Microscopic Foundations of Relativistic Fluid Dynamics

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May 2022

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05/06/2022

1 Relativistic Fluid Dynamics

The physical description of a system consisting of many degrees of freedom is in general quite complicated. However, if one is interested in the **large-distance, long-timescale** behavior of the system, it becomes possible to devise an **effective theory**, taking into account only the degrees of freedom that are relevant on these scales. This happens because, on macroscopic time and length scales, we are not able to observe the microscopic degrees of freedom of the underlying theory, but only average quantities resulting from interactions on the microscopic level. Most of the microscopic quantities vary rapidly in space and time, leading to very small changes of the average values, and are not expected to contribute to the macroscopic dynamics. On the other hand, the few variables that do vary **slowly**, such as **conserved quantities**, are expected to be relevant for the **effective description** of the system on **macroscopic length and time scales**.

Fluid dynamics is a typical example of such an effective theory. It is a **classical field theory** that describes the **macroscopic dynamics** of systems called **fluids**. A fluid is a **continuous system** in which every infinitesimal volume element is (usually) assumed to be **close to thermodynamic equilibrium** and to maintain the proximity to equilibrium throughout its evolution. In other words, in the vicinity of each point in space, we define an infinitesimal volume, called fluid element, in which the matter is taken to be homogeneous, i.e., any spatial gradients can be neglected, and is described by a finite set of thermodynamic variables and currents. This means that each fluid element must be **large enough relative to the microscopic length scales**, to guarantee the proximity to thermodynamic equilibrium, and, at the same time, it must be **small enough relative to the macroscopic length scales**, to ensure the continuum limit.

At first glance, the simultaneous applicability of the continuous (zero fluid-element volume) and thermodynamic (infinite fluid-element volume) limits might seem contradictory. However, if the microscopic and macroscopic scales of the system are **sufficiently far separated**, it is always possible to ensure the existence of a volume that is small when compared with the macroscopic scales, and large when compared with the microscopic ones. For example, considering the case of water, a fluid element with a volume of about 1 mm^3 is small enough to assure the continuous-limit approximation and large enough to enclose many molecules and to apply the thermodynamic limit. Note that, for small or rapidly changing systems, such a separation of scales may not be so clear, making it difficult to ensure the proximity to local thermal equilibrium. This is of particular importance when we apply fluid dynamics to describe the hot and dense matter formed in relativistic heavy-ion collisions, where a clear separation between microscopic and macroscopic scales does not exist. The progress in developing dissipative theories of relativistic fluid dynamics achieved in recent years has been largely driven by applying them to describe the dynamical evolution of heavy-ion collisions.

In this chapter, we discuss the basic aspects of relativistic fluid dynamics from a phenomenological perspective. It is organized as follows: In Sec. 1.1, we introduce the basic laws of thermodynamics and derive the thermodynamic relations that are useful for this book. Section 1.2 contains a brief review of relativistic ideal fluid dynamics. We derive the general form of the conserved currents of an ideal fluid and their equations of motion. Then, Sec. 1.3 shows how to introduce dissipation in fluid dynamics. Here, we explain the basic aspects of dissipative fluid dynamics and derive a covariant version of Navier-Stokes theory using the second law of thermodynamics and, also, via the gradient expansion. In Sec. 1.4 we review the problems of Navier-Stokes theory in the relativistic regime, i.e., the acausality and instability of this theory. We also explain how to render Navier-Stokes theory causal and stable, and to derive a consistent theory of fluid dynamics. Finally, in Sec. 1.5, we discuss Israel-Stewart theory and show how to derive causal fluid-dynamical equations from the second law of thermodynamics.

1.1 Thermodynamics

Thermodynamics is a theory empirically constructed to describe the thermodynamical-equilibrium state in macroscopic systems. It attempts to describe such a state in terms of a small set of **extensive quantities**, such as the total **energy**, E, **volume**, V, and (net) **number of particles**, N, of the system. Thermodynamics is based on four phenomenological laws, obtained over the years by experimental observation [1]:

Zeroth Law: Two systems that are in equilibrium with a third system are in equilibrium with each other.

First Law: Energy is conserved.

Second Law: The change in entropy of a closed thermodynamic system is always positive semi-definite.

Third Law: The difference in entropy between systems connected by a reversible process is zero in the limit of vanishing temperature.

In this lecture series, we shall make use of the **first** and **second laws of thermodynamics** and, therefore, it is convenient to discuss them in more detail. The **first law of thermodynamics** implies that small variations of the state variables, E, V, and N, must be related,

$$\delta E = \delta Q - P\delta V + \mu\delta N \, , \tag{1.1}$$

where P and μ are the **pressure** and **chemical potential**, respectively. As a conservation law, the first law of thermodynamics postulates that changes in the total energy of the system (δE) must result from mechanical work done by an external force ($-P\delta V$), from particle exchange with an external medium ($\mu\delta N$), or/and from heat exchange (δQ). The heat exchange takes into account the energy variations due to changes of internal degrees of freedom that are not described by the state variables. The heat itself is not a state variable since it can depend on the past evolution of the system and may take several values for the same thermodynamic state.

However, when dealing with time-reversible processes, it becomes possible to assign a state variable related to heat. This variable is the **entropy**, S, and is defined in terms of the heat exchange as $\delta Q = T\delta S$, with the **temperature** T being the proportionality constant. Then, when considering variations between equilibrium states that are infinitesimally close to each other, it is possible to write the first law of thermodynamics in terms of complete differentials of the state variables,

$$dE = TdS - PdV + \mu dN \,. \tag{1.2}$$

Using Eq. (1.2), it is possible to identify the **intensive quantities**, T, μ, and P as the following partial derivatives of the energy,

$$T = \left.\frac{\partial E}{\partial S}\right|_{N,V} , \qquad -P = \left.\frac{\partial E}{\partial V}\right|_{S,N} , \qquad \mu = \left.\frac{\partial E}{\partial N}\right|_{S,V} . \tag{1.3}$$

The first law of thermodynamics can also be written in terms of entropy variations, i.e.,

$$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN \,, \tag{1.4}$$

in which case the intensive variables can be obtained from partial derivatives of the entropy,

$$\frac{1}{T} = \left.\frac{\partial S}{\partial E}\right|_{N,V} , \qquad \frac{P}{T} = \left.\frac{\partial S}{\partial V}\right|_{E,N} , \qquad \frac{\mu}{T} = -\left.\frac{\partial S}{\partial N}\right|_{E,V} . \tag{1.5}$$

In the **thermodynamical limit**, the entropy is an **extensive** and **additive** function of the state variables,

$$\lambda S = S\left(\lambda E, \lambda V, \lambda N\right) \,. \tag{1.6}$$

Using this property, it is straightforward to prove that

$$S = \frac{\partial}{\partial \lambda}\left(\lambda S\right) = \left.\frac{\partial S}{\partial(\lambda E)}\right|_{\lambda N,\lambda V} E + \left.\frac{\partial S}{\partial(\lambda V)}\right|_{\lambda E,\lambda N} V + \left.\frac{\partial S}{\partial(\lambda N)}\right|_{\lambda E,\lambda V} N \,, \tag{1.7}$$

which holds for any value of λ. Taking $\lambda = 1$ and using Eq. (1.5), we derive the so-called **Euler relation**,

$$TS = E + PV - \mu N \,, \tag{1.8}$$

and using Euler's relation, combined with the first law of thermodynamics, we obtain the **Gibbs-Duhem relation**,

$$VdP = SdT + Nd\mu \,. \tag{1.9}$$

Together, Eqs. (1.2), (1.8), and (1.9) allow us to derive the thermodynamic relations satisfied by the energy, entropy, and (net) particle number **densities**, $\varepsilon \equiv E/V$, $s \equiv S/V$, and $n \equiv N/V$, respectively. They are

$$\varepsilon + P = Ts + \mu n \,, \tag{1.10}$$

$$ds = \beta d\varepsilon - \alpha dn \,, \tag{1.11}$$

$$dP = sdT + nd\mu \,, \tag{1.12}$$

where we defined the **inverse temperature** $\beta = 1/T$ and the **thermal potential** $\alpha = \mu/T$. Then, Eqs. (1.11) and (1.12) can be used to derive the relations between the intensive variables and the densities,

$$\beta = \left.\frac{\partial s}{\partial \varepsilon}\right|_n , \qquad \alpha = \left.\frac{\partial s}{\partial n}\right|_\varepsilon , \qquad s = \left.\frac{\partial P}{\partial T}\right|_\mu , \qquad n = \left.\frac{\partial P}{\partial \mu}\right|_T . \tag{1.13}$$

The **second law of thermodynamics** dictates that the entropy of an isolated system must either increase or remain constant. This implies that, if a given system is in **thermodynamic equilibrium**, i.e., if it is in a quasi-stationary state where its extensive and intensive variables no longer change, the **entropy** of this system must remain **constant** (as long as the boundary conditions imposed on the system remain fixed). On the other hand, the entropy of a system that is **out of equilibrium** must always **increase**. This is a very useful and powerful concept that will be extensively used in this chapter. As we will show later, the second law of thermodynamics can even be used to constrain and, sometimes, derive the equations of motion of a viscous fluid.

These are the basic aspects of thermodynamics that we wanted to address (for a more detailed review, see Ref. [1]). It is worth to point out that, although the thermodynamic relations specify how the macroscopic variables are related and how they change with time, they are not enough to extract the **explicit form of the equation of state**, i.e., of the function $s(\varepsilon, n)$. In order to determine the entropy density as a function of the state variables ε, n, a **microscopic description** of the matter is required, which can only be obtained from a more fundamental approach, such as statistical mechanics.

Exercise 1.1: Prove Eqs. (1.10) – (1.12).

1.2 Relativistic ideal fluid dynamics

We start our discussion of relativistic fluid dynamics by considering the most simple example: the case of an ideal fluid [2, 3]. An ideal fluid is defined by the assumption of **local thermodynamical equilibrium**, i.e., that all fluid elements must be in thermodynamic equilibrium, but not necessarily in the same thermodynamic-equilibrium state (if they are, one speaks of **global** thermodynamical equilibrium). This means that all thermodynamic state variables are functions of the space-time 4-vector $x^\mu \equiv X = (t, \mathbf{x})^T$, e.g. temperature $T(X)$, chemical potential $\mu(X)$. In addition, the fluid is described by a collective velocity field, which is also function of space-time, $\mathbf{u}(X)$. From now on, the fields T, μ, and $\mathbf{u}$ shall be referred to as **primary fluid-dynamical variables**.

1.2.1 Conserved currents in an ideal fluid

The state of a fluid is specified by the densities and currents associated to **conserved quantities**, i.e., **energy**, **momentum**, and (net) **particle number**. For a relativistic fluid, the state variables are the **energy-momentum tensor**, $T^{\mu\nu}(X)$, and the (net)

particle 4-current, $N^\mu(X)$. For an **ideal fluid**, the general form of these currents can be obtained by performing a **Lorentz transformation** to the **local rest frame** of the fluid, in which $\mathbf{u}(X) = 0$. In this frame, the energy-momentum tensor, $T^{\mu\nu}_{\mathrm{RF}}$ (the subscript RF indicates the local rest-frame form of this tensor), should have the characteristic form of a system in **static equilibrium**,

$$T^{\mu\nu}_{\mathrm{RF}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} , \tag{1.14}$$

i.e., in this frame there is **no flow of energy** ($T^{i0}_{\mathrm{RF}} = 0$), and the force per surface element is **isotropic** and equal to the **thermodynamic pressure** ($T^{ij}_{\mathrm{RF}} = \delta^{ij} P$). In the local rest frame, there should also be **no flow of particles and entropy** and, consequently, the (net) particle and entropy 4-currents in this frame, N^μ_{RF} and S^μ_{RF}, respectively, take the following simple form

$$N^\mu_{\mathrm{RF}} = (n, 0, 0, 0)^T , \tag{1.15}$$
$$S^\mu_{\mathrm{RF}} = (s, 0, 0, 0)^T . \tag{1.16}$$

The form of these tensors in a **general frame**, i.e., a frame where the fluid moves with velocity $+\mathbf{u}$, can be derived by applying a Lorentz boost with velocity $-\mathbf{u}$ to N^μ_{RF}, S^μ_{RF}, and $T^{\mu\nu}_{\mathrm{RF}}$,

$$N^\mu = \Lambda^\mu_\alpha (-\mathbf{u})\, N^\alpha_{\mathrm{RF}} , \tag{1.17}$$
$$S^\mu = \Lambda^\mu_\alpha (-\mathbf{u})\, S^\alpha_{\mathrm{RF}} , \tag{1.18}$$
$$T^{\mu\nu} = \Lambda^\mu_\alpha (-\mathbf{u})\, \Lambda^\nu_\beta (-\mathbf{u})\, T^{\alpha\beta}_{\mathrm{RF}} , \tag{1.19}$$

where we remember that the general form of a Lorentz boost is

$$\Lambda^\mu_\alpha (\mathbf{u}) = \begin{pmatrix} \gamma & -u^x & -u^y & -u^z \\ -u^x & 1 + (\gamma+1)^{-1} u^x u^x & (\gamma+1)^{-1} u^x u^y & (\gamma+1)^{-1} u^x u^z \\ -u^y & (\gamma+1)^{-1} u^y u^x & 1 + (\gamma+1)^{-1} u^y u^y & (\gamma+1)^{-1} u^y u^z \\ -u^z & (\gamma+1)^{-1} u^z u^x & (\gamma+1)^{-1} u^z u^y & 1 + (\gamma+1)^{-1} u^z u^z \end{pmatrix} , \tag{1.20}$$

with $\gamma = \sqrt{1 + \mathbf{u}\cdot\mathbf{u}}$ being the Lorentz gamma factor. The boost velocity is $-\mathbf{u}$, because an observer who sees the fluid move with velocity $+\mathbf{u}$ moves itself with velocity $-\mathbf{u}$ with respect to the fluid rest frame.

Then, using a covariant notation, the **conserved currents of an ideal fluid** can be expressed as

$$N^\mu_{\mathrm{ideal}} \equiv N^\mu_{(0)} = n u^\mu , \tag{1.21}$$
$$S^\mu_{\mathrm{ideal}} \equiv S^\mu_{(0)} = s u^\mu , \tag{1.22}$$
$$T^{\mu\nu}_{\mathrm{ideal}} \equiv T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} P , \tag{1.23}$$

where u^μ is the **velocity 4-vector**,

$$u^\mu = (\gamma, \mathbf{u})^T , \tag{1.24}$$

Note that the velocity 4-vector is constructed to satisfy the normalization condition $u^\mu u_\mu = 1$, and, therefore, has only **three independent components**. Also, in Eq. (1.23), we introduced the **projection operator** onto the **3-space orthogonal** to u^μ,

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \ , \tag{1.25}$$

where $g_{\mu\nu}$ is the space-time metric (in flat space). $\Delta^{\mu\nu}$ satisfies all the properties expected of a projector,

$$u_\mu \Delta^{\mu\nu} = u_\nu \Delta^{\mu\nu} = 0 \ , \qquad \Delta^\mu_\lambda \Delta^{\lambda\nu} = \Delta^{\mu\nu} \ , \tag{1.26}$$

and has the following trace,

$$\Delta^\mu_\mu = 3 \ . \tag{1.27}$$

Exercise 1.2: Prove Eqs. (1.21) – (1.23) by explicit computation of the right-hand sides of Eqs. (1.17) – (1.19).

