The phases of hot/dense/magnetized QCD from the lattice

Gergely Endrődi

Goethe University of Frankfurt

EMMI NQM Seminar
GSI Darmstadt, 27. June 2018
QCD phase diagram

Temperature

Magnetic field

early universe

off-central heavy-ion collisions

magnetars

Isospin chemical potential
Outline

- relevance of background magnetic fields and isospin asymmetries
- hot/magnetized QCD
  - including magnetic fields
  - including very strong magnetic fields
  - results: phase diagram
- hot/asymmetric QCD
  - including the isospin asymmetry
  - pion condensation
  - results: phase diagram
- further applications
- conclusions
Introduction
Strong interactions

- explain 99.9% of visible matter in the Universe
Strong interactions

- explain 99.9% of visible matter in the Universe

- elementary particles: quarks and gluons
Strong interactions

- explain 99.9% of visible matter in the Universe
- elementary particles: quarks and gluons
- elementary fields: $\psi(x)$ and $A_\mu(x)$
- QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \frac{1}{4} \text{Tr} F_{\mu\nu}(g_s, A)^2 + \bar{\psi} [\gamma_\mu (\partial_\mu + ig_s A_\mu) + m] \psi$$
Strong interactions

- explain 99.9% of visible matter in the Universe

- elementary particles: quarks and gluons

- elementary fields: $\psi(x)$ and $A_\mu(x)$

- QCD Lagrangian

  $$\mathcal{L}_{\text{QCD}} = \frac{1}{4} \text{Tr} F_{\mu\nu}(g_s, A)^2 + \overline{\psi}[\gamma_\mu(\partial_\mu + ig_s A_\mu) + m] \psi$$

- $g_s = \mathcal{O}(1) \leadsto \text{non-perturbative physics}$
Path integral and lattice field theory

- path integral [Feynman '48]

\[ Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( - \int d^4x \mathcal{L}_{\text{QCD}}(x) \right) \]

- discretize spacetime on a lattice with spacing \( a \) [Wilson '74]

- Monte-Carlo algorithms to generate configurations

- \( 10^9 \)-dimensional integrals \( \sim \) high-performance computing
Path integral and lattice field theory

- path integral [Feynman '48]

\[
\mathcal{Z} = \int D A_\mu \, D \bar{\psi} \, D \psi \, \exp \left( - \int d^4 x \, \mathcal{L}_{\text{QCD}} (x) \right)
\]

- discretize spacetime on a lattice with spacing \( a \) [Wilson '74]

- Monte-Carlo algorithms to generate configurations

- \( 10^9 \)-dimensional integrals \( \rightsquigarrow \) high-performance computing
QCD and external parameters

- running coupling $g_s(E)$
- relevant parameters that control the energy scale:
  - temperature $T$: excites all states
  - baryon density $n_B \propto n_u + n_d$: excites $p^+$ and $n$
  - isospin asymmetry $n_I \propto n_u - n_d$:
    - creates $p^+ - n$ asymmetry, excites $\pi^+$
  - background magnetic field $B$: forces quarks on Landau levels

(chemical potentials conjugate to densities: $\mu_B, \mu_I$)
Magnetic fields: heavy-ion collisions

- off-central events generate magnetic fields
  
  [Kharzeev, McLerran, Warringa '07]

- strength: \( B = 10^{15} \ T \approx 10^{20} B_{\text{earth}} \approx 5m_{\pi}^2 \)

- impact: chiral magnetic effect, anisotropies, elliptic flow, hydrodynamics with \( B \), ...

reviews: [Fukushima '12] [Kharzeev, Landsteiner, Schmitt, Yee '14] [Kharzeev '15]
Magnetic fields magnetars

- neutron stars with strong surface magnetic fields
  [Duncan, Thompson '92]

- strength on surface: \( B = 10^{10} \) T
- strength in core: \( B = 10^{14...16} \) T \( \approx 10^{19...21} B_{\text{earth}} \)
- impact: convectional processes, mass-radius relation, gravitational collapse/merger [Anderson et al '08], ...
Magnetic fields: early universe

