \equiv m \cdot \langle \pi \rangle \cdot \frac{1}{m^2_f f^2_\pi}
\]

- so that in leading-order chiral PT [Son, Stephanov '00]

\[
\Sigma^2_{\bar{\psi}\psi}(\mu_I) + \Sigma^2_\pi(\mu_I) = 1
\]
Condensates: old method
Pion condensate: old method

- traditional method [Kogut, Sinclair '02]
  measure full operator at nonzero $\lambda$ (via noisy estimators)

$$\Sigma_\pi \propto \left\langle \text{Tr} M^{-1} \eta_5 \tau_2 \right\rangle$$

- extrapolation very ‘steep’
Pion condensate: old method

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Pion condensate: old method

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  measure full operator at nonzero $\lambda$ (via noisy estimators)
  $$\sum_\pi \propto \left\langle \text{Tr} M^{-1} \eta_5 \tau_2 \right\rangle$$

- extrapolation very 'steep'
Chiral condensate: old method

- traditional method [Kogut, Sinclair '02]
  measure full operator at nonzero $\lambda$ (via noisy estimators)

$$\Sigma_{\bar{\psi}\psi} \propto \left\langle \text{Tr} M^{-1} \right\rangle$$

- extrapolation very ‘steep’
Pion condensate: new method
Singular value representation

▶ pion condensate

\[ \pi = \frac{\partial}{\partial \lambda} \log \det(|\mathcal{D}_\mu + m|^2 + \lambda^2) = \text{Tr} \frac{2\lambda}{|\mathcal{D}_\mu + m|^2 + \lambda^2} \]

▶ singular values

\[ |\mathcal{D}_\mu + m|^2 \psi_i = \xi_i^2 \psi_i \]

▶ spectral representation

\[ \pi = \frac{T}{V} \sum_i \frac{2\lambda}{\xi_i^2 + \lambda^2} = \int d\xi \rho(\xi) \frac{2\lambda}{\xi^2 + \lambda^2} \xrightarrow{\lambda \to 0} \pi \rho(0) \]

first derived in [Kanazawa, Wettig, Yamamoto ’11]
Singular value representation

- pion condensate

\[ \pi = \frac{\partial}{\partial \lambda} \log \det(\|\Phi_\mu + m\|^2 + \lambda^2) = \text{Tr} \frac{2\lambda}{\|\Phi_\mu + m\|^2 + \lambda^2} \]

- singular values

\[ \|\Phi_\mu + m\|^2 \psi_i = \xi_i^2 \psi_i \]

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- compare to Banks-Casher-relation at \( \mu I = 0 \)
Singular value density

- spectral densities at $\lambda/m = 0.17$
Density at zero

- scaling with $\lambda$ is improved drastically
Density at zero

- scaling with $\lambda$ is improved drastically

\[
\langle \pi \rangle_{\text{rew}} = \frac{\langle \pi W_\lambda \rangle}{\langle W_\lambda \rangle} = \exp\left[-\lambda V_4 \pi + O(\lambda^2)\right]
\]
Density at zero

- scaling with $\lambda$ is improved drastically

$$\langle \pi \rangle_{\text{rew}} = \langle \pi W_\lambda \rangle / \langle W_\lambda \rangle$$

$$W_\lambda = \exp[-\lambda V_4 \pi + O(\lambda^2)]$$
Comparison between old and new methods

- extrapolation in $\lambda$ gets almost completely flat

![Graph showing comparison between different methods with legend: direct method [Kogut, Sinclair], Banks-Casher-type method, and leading reweighting.](image)
Results
Phase boundary

- Interpolate $\rho(0)$ as function of $\mu_I$ to find phase boundary.

![Graph showing phase boundary with $T$ vs. $\mu_I / m_\pi$ relationship. The graph includes data points and shaded areas indicating the phase transition region.](image-url)
Phase boundary

- Interpolate $\rho(0)$ as a function of $\mu_I$ to find the phase boundary.

![Graph showing the phase boundary with $T$ (MeV) on the y-axis and $\mu_I / m_\pi$ on the x-axis. The graph includes data points for $24^3 \times 8$ preliminary results, with shaded regions indicating the pion condensation phase.]
Phase boundary

- Interpolate $\rho(0)$ as function of $\mu_I$ to find phase boundary

![Phase boundary diagram](image)
interpolate $\rho(0)$ as function of $\mu_I$ to find phase boundary
- Interpolate $\rho(0)$ as function of $\mu_I$ to find phase boundary

- Compare to expectations from $\chi$PT [Son, Stephanov '00]

Phase boundary

\[ \langle \bar{u}\gamma_5 d \rangle = 0 \]

\[ \langle \pi^- \rangle \neq 0 \]

\[ \langle \bar{u}\gamma_5 d \rangle \neq 0 \]
Phase boundary

- Interpolate $\rho(0)$ as function of $\mu_I$ to find phase boundary

- Compare to expectations from $\chi$PT [Son, Stephanov '00]