1.2.2 Equations of motion

The dynamical description of an ideal fluid is obtained using the **conservation laws** of **energy**, **momentum**, and (net) **particle number**. These conservation laws can be mathematically expressed in terms of the following five **continuity equations**,

$$\partial_\mu N^\mu_{(0)} = 0 \ , \tag{1.28}$$
$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \ . \tag{1.29}$$

The partial derivative $\partial_\mu \equiv \partial/\partial x^\mu$ transforms as a covariant vector under Lorentz transformations and, therefore, Eq. (1.29) transforms as a contravariant 4-vector. As a 4-vector, it is convenient to decompose this equation into a part **parallel** and a part **orthogonal** to u^μ. The component parallel to the velocity is obtained by contracting the equation of motion with u^μ, $u_\alpha \partial_\beta T^{\alpha\beta}_{(0)}$, while the component orthogonal to the velocity is obtained by contracting it with $\Delta^{\mu\nu}$, $\Delta^\mu_\alpha \partial_\beta T^{\alpha\beta}_{(0)}$. This, together with the conservation law for (net) particle number, leads to the **equations of motion of ideal fluid dynamics**,

$$u_\alpha \partial_\beta T^{\alpha\beta}_{(0)} = \dot{\varepsilon} + (\varepsilon + P)\,\theta = 0 \ , \tag{1.30}$$
$$\Delta^\mu_\alpha \partial_\beta T^{\alpha\beta}_{(0)} = (\varepsilon + P)\,\dot{u}^\mu - \nabla^\mu P = 0 \ , \tag{1.31}$$
$$\partial_\mu N^\mu_{(0)} = \dot{n} + n\theta = 0 \ , \tag{1.32}$$

where we introduced the **comoving derivative** $u^\mu \partial_\mu A \equiv \dot{A}$ of any quantity A and the **space-like gradient** $\Delta^\lambda_\mu \partial_\lambda \equiv \nabla_\mu$. We further defined the **expansion scalar**, θ as the 4-divergence of the 4-velocity,

$$\theta \equiv \nabla_\mu u^\mu \ . \tag{1.33}$$

Exercise 1.3: Prove Eqs. (1.30) – (1.32) by inserting Eqs. (1.21), (1.23) into the conservation laws (1.28), (1.29) and, in the case of energy-momentum conservation, performing the projections onto u_α and Δ^μ_α.

Note that an ideal fluid is described by four fields, ε, P, n, and u^μ, that contain, in total, **six independent degrees of freedom**. The conservation laws, on the other hand, provide only **five equations of motion**. To **close** this system of equations, we must specify the **equation of state of the fluid**, which gives the pressure as a function of the other thermodynamic variables,

$$P = P(\varepsilon, n) \ . \tag{1.34}$$

The assumption of local thermal equilibrium guarantees the existence of this function and, hence, assures that the equations of ideal fluid dynamics are always closed. In ideal fluid dynamics, the equation of state essentially defines the type of fluid that is being described – it is the only place where (some of) the microscopic properties of the system must be taken into account.

1.2.3 Covariant thermodynamics and entropy production

Using the conserved currents, $N^\mu_{(0)}$, $S^\mu_{(0)}$, and $T^{\mu\nu}_{(0)}$, we can re-write the equilibrium thermodynamic relations derived in Sec. 1.1, Eqs. (1.10), (1.11), and (1.12), in a covariant form [5, 6, 7]. For this purpose, it is convenient to introduce the following 4-vector,

$$\beta^\mu = \frac{u^\mu}{T} \ . \tag{1.35}$$

Then, following Israel and Stewart [5, 6, 7], we postulate a **covariant version** of the Gibbs-Duhem relation,

$$d(P\beta^\mu) = N^\mu_{(0)} d\alpha - T^{\mu\nu}_{(0)} d\beta_\nu \ , \tag{1.36}$$

and of Euler's relation,

$$S^\mu_{(0)} = P\beta^\mu + T^{\mu\nu}_{(0)}\beta_\nu - \alpha N^\mu_{(0)} \ . \tag{1.37}$$

Equations (1.36) and (1.37) can then be used to derive a covariant form of the first law of thermodynamics (1.11),

$$dS^\mu_{(0)} = \beta_\nu dT^{\mu\nu}_{(0)} - \alpha dN^\mu_{(0)} \ . \tag{1.38}$$

The above covariant thermodynamic relations were constructed in a such a way that, when contracted with the fluid 4-velocity, the usual thermodynamic relations, Eqs. (1.36), (1.37), and (1.38), are recovered,

$$u_\mu \left(d(P\beta^\mu) - N^\mu_{(0)} d\alpha + T^{\mu\nu}_{(0)} d\beta_\nu \right) = d(P\beta) - n d\alpha + \varepsilon d\beta = 0 \ , \tag{1.39}$$

$$u_\mu \left(S^\mu_{(0)} - P\beta^\mu - T^{\mu\nu}_{(0)}\beta_\nu + \alpha N^\mu_{(0)} \right) = s + \alpha n - \beta(\varepsilon + P) = 0 \ , \tag{1.40}$$

$$u_\mu \left(dS^\mu_{(0)} - \beta_\nu dT^{\mu\nu}_{(0)} + \alpha dN^\mu_{(0)} \right) = ds - \beta d\varepsilon + \alpha dn = 0 \ , \tag{1.41}$$

where we used that $u_\mu du^\mu = 0$. Note also that the covariant thermodynamic relations do not contain more information than the usual thermodynamic relations. The projection of Eqs. (1.36), (1.37) and (1.38) onto the 3-space orthogonal to u^μ just leads to trivial relations,

$$\Delta^\alpha_\mu \left(d\left(P\beta^\mu\right) - N^\mu_{(0)} d\alpha_0 + T^{\mu\nu}_{(0)} d\beta_\nu\right) = 0 \quad \Longrightarrow \quad 0 = 0\,, \tag{1.42}$$

$$\Delta^\alpha_\mu \left(S^\mu_{(0)} - P\beta^\mu - T^{\mu\nu}_{(0)}\beta_\nu + \alpha N^\mu_{(0)}\right) = 0 \quad \Longrightarrow \quad 0 = 0\,, \tag{1.43}$$

$$\Delta^\alpha_\mu \left(dS^\mu_{(0)} - \beta_\nu dT^{\mu\nu}_{(0)} + \alpha dN^\mu_{(0)}\right) = 0 \quad \Longrightarrow \quad 0 = 0\,. \tag{1.44}$$

The first law of thermodynamics, Eq. (1.38), leads to the following equation of motion for the entropy 4-current,

$$\partial_\mu S^\mu_{(0)} = \beta_\nu \partial_\mu T^{\mu\nu}_{(0)} - \alpha \partial_\mu N^\mu_{(0)}\,, \tag{1.45}$$

which, using the conservation of (net) particle number, energy, and momentum, $\partial_\mu N^\mu_{(0)} = \partial_\mu T^{\mu\nu}_{(0)} = 0$, implies the **conservation of entropy**,

$$\partial_\mu S^\mu_{(0)} = 0\,. \tag{1.46}$$

Note that the entropy conservation appeared naturally, as a consequence of (net) particle number and energy-momentum conservation and the first law of thermodynamics. The equation of motion for the entropy density comes directly from Eq. (1.46),

$$\partial_\mu S^\mu_{(0)} = u^\mu \partial_\mu s + s\partial_\mu u^\mu = \dot{s} + s\theta = 0\,. \tag{1.47}$$

In this section, we introduced and derived the equations of motion of ideal fluid dynamics. This was done using the conservation laws expected to be valid in a fluid and the assumption of local thermal equilibrium. In the next sections, we show how to introduce dissipation into this scheme.

Exercise 1.4: Convince yourself of the validity of Eqs. (1.39) – (1.44) by explicit calculation.

1.3 Relativistic dissipative fluid dynamics

Relativistic ideal fluid dynamics was derived using the conservation laws, the properties of the Lorentz transformation, and, most importantly, by imposing local thermodynamic equilibrium. While the conservation laws and the properties of the Lorentz transformation are always valid, the assumption of local thermodynamic equilibrium is very restrictive and is never realized in practice. Strictly speaking, a fluid can never maintain exact local thermodynamic equilibrium during the whole course of its dynamical evolution. In this section, we consider a more general theory of fluid dynamics, which attempts to take into account the dissipative processes that must happen in a fluid.

Dissipative effects originate from irreversible thermodynamic processes that occur during the motion of the fluid. In general, each fluid element is not in equilibrium with the rest of the fluid and, in order to approach global equilibrium, it will exchange heat with its surroundings. Furthermore, the fluid elements are in relative motion and can also dissipate energy by friction. All these processes must be included in order to obtain a reasonable description of a relativistic fluid.

The first to propose a covariant formulation of dissipative fluid dynamics were Eckart [4], in 1940, and, later, Landau and Lifshitz [2], in 1959. Both theories, often called **first-order theories**, are based on a **covariant formulation of Navier-Stokes theory**. At that time, Navier-Stokes theory had already become a successful theory of fluid dynamics, being able to describe a wide variety of non-relativistic fluids, from weakly coupled gases, such as air, to strongly coupled fluids, such as water. Therefore, a relativistic extension of Navier-Stokes theory was considered to be the most promising way to describe relativistic viscous fluids.

However, the situation was shown to be more subtle since the relativistic version of Navier-Stokes theory is actually intrinsically **unstable** [8, 9, 10, 11, 12]. The source of such instability is well understood and will be discussed in detail in the next chapter. It comes from the inherent acausal behavior of Navier-Stokes theory [13, 14], which allows signals to propagate with infinite speed. In non-relativistic theories this non-intuitive feature does not give rise to an intrinsic problem and can be ignored. On the other hand, in relativistic systems, where causality is a physical property that is naturally preserved, this feature leads to equations of motion that are intrinsically unstable. Nevertheless, first-order theories are an important initial step to illustrate the basic features of relativistic dissipative fluid dynamics and thus shall be reviewed in this section.

Just like for an ideal fluid, the basic equations of motion for dissipative fluids are given by the conservation laws of (net) particle number and energy-momentum,

$$\partial_\mu N^\mu = 0 , \tag{1.48}$$
$$\partial_\mu T^{\mu\nu} = 0 . \tag{1.49}$$

However, in the presence of dissipation, the energy-momentum tensor is no longer diagonal and isotropic in the local rest frame. Also, due to diffusion, we expect (net) particle-number flow to appear in the local rest frame of the fluid element. These effects must be taken into account and are introduced in fluid dynamics by adding **dissipative currents**, n^μ and $\tau^{\mu\nu}$, to the previously derived ideal currents, $N^\mu_{(0)}$ and $T^{\mu\nu}_{(0)}$,

$$N^\mu = N^\mu_{(0)} + n^\mu = n_0 u^\mu + n^\mu , \tag{1.50}$$
$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \tau^{\mu\nu} = \varepsilon_0 u^\mu u^\nu - \Delta^{\mu\nu} P_0 + \tau^{\mu\nu} , \tag{1.51}$$

where we indicated equilibrium quantities with a subscript "0". Here, n^μ is the **particle diffusion 4-current**. In order to satisfy angular-momentum conservation, $\tau^{\mu\nu}$ is defined to be a symmetric tensor, $\tau^{\mu\nu} = \tau^{\nu\mu}$. The main problem then becomes to find the dynamical or constitutive equations satisfied by such dissipative currents.

Exercise 1.5: Ignoring spin degrees of freedom, the total angular momentum tensor is defined as

$$J^{\lambda,\mu\nu} \equiv x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} \; . \tag{1.52}$$

Prove that total angular momentum conservation,

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \; , \tag{1.53}$$

requires $T^{\mu\nu}$ (and thus also $\tau^{\mu\nu}$) to be symmetric.
(Hint: use the energy-momentum conservation law (1.49).)

1.3.1 Matching conditions

The introduction of the dissipative currents renders the equilibrium variables ill-defined, since the fluid can no longer be considered to be in local thermodynamic equilibrium. In a viscous fluid, the thermodynamic variables, α_0, β_0, s_0, P_0, ..., can only be defined in terms of a **fictitious equilibrium state** (labeled by the subscript "0"), constructed such that the thermodynamic relations are valid **as if** the fluid were in local thermodynamic equilibrium. The first step to construct such an equilibrium state is to define n_0 and ε_0 as the **actual (net) particle density** n and the **actual energy density** ε in the local rest frame of the fluid. This is guaranteed by the so-called **Landau matching conditions**,

$$\varepsilon_0 \equiv \varepsilon \equiv u_\nu u_\mu T^{\mu\nu} \; , \tag{1.54}$$

$$n_0 \equiv n \equiv u_\mu N^\mu \; . \tag{1.55}$$

The matching conditions (1.54) and (1.55) are enforced by applying the following set of constraints to the dissipative currents,

$$\begin{aligned} u_\nu u_\mu \tau^{\mu\nu} &= 0 \; , \\ u_\mu n^\mu &= 0 \quad \Longleftrightarrow \quad \Delta^\mu_\lambda n^\lambda = n^\mu \; . \end{aligned} \tag{1.56}$$

Then, using ε and n we can construct our equilibrium state. The thermodynamic entropy density is determined by the equation of state of the fluid as if in thermodynamic equilibrium,

$$s_0 \equiv s_0 \left(\varepsilon, n\right) \; , \tag{1.57}$$

while the remaining thermodynamic variables, e.g., the thermodynamic pressure, temperature, and chemical potential, are defined from the thermodynamic relations derived in Sec. 1.1. The inverse temperature and the ratio of chemical potential over temperature are computed using Eq. (1.13),

$$\beta_0 = \left.\frac{\partial s}{\partial \varepsilon}\right|_n \; , \qquad \alpha_0 = \left.\frac{\partial s}{\partial n}\right|_\varepsilon \; , \tag{1.58}$$

and the thermodynamic pressure is extracted via Eq. (1.10),

$$P_0 = -\varepsilon + T_0 s_0 + \mu_0 n \; . \tag{1.59}$$

Note that, in principle, the thermodynamic pressure can also be expressed as a function of other thermodynamic variables, e.g. $P_0 = P_0(\beta_0, \alpha_0)$.

It is important to emphasize that, while the energy and (net) particle densities are physically well-defined, all other quantities (s_0, P_0, T_0, μ_0, ...) are defined only in terms of a fictitious equilibrium state and do not necessarily retain their usual physical meaning. For example, the second law of thermodynamics does not constrain the production of entropy of the fictitious state: it constrains only the production of the actual non-equilibrium entropy of the fluid – a quantity that can be rather nontrivial to construct, as will be discussed later in this chapter.