- Large-scale intergalactic magnetic fields $10 \, \mu G = 10^{-9} \, T$
- Origin in the early universe
- Generation through a phase transition: electroweak epoch $B \approx 10^{19} \, T$ [Vachaspati '91, Enqvist, Olesen '93]
Neutron to nucleon ratio in nuclei $\frac{Z}{A} \approx 0.4$ but: ‘neutron skin’ near surface

Neutron to nucleon ratio in interior of neutron stars $\frac{Z}{A} \approx 0.025$
Order parameters
Order parameters

- quark condensate $\bar{\psi} \psi$ (chiral symmetry breaking)

\[
\bar{\psi} \psi = \frac{T}{V} \frac{\partial \log Z}{\partial m} = \left\langle \text{Tr} \left\{ \frac{1}{\mathcal{D} + m} \right\} \rightangle
\]

- Polyakov loop $P$ (deconfinement)

\[
P = \left\langle \text{Tr} \mathcal{P} \exp \int_0^{1/T} A_4(x, \tau) d\tau \right\rangle
\]
Order parameters

- $T_c \leftrightarrow$ inflection point
- nature of phase transition $\leftrightarrow$ singularity in slope at $T_c$

[Aoki, Endrödi et al '06, Borsányi et al. '10]
Order parameters

- $T_c \leftrightarrow$ inflection point
- nature of phase transition $\leftrightarrow$ singularity in slope at $T_c$

[Graph showing $\Delta_{i,s}$ vs $T$ for different $N_i$ values.]

[Aoki, Endrödi et al. '06, Borsányi et al. '10]

- analytical crossover
Background magnetic fields
Impact of magnetic fields

- on quark condensate: primary

\[
\begin{align*}
\bar{\psi} & \quad \text{---} \quad \psi \\
& \quad (\text{Diagram: } B) 
\end{align*}
\]

- on Polyakov loop: secondary

\[
\begin{align*}
A_4 & \quad \text{---} \quad A_4 \\
& \quad (\text{Diagram: } B) 
\end{align*}
\]
Magnetic catalysis

- Chiral condensate $\leftrightarrow$ spectral density around 0 [Banks, Casher '80]
  \[ \bar{\psi}\psi \sim \text{tr} \hat{D}^{-1} \propto \rho(0) \]

- Large magnetic fields reduce dimensionality $3 + 1 \rightarrow 1 + 1$
  and induce degeneracy $\propto B$

- To maintain $\bar{\psi}\psi > 0$ [Gusynin et al '96]
  \[
  B = 0 \quad \rho(p)dp \sim Tp^2dp \quad \text{“strong interaction is needed”}
  \]
  \[
  B \gg m^2 \quad \rho(p)dp \sim TB\,dp \quad \text{“the weakest interaction suffices”}
  \]
Magnetic catalysis: lattice simulations

- numerical simulation of the path integral

\[ Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(-S_{\text{QCD}}) \]

- obtain condensate from

\[ \bar{\psi}\psi = \frac{1}{V_4} \frac{\partial \log Z}{\partial m} \]

- physical \( m_\pi \), continuum limit

[Bali, Bruckmann, Endrödi, Fodor, Katz, Schäfer '12]
Magnetic catalysis – zero temperature

- magnetic catalysis at zero temperature is a robust concept: \( \chi \)PT, NJL, AdS-CFT, linear \( \sigma \) model, lattice QCD, … 

[Andersen, Naylor ’14]

![Graph showing the relationship between \( \Delta (\Sigma_u + \Sigma_d) / 2 \) and \( eB \) (GeV²) at zero temperature.](image)

[Bali, Bruckmann, Endrödi, Fodor, Katz, Schäfer ’12]
Effect of magnetic fields: nonzero temperature
Phase diagram: models

- recall catalysis argument $\bar{\psi}\psi \propto \rho(0)$

- model calculations at $T > 0$:
  - magnetic catalysis for all $T$
  - $T_c(B)$ increases

- for example the PNJL model [Gatto, Ruggieri '11]
Phase diagram: models

- majority of low-energy models give the same qualitative result
  - linear sigma model + Polyakov loop [Mizher, Chernodub, Fraga '10]
  - quark-meson model + functional renormalization group [Kamikado, Kanazawa '13]
  - NJL model + Polyakov loop [Ferreira, Costa, Menezes, Providencia, Scoccola '13]