- No pion condensate above $T \approx 160$ MeV
New method for other observables
Singular value representation

- **chiral condensate**

\[
\bar{\psi}\psi = \frac{\partial}{\partial m} \log \det(|\mathcal{D}_\mu + m|^2 + \lambda^2) = \text{Tr} \frac{(\mathcal{D}_\mu + m) + (\mathcal{D}_\mu + m)^\dagger}{|\mathcal{D}_\mu + m|^2 + \lambda^2}
\]

- **singular values**

\[
|\mathcal{D}_\mu + m|^2 \psi_i = \xi_i^2 \psi_i
\]

- **spectral representation**

\[
\bar{\psi}\psi = \frac{T}{V} \sum_i 2 \text{Re} \frac{\langle \psi_i | \mathcal{D}_\mu + m | \psi_i \rangle}{\xi_i^2 + \lambda^2}
\]
Singular value representation

- spectral representation at $\lambda = 0$

$$\bar{\psi}\psi = \frac{T}{V} \sum_{i=1}^{N} 2 \text{Re} \frac{\langle \psi_i | \mathcal{D}_\mu + m | \psi_i \rangle}{\xi_i^2}$$

- convergence not visible for $N \leq 150$
Improvement

- work instead with the *difference*

\[
\delta \bar{\psi} \psi \equiv \bar{\psi} \psi (\lambda = 0) - \bar{\psi} \psi (\lambda) = \frac{2 T}{V} \sum_{i=1}^{N} \text{Re} \langle \psi_i | D_\mu + m | \psi_i \rangle \left[ \frac{1}{\xi_i^2} - \frac{1}{\xi_i^2 + \lambda^2} \right]
\]

\[
\bar{\psi} \psi (0) = \bar{\psi} \psi (\lambda) + \delta \bar{\psi} \psi
\]

- convergence already for small \( N \)
Improvement

- work instead with the difference

\[ \delta \bar{\psi} \psi \equiv \bar{\psi} \psi (\lambda = 0) - \bar{\psi} \psi (\lambda) = \frac{2 T}{V} \sum_{i=1}^{N} \text{Re} \langle \psi_{i} | D_{\mu} + m | \psi_{i} \rangle \left[ \frac{1}{\xi_{i}^{2}} - \frac{1}{\xi_{i}^{2} + \lambda^{2}} \right] \]

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noisy estimators

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Improvement

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noisy estimators

singular values

- convergence already for small \( N \)
Comparison between old and new methods

- extrapolation in $\lambda$ gets almost completely flat

![Graph showing extrapolation in $\lambda$]
Improvement

- the same strategy for the isospin density $n_I = \frac{\partial (\log \mathcal{Z})}{\partial \mu_I}$

$$\delta n_I \equiv n_I(0) - n_I(\lambda) = \frac{2T}{V} \sum_{i=1}^{N} \text{Re} \langle \psi_i | (D_{\mu} + m)^\dagger D_{\mu} | \psi_i \rangle \left[ \frac{1}{\xi_i^2} - \frac{1}{\xi_i^2 + \lambda^2} \right]$$

$n_I(0) = n_I(\lambda) + \delta n_I$

noisy estimators

singular values

- convergence already for small $N$
Comparison between old and new methods

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Results
Transition temperature

- additively renormalized chiral condensate

\[ \Sigma_{\bar{\psi}\psi}(T) - \Sigma_{\bar{\psi}\psi}(T = 0) \]

- define \( T_c \) using the temperature at a constant value (valid at low \( \mu_I \))
Check Taylor-expansion

- isospin density via Taylor-expansion at $\mu_I = 0$

\[ n_I(\mu_I) = \chi^I_2 \cdot \mu_I + \chi^I_4 \cdot \mu_I^3 + \ldots \]

using $\chi^I_{2,4}$ from [BMWc, 1112.4416]

- how far is it reliable?

- similar convergence expected for $\mu_B$
Order of the transition – fits

- fit transition region using chiral perturbation theory [Splittorff et al ’02, Endrődi ’14]
Order of the transition – fits

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Order of the transition – fits

- fit using $O(2)$ scaling [Ejiri et al ’09]

![Graph showing fit using $O(2)$ scaling](image-url)
Outlook

- order of transition?
- deconfinement/chiral symmetry breaking transition
- asymptotic-$\mu_l$ limit?
- BCS phase at large $\mu_l$?
Summary

- determine/improve observables using singular values of $\mathcal{D}_\mu + m$
- $\sim$ flat extrapolation in $\lambda$

- direct check of Taylor-expansion convergence

- phase boundary surprisingly flat for intermediate $\mu_I$
- chance to test effective theories and low-energy models