1.3.2 Tensor decomposition of $\tau^{\mu\nu}$

It is convenient to decompose $\tau^{\mu\nu}$ in terms of its **irreducible components**, i.e., a **scalar**, a **4-vector**, and a **traceless, symmetric second-rank tensor**. This tensor decomposition must respect the matching (or orthogonality) condition satisfied by $\tau^{\mu\nu}$, Eq. (1.56). For this purpose, we introduce yet another projection operator: the double-symmetric, traceless rank-4 projection operator orthogonal to u^μ,

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\nu}_{\alpha}\right) - \frac{1}{\Delta^{\lambda}_{\lambda}}\Delta^{\mu\nu}\Delta_{\alpha\beta} \; , \tag{1.60}$$

which satisfies the following properties,

$$\begin{aligned} \Delta^{(\mu\nu)(\alpha\beta)} &= \Delta^{(\alpha\beta)(\mu\nu)} \; , \\ \Delta^{\mu\nu}_{\lambda\rho}\Delta^{\lambda\rho}_{\alpha\beta} &= \Delta^{\mu\nu}_{\alpha\beta} \; , \\ u_\mu \Delta^{\mu\nu\alpha\beta} &= g_{\mu\nu}\Delta^{\mu\nu\alpha\beta} = 0 \; , \\ \Delta^{\mu\nu}_{\mu\nu} &= 5 \; . \end{aligned} \tag{1.61}$$

Then, using $\Delta^{\mu\nu}$ and $\Delta^{\mu\nu}_{\alpha\beta}$, the tensor decomposition of $\tau^{\mu\nu}$ in its irreducible form is implemented as

$$\tau^{\mu\nu} \equiv -\Pi\Delta^{\mu\nu} + 2u^{(\mu}h^{\nu)} + \pi^{\mu\nu} \; , \tag{1.62}$$

where the parentheses () denote the symmetrization of all Lorentz indices, $a^{(\mu\nu)} = (a^{\mu\nu} + a^{\nu\mu})/2$, and where we defined

$$\Pi \equiv -\frac{1}{3}\Delta_{\alpha\beta}\tau^{\alpha\beta} \; , \qquad h^\mu \equiv \Delta^\mu_\alpha u_\beta \tau^{\alpha\beta} \; , \qquad \pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta}\tau^{\alpha\beta} \; . \tag{1.63}$$

The scalar term, Π, is the **bulk-viscous pressure**, the vector term, h^μ, is the **energy-diffusion 4-current**, and the second-rank tensor, $\pi^{\mu\nu}$, is the **shear-stress tensor**. The properties of the projection operators Δ^μ_ν and $\Delta^{\mu\nu}_{\alpha\beta}$ given in Eqs. (1.26) and (1.61) imply that h^μ and $\pi^{\mu\nu}$ satisfy

$$h^\mu = \Delta^\mu_\nu h^\nu \quad \Longleftrightarrow \quad u_\mu h^\mu = 0 \; , \tag{1.64}$$

$$\pi^{\mu\nu} = \pi^{\langle\mu\nu\rangle} \quad \Longleftrightarrow \quad u_\mu \pi^{\mu\nu} = 0 \; , \tag{1.65}$$

$$\pi^\mu_\mu = 0 \; , \tag{1.66}$$

where the brackets $\langle\rangle$ denote the following projection of a second-rank tensor, $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}A^{\alpha\beta}$. In summary, the fields Π, h^μ, n^μ, and $\pi^{\mu\nu}$ are expressed in terms of N^μ and $T^{\mu\nu}$ as

$$\Pi = -P_0 - \frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} , \tag{1.67}$$

$$h^\mu = u_\alpha \Delta^\mu_\beta T^{\alpha\beta} , \tag{1.68}$$

$$n^\mu = \Delta^\mu_\alpha N^\alpha , \tag{1.69}$$

$$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle} . \tag{1.70}$$

Note that $T^{\mu\nu}$ is a symmetric second-rank tensor and, thus, N^μ and $T^{\mu\nu}$ have 14 independent components (four from N^μ and ten from $T^{\mu\nu}$). In order to satisfy their orthogonality to u^μ, n^μ and h^μ can only have three independent components each. The shear-stress tensor is symmetric, traceless, and orthogonal to u^μ and, therefore, can have only five independent components. Together with u^μ, ε, n, and Π, which have in total six independent components, we find a total of 17 independent components, three more than expected. This happened because, so far, the velocity field itself has not been specified, being introduced just as a general normalized 4-vector. The definition of the velocity field will provide the three missing constraints that will reduce the number of independent components to the correct value.

Exercise 1.6: Prove Eqs. (1.61), as well as Eqs. (1.63) by explicitly inserting Eq. (1.62) on the right-hand sides.

1.3.3 Definition of the local rest frame and equations of motion

The definition of the velocity field is an important step in deriving fluid dynamics. For ideal fluids, the local rest frame was implicitly defined as the frame in which there is no flow of energy and (net) particle number. Due to the presence of energy and particle diffusion in viscous fluids, this definition is no longer possible. From a mathematical point of view, the velocity can be defined in numerous ways. From the physical perspective, there are, however, two natural choices. The **Landau frame** [2], in which the velocity is defined by the flow of the total energy,

$$u_\mu T^{\mu\nu} = \varepsilon u^\nu , \tag{1.71}$$

and the **Eckart frame** [3, 4], in which the velocity is specified by the flow of (net) particle number,

$$N^\mu = n u^\mu . \tag{1.72}$$

If the system has more than one type of particle (or charge), the Eckart frame must be defined by selecting one of these particle (or charge) types.

Both choices of frame impose different constraints on the dissipative currents introduced in this section. In the Landau frame, the energy diffusion vanishes,

$$h^\mu = 0 , \tag{1.73}$$

while in the Eckart frame the particle diffusion is zero,

$$n^\mu = 0 \,. \tag{1.74}$$

In other words, in the Landau frame the velocity field is fixed to eliminate any diffusion of energy while in the Eckart frame it is defined to eliminate any diffusion of particles. In this lecture series, we shall always use the Landau frame, Eq. (1.71), and, therefore, the conserved currents take the following simpler form

$$N^\mu = n u^\mu + n^\mu \,, \tag{1.75}$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}\left(P_0 + \Pi\right) + \pi^{\mu\nu} \,. \tag{1.76}$$

Note that both the Landau and the Eckart choice of frame reduce the number of independent variables to 14, as advertised at the end of the last section.

As for an ideal fluid, we decompose Eq. (1.49) into a part parallel and another one orthogonal to u^μ. As shown in the last section, this is done by projecting and contracting Eq. (1.49) with u_μ and $\Delta^{\mu\nu}$, i.e., by taking $u_\alpha \partial_\beta T^{\alpha\beta}$ and $\Delta^\mu_\alpha \partial_\beta T^{\alpha\beta}$, respectively. Together with Eqs. (1.48), (1.75), and (1.76), this procedure leads to the equations of motion of the fluid,

$$u_\alpha \partial_\beta T^{\alpha\beta} = \dot{\varepsilon} + \left(\varepsilon + P_0 + \Pi\right)\theta - \pi^{\alpha\beta}\sigma_{\alpha\beta} = 0 \,, \tag{1.77}$$

$$\Delta^\mu_\alpha \partial_\beta T^{\alpha\beta} = \left(\varepsilon + P_0 + \Pi\right)\dot{u}^\mu - \nabla^\mu\left(P_0 + \Pi\right) + \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta} = 0 \,, \tag{1.78}$$

$$\partial_\mu N^\mu = \dot{n} + n\theta + \partial_\mu n^\mu = 0 \,, \tag{1.79}$$

where we defined the shear tensor

$$\sigma^{\mu\nu} = \partial^{\langle\mu} u^{\nu\rangle} = \frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}\nabla_\lambda u^\lambda \,. \tag{1.80}$$

Note that the quantities n, ε, P_0, u^μ, Π, n^μ, and $\pi^{\mu\nu}$ introduced in this section were defined from a strict mathematical perspective via the most general tensor decomposition allowed by symmetry. The conservation laws, Eqs. (1.48) and (1.49), the definition of the fictitious equilibrium state, and the definition of the velocity field are also general and valid regardless of the whether the system is in the fluid-dynamical regime (i.e., not far from local thermodynamical equilibrium). Thus, by writing down any of the above equations, we have not, by any means, derived fluid dynamics. In order to derive the complete equations of dissipative fluid dynamics, one still has to provide an **additional nine relations** that will close Eqs. (1.77), (1.78), and (1.79). In the end, this corresponds to finding **dynamical or constitutive relations** satisfied by the dissipative currents Π, n^μ, and $\pi^{\mu\nu}$. Ideal fluid dynamics, discussed in the previous section, corresponds to a trivial example of this procedure, in which the dissipative currents are simply set to zero.

Exercise 1.7: Prove Eqs. (1.77) – (1.79), extending Exercise 1.3 to dissipative fluids.

1.3.4 Relativistic Navier-Stokes theory

In the presence of dissipative currents the entropy is longer conserved. Deriving the equation for the entropy 4-current is not trivial for a viscous fluid, since *a priori* we do not know the form of this current. For now, let us take the same steps as in the ideal fluid case and see where we arrive. We start by taking Eq. (1.45),

$$\partial_\mu S^\mu_{(0)} = \beta_0 u_\nu \partial_\mu T^{\mu\nu}_{(0)} - \alpha_0 \partial_\mu N^\mu_{(0)} \; , \tag{1.81}$$

which remains valid in a viscous fluid since, as explained in Sec. 1.3.1, the equilibrium variables and, consequently, the equilibrium currents, were constructed to satisfy thermodynamic relations as if in equilibrium. Now, however, the equilibrium part of the currents are not conserved, $\partial_\mu N^\mu_{(0)} = -\partial_\mu n^\mu \neq 0$ and $u_\nu \partial_\mu T^{\mu\nu}_{(0)} = -\Pi\theta + \pi^{\mu\nu}\sigma_{\mu\nu} \neq 0$, cf. Eqs. (1.77) and (1.79), and in a viscous fluid Eq. (1.81) leads to

$$\partial_\mu S^\mu_{(0)} \;\; = \;\; \alpha_0 \partial_\mu n^\mu + \beta_0 \left(-\Pi\theta + \pi^{\mu\nu}\sigma_{\mu\nu} \right) \; . \tag{1.82}$$

By decomposing the first term on the right-hand side as $\alpha_0 \partial_\mu n^\mu = \partial_\mu \left(\alpha_0 n^\mu \right) - n^\mu \nabla_\mu \alpha_0$, Eq. (1.82) can be written in a more convenient form,

$$\partial_\mu \left(S^\mu_{(0)} - \alpha_0 n^\mu \right) = -n^\mu \nabla_\mu \alpha_0 - \beta_0 \Pi \theta + \beta_0 \pi^{\mu\nu} \sigma_{\mu\nu} \equiv Q \; . \tag{1.83}$$

It is very tempting to identify the term on the left-hand side of Eq. (1.83) as the 4-divergence of the (off-equilibrium) entropy 4-current

$$S^\mu \equiv S^\mu_{(0)} - \alpha_0 n^\mu = s_0 u^\mu - \alpha_0 n^\mu \; , \tag{1.84}$$

and the terms on the right-hand side, Q, as the source terms for entropy production. Note, however, that this is not necessarily the case. Nevertheless, this was the identification proposed by Eckart and by Landau and Lifshitz and we shall consider it here in order to derive **relativistic Navier-Stokes theory**.

Relativistic Navier-Stokes theory is then obtained by applying the second law of thermodynamics to each fluid element, i.e., by requiring that the entropy production obtained in Eq. (1.83) must always be positive semi-definite, $Q \geq 0$. The simplest way to satisfy this condition for all possible fluid configurations is to assume that the bulk-viscous pressure, the particle-diffusion 4-current, and the shear-stress tensor are **linearly proportional** to θ, $\nabla^\mu \alpha_0$, and $\sigma^{\mu\nu}$, respectively,

$$\Pi \;\; = \;\; -\zeta\theta \; , \tag{1.85}$$

$$n^\mu \;\; = \;\; \varkappa \nabla^\mu \alpha_0 \; , \tag{1.86}$$

$$\pi^{\mu\nu} \;\; = \;\; 2\eta\sigma^{\mu\nu} \; . \tag{1.87}$$

The proportionality coefficients ζ, $\varkappa$, and η are the **coefficients of bulk viscosity**, **particle diffusion**, and **shear viscosity**, respectively. Then, substituting Eqs. (1.85), (1.86), and (1.87) into Eq. (1.83), we see that the entropy production becomes a quadratic function of the dissipative currents

$$Q = \frac{\beta_0}{\zeta}\Pi^2 - \frac{1}{\varkappa} n^\mu n_\mu + \frac{\beta_0}{2\eta} \pi_{\mu\nu} \pi^{\mu\nu} \; . \tag{1.88}$$

Note that $n^\mu n_\mu$ is negative (n^μ is a space-like vector) while $\pi_{\mu\nu}\pi^{\mu\nu}$ is positive and, therefore, as long as ζ, $\varkappa$, $\eta \geq 0$, Q is, in fact, always positive semi-definite.

The equations of fluid dynamics are obtained by substituting Eqs. (1.85), (1.86), and (1.87) into the conservation laws, Eqs. (1.77), (1.78), and (1.79),

$$\dot{\varepsilon} = -\left(\varepsilon + P_0 - \zeta\theta\right)\theta + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta} \,, \tag{1.89}$$

$$\left(\varepsilon + P_0 - \zeta\theta\right)\dot{u}^\mu = \nabla^\mu P_0 - \nabla^\mu\left(\zeta\theta\right) - 2\Delta^\mu_\alpha\partial_\beta\left(\eta\sigma^{\alpha\beta}\right) \,, \tag{1.90}$$

$$\dot{n} = -n\theta - \partial_\mu\left(\varkappa\nabla^\mu\alpha_0\right) \,. \tag{1.91}$$

The above equations are known as the **relativistic Navier-Stokes equations**, as obtained by Landau and Lifshitz [2]. A similar theory was obtained independently by Eckart [4], using a different definition of the local rest frame. In this formulation, the state of a dissipative fluid remains being described by the same variables as in the case of an ideal fluid, i.e., the primary fluid-dynamical variables α_0, β_0, and u^μ. The only difference is the existence of **dissipative processes**, corresponding to new forms of particle and energy-momentum transfer, which occur due to **gradients of the primary fluid-dynamical variables**.

As already mentioned, Navier-Stokes theory is acausal and, consequently, unstable. Thus, it is unable to describe any relativistic fluid existing in Nature. The source of the acausality can be understood from the constitutive relations satisfied by the dissipative currents, Eqs. (1.85), (1.86), and (1.87). Such linear relations imply that any inhomogeneity of α_0, β_0, and u^μ, will **instantaneously** give rise to a dissipative current. This instantaneous creation of currents from (space-like) gradients of the primary fluid-dynamical variables renders the equations of motion **parabolic**. In a relativistic theory this leads to **instabilities**, as will be discussed in the next chapter.

1.3.5 Gradient expansion and Navier-Stokes theory

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Navier-Stokes theory can also be derived (and extended) via the so-called **gradient expansion** [15, 16]. In this framework, the bulk-viscous pressure, the particle-diffusion 4-current, and the shear-stress tensor are assumed to be expressable solely in terms of **powers of gradients of the primary fluid-dynamical variables** α_0, β_0, and u^μ. The dissipative currents can then be schematically written as the series

$$\Pi = \lambda_\Pi^{(1)}\mathcal{O}_1 + \lambda_\Pi^{(2)}\mathcal{O}_2 + \cdots \,, \tag{1.92}$$

$$n^\mu = \lambda_n^{(1)}\mathcal{O}_1^\mu + \lambda_n^{(2)}\mathcal{O}_2^\mu + \cdots \,, \tag{1.93}$$

$$\pi^{\mu\nu} = \lambda_\pi^{(1)}\mathcal{O}_1^{\mu\nu} + \lambda_\pi^{(2)}\mathcal{O}_2^{\mu\nu} + \cdots \,, \tag{1.94}$$

where the quantities ($\mathcal{O}_1$, $\mathcal{O}_1^\mu$, $\mathcal{O}_1^{\mu\nu}$) and ($\mathcal{O}_2$, $\mathcal{O}_2^\mu$, $\mathcal{O}_2^{\mu\nu}$) correspond to terms of first and second order in gradients of α_0, β_0, and u^μ, respectively, and the dots denote possible higher-order gradient terms.