...
Phase diagram: lattice simulations

- lattice QCD, physical $m_\pi$, continuum limit
  [Bali, Bruckmann, Endrödi, Fodor, Katz, Krieg, Schäfer, Szabó '11, '12]

- surprise: magnetic catalysis turns into inverse magnetic catalysis (IMC) around $T_c$ [Bruckmann, Endrödi, Kovács '13]
Phase diagram: lattice simulations

- lattice QCD, physical $m_\pi$, continuum limit
  [Bali, Bruckmann, Endrődi, Fodor, Katz, Krieg, Schäfer, Szabó '11, '12]

- surprise: magnetic catalysis turns into inverse magnetic catalysis (IMC) around $T_c$
  [Bruckmann, Endrődi, Kovács '13]
Phase diagram: lattice simulations

- impact on the QCD phase diagram
Phase diagram: lattice simulations

- impact on the QCD phase diagram
  [Endrődi ’15]
Phase diagram: comparison

\[ T_c(B) \] increases
\[ T_c^{(P)} \text{ and } T_c^{(\overline{\psi}\psi)} \] diverge
condensate
magnetic catalysis \( \forall T \)

\[ T \approx T_c \] inverse catalysis

\[ T_{B=0} = 175 \text{ MeV} \]
Phase diagram: comparison

![Graph showing the comparison between model and lattice simulations.](image)

<table>
<thead>
<tr>
<th></th>
<th>model</th>
<th>lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_c(B)$</td>
<td>increases</td>
<td>decreases</td>
</tr>
<tr>
<td>$T_c(P)$ and $T_c(\bar{\psi}\psi)$</td>
<td>diverge</td>
<td>converge</td>
</tr>
<tr>
<td>condensate</td>
<td>magnetic catalysis $\forall T$</td>
<td>inverse catalysis $T \approx T_c$</td>
</tr>
</tbody>
</table>

- models and lattice simulations are as different as can be
Inverse magnetic catalysis

- related to secondary effect of $B$ on gluons

- encoded in effective potential for $P$ in models

- tune this potential or coupling constants of models with $B$

[Fraga et al. '13, Ferreira et al. '14, Ayala et al. '15, '18 . . . ]
Large $B$: anisotropic effective theory
Large $B$ limit

- what happens to $\mathcal{L}_{\text{QCD}}$ at $eB \gg \Lambda^2_{\text{QCD}}, T^2$?
  - first guess: asymptotic freedom says $\alpha_s \to 0$ i.e. complete decoupling of quarks and gluons
  - but: $B$ breaks rotational symmetry and effectively reduces the dimension of the theory for quarks

- gluons also inherit this spatial anisotropy, $\kappa(B) \propto B$

[Miransky, Shovkovy 2002; Endrődi 1504.08280]

$$\mathcal{L}_{\text{QCD}} \xrightarrow{B \to \infty} \text{tr} B^2 \parallel + \text{tr} B^2 \perp + [1 + \kappa(B)] \text{tr} \mathcal{E}^2 \parallel + \text{tr} \mathcal{E}^2 \perp$$
Simulating the anisotropic effective theory

- pure (but anisotropic) gauge theory: can be simulated on the lattice [Endrődi 1504.08280]

- Polyakov loop susceptibility peak height scales with $V$

- histogram shows double peak-structure at $T_c$
Simulating the anisotropic effective theory

- pure (but anisotropic) gauge theory: can be simulated on the lattice [Endrődi 1504.08280]

- Polyakov loop susceptibility peak height scales with $V$
- histogram shows double peak-structure at $T_c$
- the transition is of first order
Critical point

- analytical crossover for $0 \leq eB \leq 3.25 \text{ GeV}^2$
- first-order transition for $B \rightarrow \infty$
  - there must be a critical point in between [Cohen, Yamamoto ’13]
- estimate: extrapolate width of susceptibility peak to 0

$$eB_{CP} \approx 10(2) \text{ GeV}^2$$
Phase diagram

The diagram illustrates the relationship between deconfinement temperature $T_c$ (in MeV) and the product of electric field strength and quark mass $eB$ (GeV$^2$). Key features include:

- **Deconfinement Transition Line**: A line indicating the transition between confined and deconfined states.
- **Prediction**: An arrow pointing to a region indicating a predicted phase transition.
- **Crossover**: A region where the transition is smooth.
- **Critical Point**: A specific point on the transition line.
- **First Order**: A transition that is abrupt or discontinuous.

The diagram provides a visual representation of the phase transitions in a high-energy physics context.
Isospin asymmetry
Isospin chemical potential

- quark chemical potentials (3-flavor)
  \[ \mu_u = \frac{\mu_B}{3} + \mu_I \quad \mu_d = \frac{\mu_B}{3} - \mu_I \quad \mu_s = \frac{\mu_B}{3} - \mu_S \]

- zero baryon number, zero strangeness, but nonzero isospin
  \[ \mu_u = \mu_I \quad \mu_d = -\mu_I \quad \mu_s = 0 \]

- pion chemical potential \( \mu_\pi = \mu_u - \mu_d = 2\mu_I \)

- isospin density \( n_I = n_u - n_d \)
Pion condensation

- QCD at low energies $\approx$ pions
  - chiral perturbation theory
- chemical potential for charged pions: $\mu_\pi$

at zero temperature

$\mu_\pi < m_\pi$ vacuum state

$\mu_\pi \geq m_\pi$ Bose-Einstein condensation

[Son, Stephanov '00]
Bose-Einstein condensate

- accumulation of bosonic particles in lowest energy state

[Anderson et al '95 JILA-NIST/University of Colorado]

- velocity distribution of Ru atoms at low temperature
  → peak at zero velocity (zero energy)
Setup on the lattice
[Brandt, Endrődi, Schmalzbauer '17]
Symmetry breaking

- QCD with light quark matrix
  \[ M = \mathcal{D} + m_{ud} \mathbb{1} \]
- Chiral symmetry (flavor-nontrivial)
  \[ SU(2)_V \]
Symmetry breaking

- QCD with light quark matrix

\[ M = \mathcal{D} + m_{ud} \mathbb{1} + \mu \gamma_0 \tau_3 \]

- chiral symmetry (flavor-nontrivial)

\[ SU(2)_V \rightarrow U(1)_{\tau_3} \]
Symmetry breaking

- QCD with light quark matrix
  \[ M = \bar{\psi} \gamma_5 \tau_1 \psi + m_{ud} \mathbb{1} + \mu \gamma_0 \tau_3 \]
- chiral symmetry (flavor-nontrivial)
  \[ SU(2)_V \rightarrow U(1)_{\tau_3} \]
- spontaneously broken by a pion condensate
  \[ \langle \pi^\pm \rangle = \langle \bar{\psi} \gamma_5 \tau_1,2 \psi \rangle \]
- a Goldstone mode appears
Symmetry breaking

- QCD with light quark matrix
  \[ M = \slashed{D} + m_{ud} 1 + \mu \gamma_0 \tau_3 + i \lambda \gamma_5 \tau_2 \]
- Chiral symmetry (flavor-nontrivial)
  \[ SU(2)_V \rightarrow U(1)_{\tau_3} \rightarrow \emptyset \]
- Spontaneously broken by a pion condensate
  \[ \langle \pi^\pm \rangle = \langle \bar{\psi} \gamma_5 \tau_{1,2} \psi \rangle \]
- A Goldstone mode appears
- Add small explicit breaking
Symmetry breaking

- QCD with light quark matrix
  \[ M = \mathcal{D} + m_{ud} \mathbb{1} + \mu \gamma_0 \tau_3 + i \lambda \gamma_5 \tau_2 \]

- chiral symmetry (flavor-nontrivial)
  \[ \text{SU}(2)_V \to \text{U}(1)_{\tau_3} \to \emptyset \]

- spontaneously broken by a pion condensate
  \[ \langle \pi^\pm \rangle = \langle \bar{\psi} \gamma_5 \tau_{1,2} \psi \rangle \]