It is important to remark that when the system exhibits a clear separation between the typical **microscopic** and **macroscopic scales**, λ and L, respectively, it may possible to truncate the expansion on the right-hand sides of Eqs. (1.92), (1.93), and (1.94). The microscopic scale can, for example, be the **mean free path** for dilute gases or the **inverse**

temperature for conformal fluids, while a macroscopic scale is given by **typical time or length scales** over which a certain **primary fluid-dynamical variable** varies, e.g., $L \sim [(\partial a)/a]^{-1}$, with $a = \alpha_0$, β_0, or u^μ. The terms $\mathcal{O}_1$, $\mathcal{O}_1^\mu$, and $\mathcal{O}_1^{\mu\nu}$ are linearly proportional to a gradient of a macroscopic variable, and thus are of order $\sim L^{-1}$. Every additional derivative brings in another inverse power of L and, thus, $\mathcal{O}_n$, $\mathcal{O}_n^\mu$, $\mathcal{O}_n^{\mu\nu} \sim L^{-n}$. The microscopic scale λ is contained in the coefficients, $\lambda_\Pi^{(n)}$, $\lambda_n^{(n)}$, and $\lambda_\pi^{(n)}$. Up to some overall power of λ (which restores the correct scaling dimension), $\lambda_\Pi^{(n)}$, $\lambda_n^{(n)}$, $\lambda_\pi^{(n)} \sim \lambda^n$. Therefore, the terms ($\mathcal{O}_1$, $\mathcal{O}_1^\mu$, $\mathcal{O}_1^{\mu\nu}$) and ($\mathcal{O}_2$, $\mathcal{O}_2^\mu$, $\mathcal{O}_2^{\mu\nu}$), multiplied by their corresponding coefficients in Eqs. (1.92), (1.93), and (1.94) are of order λ/L and $(\lambda/L)^2$, respectively. Subsequent terms would be of higher order in this ratio. This is nothing but a series in powers of the so-called **Knudsen number**

$$\mathrm{Kn} \equiv \frac{\lambda}{L} \,. \tag{1.95}$$

If $\mathrm{Kn} \ll 1$ and this series converges, the gradient expansion of the dissipative currents can be truncated at a given order and one obtains a closed macroscopic theory for them. **Ideal fluid dynamics** corresponds to the **zeroth-order truncation** of this series, i.e., when no gradient terms are considered. The **first-order truncation** of the gradient expansion is **Navier-Stokes theory**, as will be shown below. Higher-order truncations would lead to the **relativistic Burnett and super-Burnett equations** and so on. We note that the convergence of the gradient expansion is not well established and is a topic that is still being intensely investigated [17].

The first order of the gradient expansion can be obtained by constructing all possible tensors that can be formed from the first-order derivatives of α_0, β_0, and u^μ. These can be easily obtained and are

$$\partial_\mu \alpha_0 \,, \partial_\mu \beta_0 \,, \text{ and } \partial_\mu u_\nu \,. \tag{1.96}$$

Next, using these gradients one has to construct tensors that have the same properties as the dissipative currents. There must be a scalar, such as the bulk-viscous pressure, a 4-vector orthogonal to u^μ, such as the particle-diffusion 4-current, and a symmetric, traceless second-rank tensor orthogonal to u^μ, such as the shear-stress tensor. The only possibilities are

$$\text{Scalar} \;:\; \theta = \nabla_\mu u^\mu \,, \tag{1.97}$$

$$\text{Vector} \;:\; I^\mu \equiv \nabla^\mu \alpha_0 \,, \quad J^\mu \equiv \nabla^\mu \beta_0 \,, \tag{1.98}$$

$$\text{Tensor} \;:\; \sigma^{\mu\nu} \equiv \partial^{\langle\mu} u^{\nu\rangle} = \frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}\theta \,. \tag{1.99}$$

Then, the most general first-order terms allowed by symmetry are

$$\mathcal{O}_1 = \theta \,, \tag{1.100}$$

$$\mathcal{O}_1^\mu = I^\mu + \gamma J^\mu \,, \tag{1.101}$$

$$\mathcal{O}_1^{\mu\nu} = \sigma^{\mu\nu} \,, \tag{1.102}$$

where the constant γ is introduced in order to restore the correct dimension.

In order to respect the second law of thermodynamics, discussed in the previous section, the transport coefficient γ must always be zero, $\gamma = 0$. This can be seen as follows. One can rewrite the Gibbs-Duhem relation (1.12) with the help of the Euler equation (1.10) in the form

$$d\beta_0 = h_0^{-1} d\alpha_0 - \frac{\beta_0}{\varepsilon + P_0} dP_0 \,, \tag{1.103}$$

where $h_0 \equiv (\varepsilon + P_0)/n$ is the specific enthalpy. Then, using the hydrodynamic equation (1.78),

$$J^\mu = h_0^{-1} I^\mu - \beta_0 \dot{u}^\mu + \mathcal{O}_2 \,, \tag{1.104}$$

where the second-order terms involve gradients of dissipative currents or products of dissipative currents with gradients of primary fluid-dynamical variables. Employing Eqs. (1.93), (1.101) the first term in the entropy-production equation (1.83) would be $\sim -\lambda_n^{(1)} \left[\left(1 + \gamma h_0^{-1}\right) I^\mu - \gamma \beta_0 \dot{u}^\mu\right\} I_\mu$, which is in general no longer positive definite, unless $\gamma = 0$.

Therefore, the most general relations satisfied by Π, n^μ, and $\pi^{\mu\nu}$, up to first order in Kn, are

$$\begin{aligned} \Pi &= \lambda_\Pi^{(1)} \theta \,, \\ n^\mu &= \lambda_n^{(1)} I^\mu \,, \\ \pi^{\mu\nu} &= \lambda_\pi^{(1)} \sigma^{\mu\nu} \,, \end{aligned}$$

which corresponds to the relativistic Navier-Stokes theory [2], with the bulk-viscosity coefficient, the diffusion coefficient, and the shear-viscosity coefficient being identified as $\zeta \equiv -\lambda_\Pi^{(1)}$, $\varkappa \equiv \lambda_n^{(1)}$, and $\eta \equiv \lambda_\pi^{(1)}/2$, respectively.

In the framework of the gradient expansion, the relativistic Navier-Stokes theory can be extended by including terms of second order in gradients of α_0, β_0, and u^μ. In order to do so, one has to obtain all possible terms that can contribute to $\mathcal{O}_2$, $\mathcal{O}_2^\mu$, and $\mathcal{O}_2^{\mu\nu}$. These are

$$\begin{aligned} \text{Scalar}: &\quad \omega_{\mu\nu}\omega^{\mu\nu} \,,\ \sigma_{\mu\nu}\sigma^{\mu\nu} \,,\ \theta^2 \,,\ I_\mu I^\mu \,,\ J_\mu J^\mu \,,\ I_\mu J^\mu \,,\ \nabla_\mu I^\mu \,,\ \nabla_\mu J^\mu \,, \\ \text{Vector}: &\quad \sigma^{\mu\nu} I_\nu \,,\ \sigma^{\mu\nu} J_\nu \,,\ I^\mu \theta \,,\ J^\mu \theta \,,\ \omega^{\mu\nu} I_\nu \,,\ \omega^{\mu\nu} J_\nu \,,\ \Delta^\mu_\alpha \partial_\nu \sigma^{\alpha\nu} \,,\ \nabla^\mu \theta \,, \\ \text{Tensor}: &\quad \omega_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} \,,\ \theta\sigma^{\mu\nu} \,,\ \sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} \,,\ \sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} \,,\ I^{\langle\mu} I^{\nu\rangle} \,,\ J^{\langle\mu} J^{\nu\rangle} \,, \\ &\quad I^{\langle\mu} J^{\nu\rangle} \,,\ \nabla^{\langle\mu} I^{\nu\rangle} \,,\ \nabla^{\langle\mu} J^{\nu\rangle} \,, \end{aligned} \tag{1.105}$$

where we introduced the fluid **vorticity tensor**,

$$\omega^{\mu\nu} \equiv \frac{1}{2}\left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right) \,. \tag{1.106}$$

Then, the most general second-order terms allowed by symmetry are

$$\begin{aligned} \lambda_\Pi^{(2)} \mathcal{O}_2 &= \zeta_1 \omega_{\mu\nu}\omega^{\mu\nu} + \zeta_2 \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta_3 \theta^2 + \zeta_4 I_\mu I^\mu + \zeta_5 J_\mu J^\mu + \zeta_6 I_\mu J^\mu \\ &\quad + \zeta_7 \nabla_\mu I^\mu + \zeta_8 \nabla_\mu J^\mu \,, \end{aligned} \tag{1.107}$$

$$\begin{aligned} \lambda_n^{(2)} \mathcal{O}_2^\mu &= \varkappa_1 \sigma^{\mu\nu} I_\nu + \varkappa_2 \sigma^{\mu\nu} J_\nu + \varkappa_3 I^\mu \theta + \varkappa_4 J^\mu \theta + \varkappa_5 \omega^{\mu\nu} I_\nu + \varkappa_6 \omega^{\mu\nu} J_\nu \\ &\quad + \varkappa_7 \Delta^\mu_\alpha \partial_\nu \sigma^{\alpha\nu} + \varkappa_8 \nabla^\mu \theta \,, \end{aligned} \tag{1.108}$$

$$\begin{aligned} \lambda_\pi^{(2)} \mathcal{O}_2^{\mu\nu} &= \eta_1 \omega_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_2 \theta\sigma^{\mu\nu} + \eta_3 \sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \eta_4 \sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu} I^{\nu\rangle} \\ &\quad + \eta_6 J^{\langle\mu} J^{\nu\rangle} + \eta_7 I^{\langle\mu} J^{\nu\rangle} + \eta_8 \nabla^{\langle\mu} I^{\nu\rangle} + \eta_9 \nabla^{\langle\mu} J^{\nu\rangle} \,, \end{aligned} \tag{1.109}$$

where we introduced **additional transport coefficients**, ζ_i, $\varkappa_i$, and η_i. By including the above terms in the expressions for the dissipative currents, Eqs. (1.92), (1.93), and (1.94), we obtain the **relativistic Burnett equations** [16],

$$\begin{aligned}
\Pi &= -\zeta\theta + \zeta_1\omega_{\mu\nu}\omega^{\mu\nu} + \zeta_2\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta_3\theta^2 + \zeta_4 I_\mu I^\mu + \zeta_5 J_\mu J^\mu + \zeta_6 I_\mu J^\mu \\
&\quad +\zeta_7\, \nabla_\mu I^\mu + \zeta_8 \nabla_\mu J^\mu \,, \qquad (1.110) \\
n^\mu &= \varkappa I^\mu + \varkappa_1 \sigma^{\mu\nu} I_\nu + \varkappa_2 \sigma^{\mu\nu} J_\nu + \varkappa_3 I^\mu\theta + \varkappa_4 J^\mu\theta + \varkappa_5 \omega^{\mu\nu} I_\nu + \varkappa_6 \omega^{\mu\nu} J_\nu \\
&\quad +\varkappa_7 \Delta^\mu_\alpha \partial_\nu \sigma^{\alpha\nu} + \varkappa_8 \nabla^\mu\theta \,, \qquad (1.111) \\
\pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + \eta_1 \omega_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3 \sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \eta_4 \sigma^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu} I^{\nu\rangle} \\
&\quad +\eta_6 J^{\langle\mu} J^{\nu\rangle} + \eta_7 I^{\langle\mu} J^{\nu\rangle} + \eta_8 \nabla^{\langle\mu} I^{\nu\rangle} + \eta_9 \nabla^{\langle\mu} J^{\nu\rangle} \,. \qquad (1.112)
\end{aligned}$$

In this section, we showed how to extend and derive Navier-Stokes theory via the gradient expansion. Note, however, that these extensions remain **acausal** and **unstable** and have no practical purpose. As a matter of fact, the Burnett equations are unstable even in the non-relativistic regime [18].

Exercise 1.8: Check the validity of the thermodynamic relation (1.103).

1.4 Causal fluid dynamics

Many theories have been developed to incorporate dissipative effects in fluid dynamics preserving causality: **Grad-Israel-Stewart theory** [5, 6, 7, 19], **divergence-type theories** [20, 21, 22], **extended irreversible thermodynamics** [23, 24, 25, 26, 27], **Carter's theory** [28], **Öttinger-Grmela theory** [29], among others [30, 31]. In Sec. 1.5 we shall briefly review Israel and Stewart's approach.

However, before explaining Israel-Stewart theory, it is useful to discuss a more *ad hoc* approach to render Navier-Stokes theory causal and stable [32]. For the sake of simplicity, we first illustrate this method using the simple example of a transport equation that shares the same problems as Navier-Stokes theory: the **heat-conduction equation**.

1.4.1 Diffusion equation and acausality in heat conduction

The fundamental problem of the diffusion equation comes from the fact that it is a **parabolic equation**, allowing for signals that can propagate with infinite speed [25, 26, 27, 33, 34, 35]. In the non-relativistic case, this problem was first addressed by Cattaneo [36, 37] and applied to heat conduction. Cattaneo argued that the problem of acausal propagation in the diffusion equation

$$\partial_t A = D\nabla^2 A \,, \qquad (1.113)$$

can be corrected by introducing a term with a **second-order time derivative**, i.e.,

$$\tau_R \partial_t^2 A + \partial_t A = D\nabla^2 A \,, \qquad (1.114)$$

thereby converting a parabolic equation to a **hyperbolic one**. Above, we introduced the **diffusion** and the **relaxation-time coefficients**, D and τ_R, respectively. Equation (1.114) is often referred to as **telegraph equation** and, for suitable choices of D and τ_R, can lead to **causal signal propagation**. As a matter of fact, the maximum propagation speed of signals in this theory can be proven to be [38],

$$v_{\max} = \sqrt{D/\tau_R} \, . \tag{1.115}$$

Therefore, as long as $D/\tau_R \leq 1$, the telegraph equation is causal. On the other hand, in the limit $\tau_R \to 0$, in which the diffusion equation is recovered, the propagation speed diverges and the theory becomes acausal.

Next, we consider the heat-conduction problem as an example to understand the physical origin of the telegraph equation. The diffusion equation used to describe heat conduction is constructed from two basic features. One is the **energy-balance equation**,

$$\rho c \partial_t T + \nabla \cdot \mathbf{J} = 0 \, , \tag{1.116}$$

where T is the **temperature**, $\mathbf{J}$ is the **heat flux**, ρ is the **mass density**, and c is the **specific heat capacity** of the material in question. The other ingredient is **Fourier's law**,

$$\mathbf{J} = -\varpi \nabla T \, , \tag{1.117}$$

where ϖ is the **heat conductivity**. Then, Eqs. (1.116) and (1.117) can be combined to eliminate $\mathbf{J}$ from the differential equation (1.116) and describe heat conduction via the following **diffusion equation**,

$$\partial_t T = \frac{\varpi}{\rho c} \nabla^2 T \, , \tag{1.118}$$

where, for the sake of simplicity, we assumed that ρ and ϖ are spatially constant in the material.

The source of acausality cannot be the energy-balance equation, which is the direct consequence of a conservation law. Then, it must be Fourier's law. Heat conduction should always be induced by an inhomogeneous temperature profile. However, Fourier's law is only an approximation of this process. It does not contain any inertial effects and, consequently, certain perturbations in the temperature distribution are felt **instantaneously** at every point of the system, i.e., according to Eq. (1.118) **every point** in the material heats at the **same time**, no matter how distant from the heat source.

Therefore, even though very successful in non-relativistic applications, Fourier's law cannot be employed in the relativistic limit: in relativistic theories, causality is a physical principle that is naturally preserved by the Lorentz transformations and dictates that the propagation of any current must happen in a nonzero interval of time. This feature can be included in the description of heat conduction via **linear response theory**, which prescribes a more general expression for $\mathbf{J}$ [39],

$$\mathbf{J}(t) = -\int^t ds \, G(t-s) \nabla T(s) \, , \tag{1.119}$$

where $G(t)$ is the corresponding **retarded Green function**. The Green function includes the **microscopic time scales** that describe the creation of the heat flux from temperature gradients. In non-relativistic systems in which the separation between the microscopic and macroscopic time scales of the system are sufficiently large, the time dependence of the Green function can be approximated by a Dirac delta function, $G(t) \sim \delta(t)$, and we recover Fourier's law. However, when the microscopic and macroscopic time scales are not separated by orders of magnitude, the **transient dynamics** of the heat flux must be explicitly described and the time dependence of $G(t)$ must be taken into account. In relativistic systems, such microscopic time scales must always be considered in order to preserve causality. When deriving fluid dynamics in the next chapters, we shall address this topic within a more formal framework.