- a Goldstone mode appears
- add small explicit breaking

- extrapolate results \( \lambda \to 0 \)
Simulation details

- staggered light quark matrix with $\eta_5 = (-1)^{n_x+n_y+n_z+n_t}$

\[ M = \begin{pmatrix} \mathcal{D}(\mu_I) + m & \lambda \eta_5 \\ -\lambda \eta_5 & \mathcal{D}(\mu_I) + m \end{pmatrix} \]

- we have $\gamma_5 \tau_1$-hermiticity

\[ \eta_5 \tau_1 M \tau_1 \eta_5 = M^\dagger \]

- determinant is real and positive

\[ \det M = \det(|\mathcal{D}(\mu_I) + m|^2 + \lambda^2) > 0 \]

- partition function via Monte-Carlo

\[ Z = \int \mathcal{D}U \det(|\mathcal{D}(\mu_I) + m|^2 + \lambda^2) \det(\mathcal{D}(0) + m_s) e^{-S_g} \]

\[ \text{light quarks} \quad \text{strange quark} \quad \text{gluons} \]
Need for improvement

- simulations at $\lambda > 0$ are far away from desired $\lambda \rightarrow 0$ limit
Need for improvement

- simulations at $\lambda > 0$ are far away from desired $\lambda \rightarrow 0$ limit

- improvement using the singular value representation of $M$
Singular value representation

- pion condensate
  \[ \Sigma_\pi = \frac{\partial}{\partial \lambda} \log \det(|\Phi(\mu_I) + m|^2 + \lambda^2) = \text{Tr} \frac{2\lambda}{|\Phi(\mu_I) + m|^2 + \lambda^2} \]

- singular values
  \[ |\Phi(\mu_I) + m|^2 \psi_i = \xi_i^2 \psi_i \]

- spectral representation
  \[ \Sigma_\pi = \frac{T}{V} \sum_i \frac{2\lambda}{\xi_i^2 + \lambda^2} \xrightarrow{V \to \infty} \int d\xi \rho(\xi) \frac{2\lambda}{\xi^2 + \lambda^2} \xrightarrow{\lambda \to 0} \pi \rho(0) \]
  first derived in [Kanazawa, Wettig, Yamamoto ’11]
Singular value representation

- **pion condensate**

  \[ \Sigma_\pi = \frac{\partial}{\partial \lambda} \log \det \left( |\Phi(\mu_I) + m|^2 + \lambda^2 \right) = \text{Tr} \frac{2\lambda}{|\Phi(\mu_I) + m|^2 + \lambda^2} \]

- **singular values**

  \[ |\Phi(\mu_I) + m|^2 \psi_i = \xi_i^2 \psi_i \]

- **spectral representation**

  \[ \Sigma_\pi = \frac{T}{V} \sum_i \frac{2\lambda}{\xi_i^2 + \lambda^2} \xrightarrow{V \to \infty} \int \, d\xi \rho(\xi) \frac{2\lambda}{\xi^2 + \lambda^2} \xrightarrow{\lambda \to 0} \pi \rho(0) \]

  first derived in [Kanazawa, Wettig, Yamamoto ’11]

- **compare to Banks-Casher-relation at \( \mu_I = 0 \)**
**Singular value density**

- integrated spectral density

\[ N(\xi) = \int_0^\xi d\xi' \rho(\xi'), \quad \rho(0) = \lim_{\xi \to 0} \frac{N(\xi)}{\xi} \]
Singular value density

- integrated spectral density

\[ N(\xi) = \int_{0}^{\xi} d\xi' \rho(\xi'), \quad \rho(0) = \lim_{\xi \to 0} N(\xi) / \xi \]

- compare \( \rho(0) \) to velocity distribution around zero

\[ \frac{a^3 \langle N(\xi) \rangle}{\xi} \]

\[ \xi / m_{ud} \]

\[ \mu_1 / m_\pi = 0.55 \]
\[ 0.48 \]
\[ 0.38 \]

\[ T = 113 \text{ MeV} \]
Singular value density

- integrated spectral density

\[ N(\xi) = \int_0^\xi d\xi' \rho(\xi'), \quad \rho(0) = \lim_{\xi \to 0} \frac{N(\xi)}{\xi} \]