The simplest choice for $G(t)$ is the exponential *ansatz* (this choice can actually be derived in the framework of kinetic theory) [40],

$$G(t) = \frac{\varpi}{\tau_R} e^{-t/\tau_R} , \tag{1.120}$$

where τ_R is the **heat-flux relaxation time**. Then, by taking the time derivative of Eq. (1.119) and substituting this *ansatz*, we obtain the following equation of motion for $\mathbf{J}$,

$$\tau_R \partial_t \mathbf{J}(t) + \mathbf{J}(t) = -\varpi \nabla T(t) . \tag{1.121}$$

This equation is often referred to as **Maxwell-Cattaneo equation**. In this theory, the heat flux is not created instantaneously from temperature inhomogeneities. For example, when $\nabla T = 0$,

$$\tau_R \partial_t \mathbf{J}(t) + \mathbf{J}(t) = 0 , \tag{1.122}$$

the heat flux does not vanish instantaneously, as happened in Fourier's law, but **relaxes exponentially to zero** on times scales given by the heat-flux relaxation time τ_R.

By using the energy-balance equation (1.116), we can eliminate $\mathbf{J}$ from the divergence of the Maxwell-Cattaneo equation (1.121), and obtain the **telegraph equation** for the temperature T,

$$\tau_R \partial_t^2 T + \partial_t T = \frac{\varpi}{\rho c} \nabla^2 T . \tag{1.123}$$

Note that, in this case, the diffusion equation becomes the **asymptotic limit** of the telegraph equation, attained only for times much longer than the heat-flux relaxation time (as long as ∇T varies slowly in time). As mentioned before, Eq. (1.123) is **causal** as long as

$$\sqrt{\frac{\varpi}{\rho c \tau_R}} \leqslant 1 . \tag{1.124}$$

1.4.2 Transient theory of fluid dynamics

The same idea as explained in the previous section can be applied to render Navier-Stokes theory causal. In Sec. 1.3.4, Navier-Stokes theory was constructed from the conservation laws of (net) particle number, energy, and momentum, and the constitutive relations satisfied by the dissipative currents, Eqs. (1.85), (1.86), and (1.87). The conservation laws come from general physical principles, valid even outside the fluid-dynamical regime, and cannot be modified. Therefore, in order to improve on Navier-Stokes theory, we must extend the constitutive relations (1.85), (1.86), and (1.87). We have already argued that such relations are the source of causality violations in Navier-Stokes theory since, as happens with Fourier's law, they allow for the **instantaneous creation** of dissipative currents from gradients of α_0, β_0, and u^μ. In this section, we show how to correct this unphysical behavior using the same arguments as previously applied to heat conduction: the inclusion of a **time delay** in the creation of the dissipative currents from gradients of the primary fluid-dynamical variables.

In the heat-conduction case, inertial effects on the creation of heat flux from temperature inhomogeneities were included by introducing a term with a **first-order time derivative** of $\mathbf{J}$ in Fourier's law, giving rise to a **causal transport equation** for heat conduction, Eq. (1.122). Similarly, relaxation effects can be introduced in Navier-Stokes theory by adding a term of **first order in the comoving derivative of each dissipative current** to the constitutive relation satisfied by this current, i.e., $\dot{\Pi}$, $\Delta^\mu_\alpha \dot{n}^\alpha$, and $\Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta}$. Then, instead of Navier-Stokes theory, we obtain the following transport equations for Π, n^μ, and $\pi^{\mu\nu}$,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \dots \,, \tag{1.125}$$

$$\tau_n \, \Delta^\mu_\alpha \dot{n}^\alpha + n^\mu = \varkappa \, \nabla^\mu \alpha_0 + \dots \,, \tag{1.126}$$

$$\tau_\pi \, \Delta^{\mu\nu}_{\alpha\beta} \dot{\pi}^{\alpha\beta} + \pi^{\mu\nu} = 2\eta \, \sigma^{\mu\nu} + \dots \,, \tag{1.127}$$

where we introduced the **bulk relaxation time**, τ_Π, the **diffusion relaxation time**, τ_n, and the **shear relaxation time**, τ_π. The dots denote possible nonlinear terms involving the fluid-dynamical quantities and their gradients.

In this formulation, the fluid-dynamical dissipative currents appear as **independent dynamical variables**, which **relax** to the values of Navier-Stokes theory on **characteristic time scales** given by the relaxation times τ_Π, τ_n, and τ_π. Thus, unlike for the gradient expansion, the dissipative currents in this theory do not have to be zero in the absence of gradients. Instead, they **decay to zero** on the time scales given by the relaxation times. This type of formalism was referred to as **transient theory of fluid dynamics** by Israel and Stewart [5], since it describes this relaxation (or transient) process of each dissipative current towards its respective (asymptotic) Navier-Stokes value.

One of the features of transient fluid dynamics is that it reduces to Navier-Stokes theory in the limit of vanishing relaxation times. In other words, in Navier-Stokes theory the dissipative currents relax instantaneously to their Navier-Stokes values (also referred to as fluid-dynamical forces), which leads to a violation of causality. In many non-relativistic fluids, the relaxation times are very short, and such transient dynamics can be neglected. In this case, the dissipative currents can actually be well approximated by their Navier-Stokes solution. Nevertheless, in the relativistic case, this approximation is not possible since it will render the equations of motion parabolic and unstable.

For fluid dynamics to be causal it is therefore **necessary** that the relaxation times assume a non-zero value, but this is **not sufficient**. As will be shown in the next chapter (see also Refs. [13, 14]) causality imposes a **stronger** constraint for transient theories: the ratio of the relaxation times to their respective viscosity coefficients must **exceed certain values**. It will be also shown that, for relativistic fluids, **causality implies stability** of the fluid-dynamical equations.

Equations (1.125) – (1.127) correspond to the type of relativistic fluid-dynamical theory that we expect to derive from microscopic theory, including all the nonlinear and higher-order terms that might appear on the right-hand side. This will be shown explicitly in the later chapters of this lecture series.

1.5 Transient thermodynamics and Israel-Stewart theory

It is also possible to derive causal fluid-dynamical equations, with the same structure as Eqs. (1.125) – (1.127), from the **second law of thermodynamics**. In this section, we review this derivation as first proposed by Israel and Stewart [5, 7]. The main idea of their approach is to apply the second law of thermodynamics to a more general expression of the non-equilibrium entropy 4-current. In equilibrium, the entropy 4-current was expressed exactly in terms of the primary fluid-dynamical variables, α_0, β_0, and u^μ. When the fluid deviates from equilibrium, the situation becomes more complicated. Strictly speaking, the entropy 4-current should depend on a very large number of independent dynamical variables (for a dilute gas, these correspond to all the moments of the Boltzmann equation) that are needed in order to characterize the complicated state of a non-equilibrium system. However, it is reasonable to assume that, as the system approaches equilibrium, the number of dynamical variables needed to describe the state of the fluid gradually decreases, until it reaches the variables required by the equilibrium state, α_0, β_0, and u^μ.

In the previous section, we showed that, in order to render the fluid-dynamical equations causal, the dissipative currents must be promoted to **independent dynamical variables**. Therefore, we expect that a more realistic description of the entropy 4-current can be obtained by considering it to be a function not only of the primary fluid-dynamical variables, but also of the dissipative currents,

$$S^\mu = S^\mu(\alpha_0, \beta_0, u^\mu, \Pi, n^\mu, \pi^{\mu\nu}) \ . \tag{1.128}$$

Mathematically, it is further assumed that the entropy 4-current has the following properties: (i) it is **additive**; (ii) it is a **convex function** of the equilibrium variables **and** the dissipative currents; and (iii) the corresponding **entropy production is locally positive**. We remark that, while here these properties enter as hypothesis, they can be rigorously derived in the framework of kinetic theory [5, 6, 7].

Then, the entropy 4-current can be expanded in terms of **powers of the dissipative currents** around a (fictitious) equilibrium state [5, 7],

$$S^\mu = S^\mu_{(0)} - \alpha_0 n^\mu + Q^\mu + \mathcal{O}_3 \ , \tag{1.129}$$

where $\mathcal{O}_3$ denotes terms of third order or higher in the dissipative currents and

$$Q^\mu \equiv -\frac{1}{2} u^\mu \left(\delta_0 \Pi^2 - \delta_1 n_\alpha n^\alpha + \delta_2 \pi_{\alpha\beta}\pi^{\alpha\beta}\right) - \gamma_0 \Pi n^\mu - \gamma_1 \pi^\mu_\nu n^\nu \tag{1.130}$$

is of second order, $Q^\mu \sim \mathcal{O}_2$. The expansion coefficients, δ_0, δ_1, δ_2, γ_0, and γ_1, are complicated functions of the temperature and chemical potential of the (fictitious) equilibrium state and can only be obtained by matching this expansion with the underlying microscopic theory. Note that the entropy 4-current used to derive relativistic Navier-Stokes theory is recovered by taking $Q^\mu = 0$. It is important to remember that Q^μ is not orthogonal to the fluid 4-velocity and, consequently,

$$s \equiv u_\mu S^\mu = s_0 + u_\mu Q^\mu \neq s_0 \ . \tag{1.131}$$

That is, the non-equilibrium entropy density in the local rest frame, s, does **not** correspond to the entropy density computed using the (fictitious) equilibrium state, $s_0\,(\varepsilon, n)$.

The existence of second-order contributions to the entropy 4-current will affect all previous conclusions drawn from the second law of thermodynamics, which can then be understood to be valid only up to **first order** in the dissipative currents (hence the name first-order theory). Next, we re-calculate the entropy production using the more general entropy 4-current introduced in Eq. (1.129),

$$\partial_\mu S^\mu = \beta_0 u_\nu \partial_\mu T^{\mu\nu}_{(0)} - \alpha_0 \partial_\mu N^\mu_{(0)} - \partial_\mu\left(\alpha_0 n^\mu\right) + \partial_\mu Q^\mu \ , \tag{1.132}$$

where we employed Eq. (1.45). The conservation laws (1.77) and (1.79) lead to the following result,

$$\partial_\mu S^\mu = -\beta_0 \Pi\theta + \beta_0 \pi^{\mu\nu}\sigma_{\mu\nu} - n^\mu \nabla_\mu \alpha_0 + \partial_\mu Q^\mu \ . \tag{1.133}$$

Using Eq. (1.130), we can derive all terms originating from $\partial_\mu Q^\mu$,

$$\begin{aligned}\partial_\mu Q^\mu \ = \ & -\delta_0 \Pi\dot{\Pi} + \delta_1 n_\mu \dot{n}^\mu - \delta_2 \pi_{\mu\nu}\dot{\pi}^{\mu\nu} - \frac{1}{2}\left(\Pi^2 \dot{\delta}_0 - n_\mu n^\mu \dot{\delta}_1 + \pi_{\mu\nu}\pi^{\mu\nu}\dot{\delta}_2\right) \\ & -\frac{1}{2}\left(\delta_0 \Pi^2 - \delta_1 n_\mu n^\mu + \delta_2 \pi_{\mu\nu}\pi^{\mu\nu}\right)\theta - \gamma_0 \Pi \partial_\mu n^\mu - \gamma_0 n^\mu \nabla_\mu \Pi - \Pi n^\mu \nabla_\mu \gamma_0 \\ & -\gamma_1 \pi_{\mu\nu}\nabla^{\langle\mu} n^{\nu\rangle} - \pi_{\mu\nu} n^{\langle\mu}\nabla^{\nu\rangle}\gamma_1 - \gamma_1 n_\nu \partial_\mu \pi^{\mu\nu} \ . \end{aligned} \tag{1.134}$$

Then, substituting Eq. (1.134) into Eq. (1.133), we obtain the more general entropy-production equation

$$\begin{aligned}\partial_\mu S^\mu = \ & \beta_0 \Pi\left(-\theta - \frac{\delta_0}{\beta_0}\dot{\Pi} - \frac{1}{2\beta_0}\Pi\dot{\delta}_0 - \frac{1}{2\beta_0}\delta_0 \Pi\theta - \frac{\gamma_0}{\beta_0}\partial_\mu n^\mu - \frac{1-r}{\beta_0} n^\mu \nabla_\mu \gamma_0\right) \\ & + n_\mu\left(-\nabla^\mu \alpha_0 + \delta_1 \Delta^\mu_\alpha \dot{n}^\alpha + \frac{n^\mu}{2}\dot{\delta}_1 + \frac{\delta_1}{2} n^\mu \theta - \gamma_0 \nabla^\mu \Pi - r\Pi\nabla^\mu \gamma_0 - \gamma_1 \partial_\nu \pi^{\mu\nu} - y\pi^{\mu\nu}\nabla_\nu \gamma_1\right) \\ & + \beta_0 \pi_{\mu\nu}\left(\sigma^{\mu\nu} - \frac{\delta_2}{\beta_0}\Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} - \frac{1}{2\beta_0}\pi^{\mu\nu}\dot{\delta}_2 - \frac{1}{2\beta_0}\delta_2 \pi^{\mu\nu}\theta - \frac{\gamma_1}{\beta_0}\nabla^{\langle\mu} n^{\nu\rangle} - \frac{1-y}{\beta_0} n^{\langle\mu}\nabla^{\nu\rangle}\gamma_1\right) \ , \end{aligned} \tag{1.135}$$

where r, y are arbitrary constants.

As argued before, the only way to explicitly satisfy the second law of thermodynamics is to assure that the entropy production is a **positive semi-definite quadratic function of the dissipative currents**, i.e.,

$$\partial_\mu S^\mu \equiv \beta_0 \varpi_\Pi \Pi^2 - \varpi_n n_\mu n^\mu + \beta_0 \varpi_\pi \pi_{\mu\nu}\pi^{\mu\nu} \ , \tag{1.136}$$

where ϖ_Π, ϖ_n, $\varpi_\pi \geq 0$. This further implies that the dissipative currents must satisfy the following **dynamical** equations

$$\begin{aligned}
\frac{\delta_0}{\beta_0}\dot{\Pi} + \varpi_\Pi \Pi &= -\theta - \frac{1}{2\beta_0}\Pi\dot{\delta}_0 - \frac{1}{2\beta_0}\delta_0\Pi\theta - \frac{\gamma_0}{\beta_0}\partial_\mu n^\mu - \frac{1-r}{\beta_0}n^\mu\nabla_\mu\gamma_0 \ , \\
\delta_1\Delta^\mu_\alpha\dot{n}^\alpha + \varpi_n n^\mu &= \nabla^\mu\alpha_0 - \frac{1}{2}n^\mu\dot{\delta}_1 - \frac{\delta_1}{2}n^\mu\theta + \gamma_0\nabla^\mu\Pi + r\Pi\nabla^\mu\gamma_0 + \gamma_1\partial_\nu\pi^{\mu\nu} \\
&\quad + y\pi^{\mu\nu}\nabla_\nu\gamma_1 \ , \\
\frac{\delta_2}{\beta_0}\Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} + \varpi_\eta\pi^{\mu\nu} &= \sigma^{\mu\nu} - \frac{1}{2\beta_0}\pi^{\mu\nu}\dot{\delta}_2 - \frac{1}{2\beta_0}\delta_2\pi^{\mu\nu}\theta - \frac{\gamma_1}{\beta_0}\nabla^{\langle\mu}n^{\nu\rangle} - \frac{1-y}{\beta_0}n^{\langle\mu}\nabla^{\nu\rangle}\gamma_1 \ ,
\end{aligned} \tag{1.137}$$

which are relaxation-type equations, similar to those conjectured in the last section, i.e., Eqs. (1.125) – (1.127). By comparison with those equations, we find ϖ_Π, ϖ_n, and ϖ_π to be related to the viscosity and diffusion coefficients,

$$\varpi_\Pi = \frac{1}{\zeta} \ , \quad \varpi_n = \frac{1}{\varkappa} \ , \quad \varpi_\pi = \frac{1}{2\eta} \ , \tag{1.138}$$

and identify the relaxation times as

$$\tau_\Pi = \zeta\frac{\delta_0}{\beta_0} \ , \quad \tau_n = \varkappa\delta_1 \ , \quad \tau_\pi = 2\eta\frac{\delta_2}{\beta_0} \ . \tag{1.139}$$

Since the relaxation times must be positive, the expansion coefficients δ_0, δ_1, and δ_2 must all be larger than zero. Furthermore, we found some of the nonlinear terms that may appear as source terms in the transient equations of motion for the dissipative currents. We shall see in the next chapters, when we derive the equations of fluid dynamics from microscopic theory, that there are still other nonlinear terms that are missing in this type of derivation. A derivation from microscopic theory also allows to uniquely fix the, as of yet, arbitrary constants r, y.

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2 Linear Stability and Causality

It was emphasized in the previous chapter that relativistic generalizations of Navier-Stokes theory, derived by Landau and Lifshitz [1] and, independently, by Eckart [2], are ill-defined, containing unphysical instabilities when perturbed around an arbitrary global-equilibrium state [3, 4, 5]. Such instabilities are intrinsically related to the acausal nature of Navier-Stokes theory, which allows for perturbations that propagate with an infinite speed. These fundamental problems prohibit the application of relativistic Navier-Stokes theory to describe any practical fluid-dynamical problem, may it be in the description of neutron-star mergers or in the description of the quark-gluon plasma produced in heavy-ion collisions.

In order to address these fundamental issues, Israel and Stewart constructed stable and causal theories of relativistic fluid dynamics, following the procedure initially developed by H. Grad [6] for non-relativistic systems. Israel and Stewart performed this task in two distinct ways [7]. The first is a phenomenological derivation, based on the second law of thermodynamics, which was discussed in detail in Chapter 1. The second is a microscopic derivation starting from the relativistic Boltzmann equation, which will be discussed thoroughly in one of the following chapters. Similar theories have been widely developed in the past decades [8, 9, 10, 11, 12, 13, 14, 15, 16], but all carry the same fundamental aspect: in contrast to Navier-Stokes theory, such causal theories of fluid dynamics include in their description the transient dynamics of the non-conserved dissipative currents. For this reason, they were initially named by Israel and Stewart as **transient fluid dynamics** (nowadays, they are often referred to as **second-order theories**).

However, it is important to remark that the theory formulated by Israel and Stewart is **not guaranteed** to be causal and stable. As was first shown by Hiscock, Lindblom, and later by Olson, such transient theories of fluid dynamics are **only causal and stable** if their **transport coefficients satisfy certain conditions** [17, 18, 19]. Such conclusions were obtained by analysing the properties of the theory in the linear regime and by imposing that the perturbations around a global-equilibrium state are stable and propagate subluminally. Such stability analyses were more recently revisited in Ref. [4], including only the effects of bulk viscosity, and in Ref. [5], which included the effects of both shear and bulk viscosity. In both these papers, constraints for the shear and bulk relaxation times were explicitly derived. Nowadays, the causality of relativistic fluid-dynamical theories has been investigated even in the nonlinear regime [20, 21] (including the effects of shear and bulk viscosity), where more general inequalities required to ensure causality were derived. In the latter case, the inequalities constrain not only the transport coefficients, but also the values of the dissipative currents (in the linear regime, the inequalities derived in Ref. [21] reduce to those derived in Refs. [4, 5]). Such constraints are relevant for, e.g., fluid-dynamical applications to heavy-ion collisions, since the transport coefficients of QCD matter are not precisely known (often, they are completely unknown) and such fundamental constraints on transport coefficients (and the values of the dissipative currents) can be extremely useful.

In this chapter we perform a **linear stability analysis** of **relativistic Navier-Stokes theory** and of **Israel-Stewart theory** around a general global-equilibrium state. We then demonstrate explicitly the results described above. First, we prove that the general global-equilibrium state in relativistic Navier-Stokes theory is unstable, due to the appearance of so-called non-hydrodynamic modes that grow exponentially on microscopic time scales. We then demonstrate that the Israel-Stewart theory remains stable, as long as the transport coefficients of the linearized theory satisfy certain constraints.

This chapter is organized as follows. In Sec. 2.1, we first write the fluid-dynamical equations by approximating them to linear order in perturbations around a global-equilibrium background. In Sec. 2.2 we transform these equations into Fourier space, with the purpose of determining the dispersion relations satisfied by the perturbations. Section 2.3 discusses these dispersion relations for ideal fluid dynamics, where one just has stable sound modes. In Sec. 2.4 we investigate the causality and stability of the linearized fluid-dynamical equations in the Navier-Stokes limit. Although these equations appear to be stable in a static background, they will become **unstable when the perturbations are performed on a moving background**. Section 2.5 then discusses stability and causality of transient theories of relativistic fluid dynamics. **Both in a static as well as in a moving background these theories are causal and stable**, provided that certain **asymptotic causality conditions are fulfilled**. A summary of our results concludes this chapter in Sec. 2.6.

2.1 Fluid-dynamical equations linearized around global equilibrium

In this section, we linearize the fluid-dynamical equations described in the previous chapter around a global-equilibrium state. In their linearized form, the equations simplify considerably and some of their properties can be studied systematically. In particular, our goal is to discuss the stability of relativistic fluids and to verify under which circumstances acausal modes appear in the linearized theory. Here, we consider small fluid-dynamical perturbations around a global-equilibrium state, with inverse temperature $\beta_0 \equiv 1/T_0$, thermal potential $\alpha_0 = \beta_0 \mu_0$, where μ_0 is the chemical potential, and a velocity u_0^μ, satisfying the normalization condition, $u_{0\mu} u_0^\mu = 1$. The pressure is obtained using an arbitrary equation of state, $P_0 \equiv P(\beta_0, \alpha_0)$. Naturally, in this state the dissipative currents appearing in the net-charge 4-current, N^μ, and the energy-momentum tensor, $T^{\mu\nu}$, all vanish: $\Pi_0 = \pi_0^{\mu\nu} = n_0^\mu = h_0^\mu = 0$.

In Chapter 1, we derived the equations of motion of a relativistic fluid and discussed two possible definitions of the fluid 4-velocity: one proposed by Landau [1] and another by Eckart [2]. We note that the prescription chosen for the velocity field can affect some aspects of the linear stability analyses [19]. Here, we perform our analysis using Landau's prescription for the velocity field, i.e., $T^{\mu\nu} u_\nu = \varepsilon u^\mu$. In this case, the continuity equations related to energy, momentum, and net-charge conservation are given by

$$D\varepsilon + (\varepsilon + P + \Pi)\,\theta - \pi^{\alpha\beta}\sigma_{\alpha\beta} = 0\ , \tag{2.1}$$

$$(\varepsilon + P + \Pi)\,Du^{\mu} - \nabla^{\mu}(P + \Pi) - \pi^{\mu\beta}Du_{\beta} + \Delta^{\mu}_{\alpha}\nabla_{\beta}\pi^{\alpha\beta} = 0\ , \tag{2.2}$$

$$Dn + n\theta - n^{\mu}Du_{\mu} + \nabla_{\mu}n^{\mu} = 0\ . \tag{2.3}$$

Here, ε is the energy density, n is the net-charge density, $D \equiv u^{\mu}\partial_{\mu}$ is the comoving derivative, $\theta \equiv \partial_{\mu}u^{\mu}$ is the expansion scalar, $\sigma^{\mu\nu} \equiv \partial^{\langle\mu}u^{\nu\rangle}$ is the shear tensor, $\Delta^{\mu}_{\nu} = g^{\mu}_{\nu} - u^{\mu}u_{\nu}$ is the projection operator onto the 3-space orthogonal to u^{μ}, and $\nabla^{\mu} \equiv \partial^{\langle\mu\rangle}$, $A^{\langle\mu\rangle} \equiv \Delta^{\mu}_{\nu}A^{\nu}$, $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}A^{\alpha\beta}$, with $\Delta^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}$.

As discussed in Sec. 1.4.2, in transient theories of fluid dynamics the dissipative currents are determined from **dynamical evolution equations**, and not by **static constitutive relations**. These equations of motion were already derived in the previous chapter and are given by

$$\tau_{\Pi}D\Pi + \Pi = -\zeta\theta + \cdots\ , \tag{2.4}$$

$$\tau_{n}\Delta^{\mu}_{\alpha}Dn^{\alpha} + n^{\mu} = \varkappa\nabla^{\mu}\alpha\ + \cdots\ , \tag{2.5}$$

$$\tau_{\pi}\Delta^{\mu\nu}_{\alpha\beta}D\pi^{\alpha\beta} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots\ . \tag{2.6}$$

Above, the dots indicate possible second-order terms [16, 22, 23] which will be neglected in the following linear stability analysis. Note that most (but not all) second-order terms are nonlinear and do not contribute in the linearized regime (at least when considering perturbations around an equilibrium state). For an analysis of the influence of some of the second-order terms that do contribute in the linear stability analysis, see Ref. [24].

We consider perturbations of all fluid-dynamical fields around the global-equilibrium state described above,

$$\varepsilon = \varepsilon_0 + \delta\varepsilon\ , \tag{2.7}$$

$$n = n_0 + \delta n\ , \tag{2.8}$$

$$u^{\mu} = u^{\mu}_0 + \delta u^{\mu}\ , \tag{2.9}$$

$$\Pi = \delta\Pi\ , \tag{2.10}$$

$$n^{\mu} = \delta n^{\mu}\ , \tag{2.11}$$

$$\pi^{\mu\nu} = \delta\pi^{\mu\nu}\ , \tag{2.12}$$

where $n_0 \equiv n(\beta_0, \alpha_0)$ and $\varepsilon_0 \equiv \varepsilon(\beta_0, \alpha_0)$ are constants. The linearization procedure consists of substituting Eqs. (2.7) – (2.12) into the exact fluid-dynamical equations (2.1) – (2.6) and only retaining the terms that are linear, i.e., of first order, in the deviations from the equilibrium state.

Since the fluid 4-velocity is normalized, i.e., $u_{\mu}u^{\mu} = 1$, it is straightforward to demonstrate that the perturbations of the fluid velocity satisfy

$$\delta u_{\mu}u^{\mu}_0 = \mathcal{O}_2 \approx 0\ , \tag{2.13}$$

where $\mathcal{O}_2$ denotes terms that are of second order or higher in perturbations of the fluid-dynamical fields. That is, up to first order in perturbations, the fluctuations of the fluid

4-velocity are orthogonal to the background 4-velocity. Similarly, it is possible to obtain orthogonality relations satisfied by the dissipative currents, δn^μ and $\delta\pi^{\mu\nu}$. As shown in the previous chapter, the dissipative currents are constructed to be orthogonal to the 4-velocity field, $u_\mu n^\mu = u_\mu \pi^{\mu\nu} = 0$. Since n^μ and $\pi^{\mu\nu}$ are at least of first order (in this power-counting scheme), these relations lead to

$$u^0_\mu \delta\pi^{\mu\nu} = \mathcal{O}^\nu_2 \approx 0 \,, \tag{2.14}$$
$$u^0_\mu \delta n^\mu = \mathcal{O}_2 \approx 0 \,. \tag{2.15}$$

Thus, the perturbations of the dissipative currents are, to first order, also orthogonal to the background velocity field. Due to these orthogonality relations, it is convenient to introduce a projection operator onto the 3-space orthogonal to the **background** velocity,

$$\Delta^{\mu\nu}_0 \equiv g^{\mu\nu} - u^\mu_0 u^\nu_0 \,, \tag{2.16}$$

and, similarly, a rank-4 symmetric, traceless projection operator

$$\Delta^{\mu\nu\alpha\beta}_0 \equiv \frac{1}{2}\left(\Delta^{\mu\alpha}_0 \Delta^{\nu\beta}_0 + \Delta^{\mu\beta}_0 \Delta^{\nu\alpha}_0\right) - \frac{1}{3}\Delta^{\mu\nu}_0 \Delta^{\alpha\beta}_0 \,. \tag{2.17}$$

Finally, we define a comoving derivative relative to the background velocity field, $D_0 \equiv u^\mu_0 \partial_\mu$.

Using this notation, the equations of motion for the energy density, local velocity field, and net-charge density, linearized around the global-equilibrium state, become

$$D_0 \delta\varepsilon + (\varepsilon_0 + P_0)\,\partial_\mu \delta u^\mu = \mathcal{O}_2 \approx 0 \,, \tag{2.18}$$
$$(\varepsilon_0 + P_0)\, D_0 \delta u^\mu - \Delta^{\mu\nu}_0 \partial_\nu (\delta P + \delta\Pi) + \partial_\nu \delta\pi^{\mu\nu} = \mathcal{O}^\mu_2 \approx 0 \,, \tag{2.19}$$
$$D_0 \delta n + n_0 \partial_\mu \delta u^\mu + \partial_\mu \delta n^\mu = \mathcal{O}_2 \approx 0 \,, \tag{2.20}$$

while the linearized equations for the dissipative currents are

$$\tau_\Pi D_0 \delta\Pi + \delta\Pi + \zeta \partial_\mu \delta u^\mu = \mathcal{O}_2 \approx 0 \,, \tag{2.21}$$
$$\tau_n D_0 \delta n^\mu + \delta n^\mu - \varkappa \Delta^{\mu\nu}_0 \partial_\nu \delta\alpha = \mathcal{O}^\mu_2 \approx 0 \,, \tag{2.22}$$
$$\tau_\pi D_0 \delta\pi^{\mu\nu} + \delta\pi^{\mu\nu} - 2\eta \Delta^{\mu\nu\alpha\beta}_0 \partial_\alpha \delta u_\beta = \mathcal{O}^{\mu\nu}_2 \approx 0 \,, \tag{2.23}$$

where ζ, $\varkappa$, η, τ_Π, τ_n, and τ_π are the transport coefficients as functions of the *background* temperature and thermal potential. We note that, in the linear regime, pressure perturbations can be expressed as

$$\delta P = \left.\frac{\partial P_0}{\partial \alpha_0}\right|_{\beta_0} \delta\alpha + \left.\frac{\partial P_0}{\partial \beta_0}\right|_{\alpha_0} \delta\beta + \mathcal{O}_2 = \frac{n_0}{\beta_0}\delta\alpha - \frac{h_0 n_0}{\beta_0}\delta\beta + \mathcal{O}_2 \,, \tag{2.24}$$

where we have used Eq. (1.103). Similar relations apply to energy-density and net-charge density fluctuations.

Exercise 2.1: Prove Eqs. (2.18) – (2.23).