- compare \( \rho(0) \) to velocity distribution around zero

- Bose-Einstein condensation!
Results: phase transition
Condensates

- pion and chiral condensate after $\lambda \to 0$ extrapolation

- read off chiral crossover $T_{pc}(\mu_I)$ and pion condensation boundary $\mu_{I,c}(T)$
Order of the transition

- Volume scaling of order parameter shows typical sharpening
- Collapse according to $O(2)$ critical exponents [Ejiri et al. '09]
Order of the transition

- volume scaling of order parameter shows typical sharpening
- collapse according to $O(2)$ critical exponents [Ejiri et al '09]
- indications for a second order phase transition at $\mu_I = m_\pi / 2$, in the $O(2)$ universality class
Continuum extrapolations

- compare (pseudo)critical temperatures for different lattice spacings \( a = 1/(N_t T) \)
- take continuum limit \( a \to 0 \) \( (N_t \to \infty) \)
Phase diagram

- meeting point of chiral crossover and pion condensation boundary: *pseudo-triple* point

at $T_{pt} = 151(7)$ MeV, $\mu_{I,pt} = 70(5)$ MeV
Deconfinement vs chiral symmetry breaking

- Polyakov loop contour lines apparently insensitive to pion condensation boundary
- existence of a condensed but deconfined phase?
Conjectured phase diagram

- Hadronic phase
- Quark-gluon plasma
- BCS phase
- BEC phase

BCS phase expected on general grounds for high $\mu$ [Son, Stephanov '00]
Conjectured phase diagram

hadronic phase

quark-gluon plasma

BCS phase

\[ T \]

\[ T_d \]

\[ m_{\pi}/2 \]

BEC phase

\[ \mu_i \]
Conjectured phase diagram

- BCS phase expected on general grounds for high $\mu_i$
  
  [Son, Stephanov '00]
Potential applications
Magnetic fields in the early universe?

- large-scale intergalactic magnetic fields \(10 \, \mu G = 10^{-9} \, T\) origin in the early universe
- generation through a phase transition: electroweak epoch \(B \approx 10^{19} \, T \approx 600 \, \text{GeV}^2/e\) [Vachaspati '91, Enqvist, Olesen '93]
- how large is \(B\) that survives until the QCD epoch?
Magnetic fields in the early universe?

- large-scale intergalactic magnetic fields $10 \mu G = 10^{-9} \text{T}$
  origin in the early universe
- generation through a phase transition: electroweak epoch
  $B \approx 10^{19} \text{T} \approx 600 \text{GeV}^2/e$ [Vachaspati '91, Enqvist, Olesen '93]
- how large is $B$ that survives until the QCD epoch?
Pion condensation in the early universe?

- weak equilibrium
  \[ u \leftrightarrow d \, e^- \bar{\nu}_e \]
  (simplicity: one family of particles)

- charge neutrality \( n_Q = 0 \), baryon symmetry \( n_B = 0 \)
  but nonzero lepton number \( n_L \neq 0 \)

- chemical potentials \( \mu_Q \), \( \mu_B \) and \( \mu_L \)

- can \( \mu_Q = 2\mu_l > m_\pi \) be reached? for sufficiently large \( n_L \), yes
  [Abuki, Brauner, Warringa '09]
Pion condensation in compact stars?

- equation of state for isospin-dense system at low $T$, neutralized by leptons
  [Brandt, Endrődi, Fraga, Hippert, Schaffner-Bielich, Schmalzbauer '18]
- solve Tollman-Oppenheimer-Volkov equations for gravitational stability

- weak decays under investigation
Magnetic fields in heavy-ion collisions?

- off-central events generate magnetic fields
  [Kharzeev, McLerran, Warringa '07]

- test charge-dependence in RHIC isobar run [bnl.gov]
CME-sensitive observables for nonzero density

- correlation of topology and electric polarization
  
  [Bali, Bruckmann, Endrődi, Fodor, Katz, Schäfer ’14]

\[ D_u(\Delta) = \int d^4x \langle q_{\text{top}}(x)E_z(x + \Delta) \rangle \]

- correlations affected by isospin chemical potential?
Summary

- phase diagram for strong background magnetic fields
- phase diagram for nonzero isospin-asymmetry
- new Banks-Casher-type relation
  → establish pion condensation
  → improve various observables