2.2 Linearized fluid-dynamical equations in Fourier space

In order to investigate the (propagating and exponentially in- or decreasing) modes of the fluid-dynamical equations, it is convenient to Fourier transform the space-time dependence of the linearized equations. We define our Fourier transformation using the following convention,

$$\tilde{A}(K) = \int d^4x \exp\left(-ix_\mu k^\mu\right) A(X) \, , \tag{2.25}$$

$$A(X) = \int \frac{d^4k}{(2\pi)^4} \exp\left(ix_\mu k^\mu\right) \tilde{A}(K) \, . \tag{2.26}$$

Here, $k^\mu \equiv K = (\omega, \mathbf{k})^T$ is the wavenumber 4-vector, with ω being the frequency and $\mathbf{k}$ the wavenumber 3-vector of the fluctuation, and $x^\mu \equiv X = (t, \mathbf{x})^T$ is the usual space-time coordinate 4-vector.

Let us define the covariant variables

$$\Omega \equiv u_0^\mu k_\mu \, , \quad \kappa^\mu \equiv \Delta_0^{\mu\nu} k_\nu \, , \tag{2.27}$$

where Ω is the **frequency of the perturbations in the local rest frame of the background fluid** while κ^μ is the corresponding **wavenumber 4-vector**. The linearized conservation laws (2.18) – (2.20) can then be written in Fourier space in the simple form

$$\Omega\delta\tilde{\varepsilon} + \left(\varepsilon_0 + P_0\right)\kappa_\mu \delta\tilde{u}^\mu = 0 \, , \tag{2.28}$$

$$\left(\varepsilon_0 + P_0\right)\Omega\delta\tilde{u}^\mu - \kappa^\mu\left(\delta\tilde{P} + \delta\tilde{\Pi}\right) + \kappa_\nu \delta\tilde{\pi}^{\mu\nu} = 0 \, , \tag{2.29}$$

$$\Omega\delta\tilde{n} + n_0\kappa_\mu\delta\tilde{u}^\mu + \kappa_\mu\delta\tilde{n}^\mu = 0 \, , \tag{2.30}$$

and the linearized equations of motion (2.21) – (2.23) for the dissipative currents become

$$\left(1 + i\tau_\Pi\Omega\right)\delta\tilde{\Pi} = -i\zeta\kappa_\mu\delta\tilde{u}^\mu \, , \tag{2.31}$$

$$\left(1 + i\tau_n\Omega\right)\delta\tilde{n}^\mu = i\varkappa\kappa^\mu\delta\tilde{\alpha} \, , \tag{2.32}$$

$$\left(1 + i\tau_\pi\Omega\right)\delta\tilde{\pi}^{\mu\nu} = i\eta\left(\kappa^\mu\delta\tilde{u}^\nu + \kappa^\nu\delta\tilde{u}^\mu - \frac{2}{3}\Delta_0^{\mu\nu}\kappa_\lambda\delta\tilde{u}^\lambda\right) \, . \tag{2.33}$$

We further define the scalar κ as the modulus of κ^μ, $\kappa_\mu\kappa^\mu \equiv -\kappa^2$. Note that the usual dispersion relation of perturbations in a fluid, $\omega\left(\mathbf{k}\right)$ (with $\omega \equiv k^0$), is not a Lorentz scalar and will change depending on the magnitude of the background velocity u_0^μ.

2.2.1 Tensor decomposition in Fourier space

It is also convenient to decompose the perturbations into components **parallel** and **orthogonal** to κ^μ. For this purpose we introduce another **projection operator**, now onto the 3-space orthogonal to $\hat{\kappa}^\mu$,

$$\Delta_\kappa^{\mu\nu} = g^{\mu\nu} + \hat{\kappa}^\mu\hat{\kappa}^\nu \, , \tag{2.34}$$

where $\hat{\kappa}^\mu \equiv \kappa^\mu/\kappa$ is a space-like 4-vector normalized to $\hat{\kappa}_\mu\hat{\kappa}^\mu = -1$. Within this scheme, an arbitrary 4-vector A^μ is decomposed as

$$A^\mu = A_\parallel \hat{\kappa}^\mu + A_\perp^\mu \,, \tag{2.35}$$

with components $A_\parallel \equiv -\hat{\kappa}_\mu A^\mu$ and $A_\perp^\mu \equiv \Delta_\kappa^{\mu\lambda} A_\lambda$. We shall refer to $A_\parallel$ as the **longitudinal** component of the corresponding 4-vector and $A_\perp^\mu$ as its **transverse** component. Similarly, an arbitrary traceless, symmetric second-rank tensor, $A^{\mu\nu}$, is decomposed as

$$A^{\mu\nu} = A_\parallel \hat{\kappa}^\mu \hat{\kappa}^\nu + \frac{A_\parallel}{3}\Delta_\kappa^{\mu\nu} + A_\perp^\mu \hat{\kappa}^\nu + A_\perp^\nu \hat{\kappa}^\mu + A_\perp^{\mu\nu} \,. \tag{2.36}$$

with $A_\parallel \equiv \hat{\kappa}^\mu\hat{\kappa}^\nu A_{\mu\nu}$, $A_\perp^\lambda \equiv -\hat{\kappa}^\nu \Delta_\kappa^{\mu\lambda} A_{\mu\nu}$, and $A_\perp^{\mu\nu} \equiv \Delta_\kappa^{\mu\nu\alpha\beta} A_{\alpha\beta}$, where we defined the rank-4 symmetric, traceless projection operator $\Delta_\kappa^{\mu\nu\alpha\beta} \equiv \frac{1}{2}(\Delta_\kappa^{\mu\alpha}\Delta_\kappa^{\nu\beta} + \Delta_\kappa^{\mu\beta}\Delta_\kappa^{\nu\alpha}) - \frac{1}{3}\Delta_\kappa^{\mu\nu}\Delta_\kappa^{\alpha\beta}$. In this case, $A_\parallel$ is the **longitudinal** component of the tensor, $A_\perp^\mu$ corresponds to its **partially transverse** component, and $A_\perp^{\mu\nu}$ is its **fully transverse** (traceless) component.

2.2.2 Longitudinal and transverse components

We now project Eqs. (2.28) – (2.33) onto their components longitudinal and transverse to $\hat{\kappa}^\mu$. This is useful because the longitudinal and transverse projections of the equations **decouple** from each other, and can be solved **independently** to obtain the dispersion relations satisfied by the perturbations. We first consider the equations for the longitudinal components of the fluctuations.

First, we note that Eqs. (2.28), (2.30), and (2.31) are already written in terms of longitudinal perturbations since they describe perturbations of scalar quantities, i.e., energy density, net-charge density, and bulk-viscous pressure. The longitudinal component of Eqs. (2.29) and (2.32) are obtained by projecting each equation onto $\hat{\kappa}_\mu$, while the longitudinal component of Eq. (2.33) is obtained by projecting it onto $\hat{\kappa}_\mu\hat{\kappa}_\nu$. The result is a set of coupled equations for the perturbations $\delta\tilde{\varepsilon}$, $\delta\tilde{P}$, $\delta\tilde{n}$, $\delta\tilde{\alpha}$, $\delta\tilde{u}_\parallel$, $\delta\tilde{\Pi}$, $\delta\tilde{n}_\parallel$, and $\delta\tilde{\pi}_\parallel$,

$$\Omega\delta\tilde{\varepsilon} - \kappa\left(\varepsilon_0 + P_0\right)\delta\tilde{u}_\parallel = 0 \,, \tag{2.37}$$

$$\left(\varepsilon_0 + P_0\right)\Omega\delta\tilde{u}_\parallel - \kappa\left(\delta\tilde{P} + \delta\tilde{\Pi} + \delta\tilde{\pi}_\parallel\right) = 0 \,, \tag{2.38}$$

$$\Omega\delta\tilde{n} - \kappa\left(n_0\delta\tilde{u}_\parallel + \delta\tilde{n}_\parallel\right) = 0 \,, \tag{2.39}$$

$$\left(1 + i\tau_\Pi\Omega\right)\delta\tilde{\Pi} - i\zeta\kappa\delta\tilde{u}_\parallel = 0 \,, \tag{2.40}$$

$$\left(1 + i\tau_n\Omega\right)\delta\tilde{n}_\parallel - i\varkappa\kappa\delta\tilde{\alpha} = 0 \,, \tag{2.41}$$

$$\left(1 + i\tau_\pi\Omega\right)\delta\tilde{\pi}_\parallel - \frac{4}{3}i\eta\kappa\delta\tilde{u}_\parallel = 0 \,. \tag{2.42}$$

Exercise 2.2: Prove Eqs. (2.37) – (2.42).

The fluctuations of energy density, net-charge density, and pressure can be converted into fluctuations of inverse temperature and thermal potential, $\delta\tilde{\beta}$ and $\delta\tilde{\alpha}$, respectively. Solving these equations leads to **six different modes** of the theory, two of them related to the **propagation of sound waves**, one related to the **diffusion of net-charge density fluctuations**, and the remaining three being new non-propagating modes due to **fluctuations of the dissipative currents** that are not allowed in Navier-Stokes theory. The latter modes are **non-hydrodynamic**, i.e., they are modes that do not vanish when the wavenumber is taken to zero, and describe the **relaxation of the dissipative currents back to global equilibrium**. Such modes were thought to never exist in fluid-dynamical theories, but we shall see that they are **required in the relativistic regime in order to restore causality** – and that they even appear in Navier-Stokes theory in the relativistic regime.

The equation of motion for the transverse component of the velocity field is obtained by projecting Eq. (2.29) with $\Delta^{\lambda}_{\kappa\mu}$ while the equation of motion for the partially transverse component of the shear-stress tensor is obtained by projecting Eq. (2.33) with $\hat{\kappa}_{\mu}\Delta^{\lambda}_{\kappa\nu}$. The resulting equations are coupled,

$$(\varepsilon_0 + P_0)\,\Omega\delta\tilde{u}^{\lambda}_{\perp} - \kappa\delta\tilde{\pi}^{\lambda}_{\perp} = 0\,, \tag{2.43}$$

$$(1 + i\tau_{\pi}\Omega)\,\delta\tilde{\pi}^{\lambda}_{\perp} - i\eta\kappa\delta\tilde{u}^{\lambda}_{\perp} = 0\,. \tag{2.44}$$

Exercise 2.3: Prove Eqs. (2.43) and (2.44).

Such transverse modes are usually referred to as **shear modes**, since they do not display any contributions from bulk-viscous pressure or diffusion 4-current. They describe the **diffusion of the velocity field** and, as long as the relaxation times remain finite, they also contain a non-hydrodynamic mode. Since $\delta\tilde{u}^{\mu}_{\perp}$ and $\delta\tilde{\pi}^{\mu}_{\perp}$ carry a total of four independent degrees of freedom, each mode obtained from the above equations will have a twofold degeneracy.

Finally, we have the two equations of motion for the transverse fluctuation of the diffusion 4-current and the fully transverse fluctuation of the shear-stress tensor, also containing two modes each,

$$(1 + i\tau_{n}\Omega)\,\delta\tilde{n}^{\lambda}_{\perp} = 0\,, \tag{2.45}$$

$$(1 + i\tau_{\pi}\Omega)\,\delta\tilde{\pi}^{\alpha\beta}_{\perp} = \frac{2}{9} i\eta\kappa\delta\tilde{u}_{\parallel}\left(\Delta^{\alpha\beta}_{\kappa} - 3u^{\alpha}_{0}u^{\beta}_{0}\right)\,. \tag{2.46}$$

The first equation is obtained by projecting Eq. (2.32) with $\Delta^{\lambda}_{\kappa\mu}$ while the second equation is obtained by projecting Eq. (2.33) with $\Delta^{\alpha\beta}_{\kappa\mu\nu}$.

Exercise 2.4: Prove Eqs. (2.45) and (2.46).

Equation (2.45) gives rise to a twofold degenerate mode,

$$\Omega = \frac{i}{\tau_n}\,, \tag{2.47}$$

which, in the rest frame of the global-equilibrium background, $u_0^\mu = (1,0,0,0)^T$, has no wavenumber dependence, i.e., it is non-propagating. Such a mode is obviously **non-hydrodynamic**, i.e., it has a frequency that does not vanish when the wavenumber is taken to zero in the local rest frame. It is purely imaginary, with positive imaginary part, and thus related to the exponential damping of the dissipative currents towards the equilibrium state. Such modes are rather simple and will not be discussed further.

Equation (2.46) also gives rise to a twofold degenerate mode. With the help of Eq. (2.42) we can write this equation as

$$\delta\tilde{\pi}_\perp^{\alpha\beta} = \frac{1}{6}\,\delta\tilde{\pi}_\parallel \left(\Delta_\kappa^{\alpha\beta} - 3u_0^\alpha u_0^\beta\right) , \tag{2.48}$$

which shows that the fully transverse fluctuations of $\pi^{\alpha\beta}$ simply follow the time dependence of the longitudinal one. In the next sections, we discuss the solutions of the longitudinal modes and the remaining transverse modes for ideal fluids, viscous fluids in the Navier-Stokes limit, and for the complete transient fluid-dynamical equations.

2.3 Ideal fluid dynamics

The ideal-fluid limit is obtained by setting the viscosity and relaxation-time coefficients to zero. In this case, one obtains the following set of equations for the longitudinal perturbations

$$\Omega\delta\tilde{\varepsilon} - (\varepsilon_0 + P_0)\,\kappa\delta\tilde{u}_\parallel = 0 , \tag{2.49}$$

$$(\varepsilon_0 + P_0)\,\Omega\delta\tilde{u}_\parallel - \kappa\delta\tilde{P} = 0 , \tag{2.50}$$

$$\Omega\delta\tilde{n} - n_0\kappa\delta\tilde{u}_\parallel = 0 , \tag{2.51}$$

and one equation for the transverse velocity perturbation

$$(\varepsilon_0 + P_0)\,\Omega\delta\tilde{u}_\perp^\mu = 0 . \tag{2.52}$$

The set of equations for the longitudinal perturbations can be cast into the following matrix form,

$$\begin{pmatrix} \Omega \left.\frac{\partial\varepsilon_0}{\partial\beta_0}\right|_{\alpha_0} & \Omega \left.\frac{\partial\varepsilon_0}{\partial\alpha_0}\right|_{\beta_0} & -(\varepsilon_0+P_0)\,\kappa \\ \kappa\frac{h_0 n_0}{\beta_0} & -\kappa\frac{n_0}{\beta_0} & (\varepsilon_0+P_0)\,\Omega \\ \Omega \left.\frac{\partial n_0}{\partial\beta_0}\right|_{\alpha_0} & \Omega \left.\frac{\partial n_0}{\partial\alpha_0}\right|_{\beta_0} & -n_0\kappa \end{pmatrix} \begin{pmatrix} \delta\tilde{\beta} \\ \delta\tilde{\alpha} \\ \delta\tilde{u}_\parallel \end{pmatrix} = 0 , \tag{2.53}$$

where the energy-density, net-charge density, and pressure perturbations were decomposed in terms of inverse temperature and thermal-potential perturbations. Note that, for the pressure perturbation, we have used Eq. (2.24).

The solution for the transverse mode is simply $\Omega = 0$. The remaining modes from the longitudinal fluctuations are obtained by finding the roots of the determinant of the matrix in Eq. (2.53), leading to the following equation

$$\Omega\left(\Omega^2 - c_s^2\kappa^2\right) = 0 , \tag{2.54}$$

where we identify the square of the **sound velocity** as

$$c_s^2 = \left.\frac{\partial P_0}{\partial \varepsilon_0}\right|_{s_0/n_0} = \frac{n_0}{\beta_0} \frac{h_0^{-1} \left.\frac{\partial \varepsilon_0}{\partial \beta_0}\right|_{\alpha_0} - \left.\frac{\partial n_0}{\partial \beta_0}\right|_{\alpha_0} + \left.\frac{\partial \varepsilon_0}{\partial \alpha_0}\right|_{\beta_0} - h_0 \left.\frac{\partial n_0}{\partial \alpha_0}\right|_{\beta_0}}{\left.\frac{\partial \varepsilon_0}{\partial \beta_0}\right|_{\alpha_0} \left.\frac{\partial n_0}{\partial \alpha_0}\right|_{\beta_0} - \left.\frac{\partial \varepsilon_0}{\partial \alpha_0}\right|_{\beta_0} \left.\frac{\partial n_0}{\partial \beta_0}\right|_{\alpha_0}} . \tag{2.55}$$

Exercise 2.5:

(i) Prove Eq. (2.54) by explicitly computing the determinant of the coefficient matrix in Eq. (2.53).

(ii) Prove Eq. (2.55) by applying the chain rule for Jacobi determinants, i.e.,

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(a,b)} \frac{\partial(a,b)}{\partial(u,v)} . \tag{2.56}$$

In the **rest frame** of the global-equilibrium state, i.e., $u_0^\mu = (1,0,0,0)^T$, the scalars Ω and κ reduce to the usual frequency and wavenumber of the perturbations, respectively, $\Omega = \omega$ and $\kappa = k$. In this case, one recovers the well-known expressions for the modes of an ideal fluid,

$$\omega(k) = 0 \; , \quad \omega(k) = \pm c_s k \; , \tag{2.57}$$

i.e., a **static mode** and two **sound modes**, travelling with velocity c_s forward and backward with respect to the fluid at rest. The static mode appears because we included net-charge fluctuations in the analysis. If one considers a system with zero net charge, the static mode disappears, with the sound modes remaining in the same form as obtained above.

For a **nonzero background velocity** (without loss of generality, we can always set the velocity to point into the x–direction), $u_0^\mu = (\gamma, \gamma V, 0, 0)^T$, we have $\Omega = \gamma\omega - \gamma V k^x$ and $\kappa^2 = \Omega^2 - k_\mu k^\mu = \gamma^2 (\omega V - k^x)^2 + k_T^2$. Naturally, a nonzero background velocity introduces an **anisotropy** in the system, with the modes no longer depending on the modulus of the wavenumber, but displaying a separate dependence on k^x and $k_T \equiv \sqrt{k_y^2 + k_z^2}$. For the sake of simplicity, we consider **waves traveling only in the x–direction**, i.e., with $k_T = 0$, in which case the frequency will depend solely on $k \equiv k^x$. Then the dispersion relation is obtained by solving the equation

$$(\omega - Vk) \left[\left(1 - c_s^2 V^2\right) \omega^2 - 2\left(1 - c_s^2\right) V k \omega + \left(V^2 - c_s^2\right) k^2 \right] = 0 \; . \tag{2.58}$$

This equation has three solutions,

$$\omega(k) = Vk \; , \tag{2.59}$$

$$\omega_\pm(k) = \frac{V \pm c_s}{1 \pm V c_s} k \; . \tag{2.60}$$

We see that the frequencies of oscillations are both linear in k. As expected, the mode that previously vanished now starts to move with the background velocity. Meanwhile, each sound mode has now a different propagation speed, expressed by the **relativistic velocity-addition rule**

$$v_s^{\pm} = \frac{V \pm c_s}{1 \pm V c_s} .$$

One mode is moving **in** the direction of the unperturbed fluid and the other in the **opposite** direction. Naturally, the waves propagating against the flow of the background will propagate slower than those moving in the same direction as the background fluid.

Exercise 2.6: Check that Eq. (2.60) is indeed a solution of Eq. (2.58).

05/11/2022

2.4 Relativistic Navier-Stokes theory

The modes related to relativistic Navier-Stokes theory are obtained from the expressions derived in the previous section by taking the limit of vanishing relaxation times, $\tau_\pi = \tau_n = \tau_\Pi = 0$. In this limit, Eqs. (2.40), (2.41), (2.42), and (2.44) dictate that the fluctuations of the dissipative currents are expressed in terms of fluctuations of velocity and thermal potential in the following way,

$$\delta\tilde{\Pi} = i\zeta\kappa\delta\tilde{u}_{\parallel} , \tag{2.61}$$

$$\delta\tilde{n}_{\parallel} = i\varkappa\kappa\delta\tilde{\alpha} , \tag{2.62}$$

$$\delta\tilde{\pi}_{\parallel} = \frac{4}{3} i\eta\kappa\delta\tilde{u}_{\parallel} , \tag{2.63}$$

$$\delta\tilde{\pi}_{\perp}^{\lambda} = i\eta\kappa\delta\tilde{u}_{\perp}^{\lambda} . \tag{2.64}$$

The equations of motion (2.37) – (2.39) for the longitudinal perturbations then reduce to

$$\Omega\delta\tilde{\varepsilon} - \kappa\left(\varepsilon_0 + P_0\right)\delta\tilde{u}_{\parallel} = 0 , \tag{2.65}$$

$$\left(\varepsilon_0 + P_0\right)\Omega\delta\tilde{u}_{\parallel} - \kappa\delta\tilde{P} - i\kappa^2\left(\zeta + \frac{4}{3}\eta\right)\delta\tilde{u}_{\parallel} = 0 , \tag{2.66}$$

$$\Omega\delta\tilde{n} - \kappa n_0\delta\tilde{u}_{\parallel} - i\kappa^2\varkappa\delta\tilde{\alpha} = 0 , \tag{2.67}$$

while the transverse modes (2.43) satisfy the simple relation

$$\left(\Omega - i\kappa^2\frac{\eta}{\varepsilon_0 + P_0}\right)\delta\tilde{u}_{\perp}^{\mu} = 0 . \tag{2.68}$$

There are no transverse perturbations of the diffusion current, while on account of Eq. (2.48) the fully transverse perturbations of the shear-stress tensor follow that of the longitudinal one, Eq. (2.63).

2.4.1 Transverse modes

Let us first discuss the transverse (i.e., shear) modes since their dispersion relations are more simple and easier to obtain. From Eq. (2.68), we can immediately infer that Ω and κ obey a **diffusion-type dispersion relation**

$$\Omega = i\tau_\eta \kappa^2 \, . \tag{2.69}$$

where we defined

$$\tau_\eta \equiv \frac{\eta}{\varepsilon_0 + P_0} \, . \tag{2.70}$$

We note that τ_η is a time scale that naturally appears in the shear modes of Navier-Stokes theory and encompasses all effects of dissipation on the transverse perturbations. We shall soon demonstrate that this time scale is also related to the exponential growth of unstable non-hydrodynamic modes that will appear when perturbing the theory in a moving background. These unstable modes appear only due to the parabolic nature of the relativistic Navier-Stokes theory, which ends up generating new modes in boosted frames.

When $u_0^\mu = (1,0,0,0)^T$, one simply obtains a non-propagating, exponentially damped mode with twofold degeneracy,

$$\omega(k) = i\tau_\eta k^2 \, . \tag{2.71}$$

This relation is the same as the one obtained by solving the diffusion equation, with τ_η playing the role of the diffusion coefficient. That is, the shear modes simply describes a diffusion process of the velocity field.

Now we consider propagation of shear perturbations in a moving background. As done for the case of an ideal fluid, we choose our coordinate system in such a way that the velocity of the background is in the x–direction, i.e., $u_0^\mu = (\gamma, \gamma V, 0, 0)^T$. We also only consider the case of perturbations that travel in the x–direction, i.e., $k^\mu = (\omega, k, 0, 0)^T$. The dispersion relation (2.69) then becomes

$$\begin{aligned} \gamma(\omega - Vk) &= i\tau_\eta \gamma^2 (\omega V - k)^2 \\ \Longleftrightarrow \quad 0 &= \omega^2 V^2 + \left(\frac{i}{\gamma \tau_\eta} - 2Vk \right) \omega - \frac{i}{\gamma\tau_\eta} Vk + k^2 \, . \end{aligned} \tag{2.72}$$

A **nonzero background velocity** has the effect of **mixing the contributions of frequency**, ω, and **wavenumber**, k, which are contained in the Lorentz scalars Ω and κ. Therefore, for a moving background the quadratic term in κ in the dispersion relation will not only carry contributions that are quadratic in k, as in the $V = 0$ case, but will also carry contributions that are **quadratic** in ω. This leads to the appearance of the quadratic term $\omega^2 V^2$ in the dispersion relation, producing an **additional transverse mode** in the theory. The two solutions are

$$\omega(k) = \frac{1 + 2i\gamma V \tau_\eta k \pm \sqrt{1 + 4iV\tau_\eta k/\gamma}}{2i\gamma V^2 \tau_\eta} \, . \tag{2.73}$$

The analytic solution in the limit of small wavenumbers, $k \to 0$, is

$$\omega\,(k \to 0) = \begin{cases} -\dfrac{i}{\gamma V^2 \tau_\eta} \, , \\ Vk + i\tau_\eta k^2/\gamma^3 \, . \end{cases} \tag{2.74}$$

The second mode corresponds to the one of Eq. (2.71), since it tends to $ik^2\tau_\eta$ when the background velocity goes to zero. On the other hand, the first mode is intrinsically new and is non-hydrodynamic, non-propagating, and, because of a negative imaginary part, **unstable**. This mode describes perturbations that **grow exponentially** on time scales of the order of $\gamma V^2 \tau_\eta$. This additional mode does **not** appear when the velocity of the unperturbed fluid is **zero** and, for this reason, this problematic feature of Navier-Stokes theory remains largely unperceived in the literature. Nevertheless, the emergence of unstable non-hydrodynamic modes in relativistic Navier-Stokes theory is a fundamental issue that must be fixed, in order to obtain a relativistic theory of fluid dynamics. We remark that this is a problem of the relativistic version of Navier-Stokes theory – the nonrelativistic version of the theory is stable, even when perturbed around a **moving** global-equilibrium state.

Exercise 2.7: Prove Eqs. (2.73) and (2.74).

2.4.2 Longitudinal modes

Next, we consider the longitudinal modes. For the sake of simplicity, we restrict our analysis to the limit of vanishing net-charge fluctuations. (This case is treated in Ref. [24] for a vanishing net background charge). In this case, Eqs. (2.65) and (2.66) simplify to

$$\begin{pmatrix} \Omega & -\kappa \\ -c_s^2\kappa & \Omega - i\tau_{\text{eff}}\kappa^2 \end{pmatrix} \begin{pmatrix} \dfrac{\delta\tilde{\varepsilon}}{\varepsilon_0 + P_0} \\ \delta\tilde{u}_\parallel \end{pmatrix} = 0 \, , \tag{2.75}$$

where we used the definition of sound velocity at zero chemical potential, $c_s^2 = dP_0/d\varepsilon_0$, and defined the effective time scale

$$\tau_{\text{eff}} \equiv \frac{\zeta + \frac{4}{3}\eta}{\varepsilon_0 + P_0} \, , \tag{2.76}$$

which plays a very similar role as the variable τ_η that appeared in the previous subsection when discussing the shear modes. As before, the dispersion relations satisfied by the perturbations are found by setting the determinant of the matrix in Eq. (2.75) to zero. The equation satisfied by Ω and κ then is

$$\Omega^2 - i\tau_{\text{eff}}\Omega\kappa^2 - c_s^2\kappa^2 = 0 \, . \tag{2.77}$$

In the case where the velocity of the unperturbed system is zero, i.e., $u_0^\mu = (1,0,0,0)^T$, one simply has $\Omega = \omega$ and $\kappa = k$ and Eq. (2.77) simplifies to

$$\omega^2 - i\tau_{\text{eff}}\omega k^2 - c_s^2 k^2 = 0 \quad \Longrightarrow \quad \omega(k) = i\frac{\tau_{\text{eff}}}{2}k^2 \pm k\sqrt{c_s^2 - \frac{\tau_{\text{eff}}^2 k^2}{4}} \, . \tag{2.78}$$

These are the sound modes of the theory and they reduce to the solution found for ideal fluids when $\tau_{\text{eff}} = 0$. Defining the critical wavenumber

$$k_c^{\text{NS}} = 2c_s/\tau_{\text{eff}} \, , \tag{2.79}$$

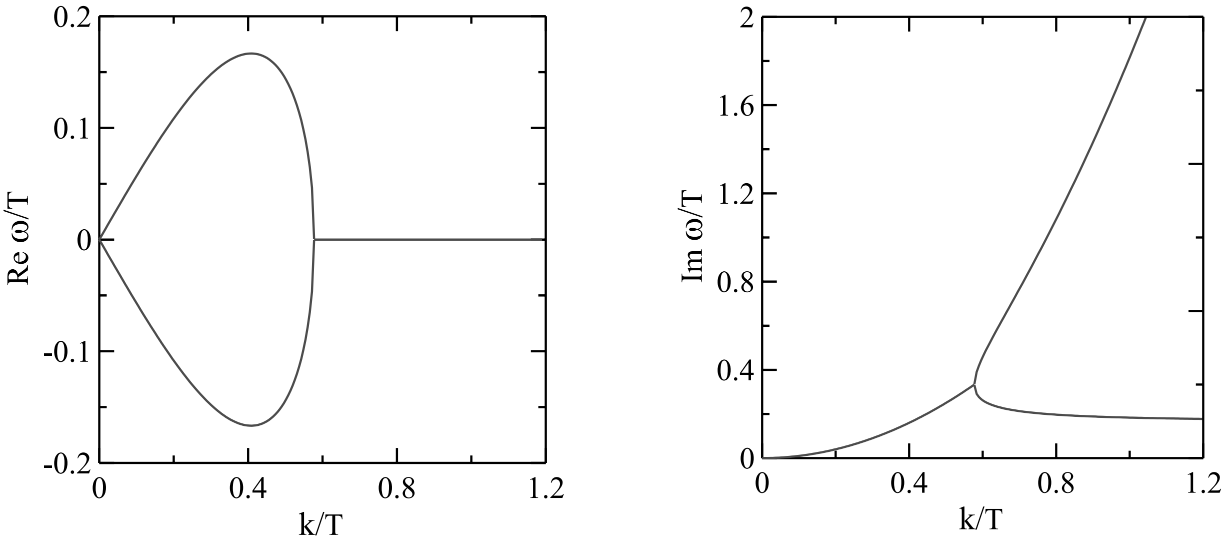

Figure 2.1: The real parts (left panel) and the imaginary parts (right panel) of the dispersion relations for the longitudinal modes of Navier-Stokes theory for a static background. We set $\tau_{\text{eff}} = 2\beta_0$.

we see that for $k < k_c^{\text{NS}}$ the sound modes have a part propagating with the reduced speed of sound $c_{s,\text{eff}} \equiv \sqrt{c_s^2 - \tau_{\text{eff}}^2 k^2/4} \leq c_s$. They also have a non-propagating part which describes how such modes are damped by viscosity. On the other hand, for wavenumbers larger than $k \geq k_c^{\text{NS}}$, the frequencies become purely imaginary and there are no propagating modes. In Fig. 2.1 we show the real (left panel) and imaginary (right panel) parts of $\omega(k)$ in units of temperature, for $\tau_{\text{eff}} = 2\beta_0$.

The solution at small wavenumbers, $k \to 0$, can be written in the simple form

$$\omega(k) = \pm c_s k + i\frac{\tau_{\text{eff}}}{2}k^2 + \mathcal{O}(k^3) \ , \tag{2.80}$$

from which we conclude that, to leading order, the sound modes propagate with the usual speed of sound c_s, damped by the effective specific viscosity $\tau_{\text{eff}}/2$. On the other hand, at large wavenumbers, $k \to \infty$,

$$\omega(k) = i\frac{\tau_{\text{eff}}}{2}k^2\left(1 \pm 1\right) + \mathcal{O}(1) \ . \tag{2.81}$$

That is, for large values of wavenumbers, the mode is either zero or appears to be purely diffusive, $\omega(k) \sim i\tau_{\text{eff}}k^2$.

For the same moving background as considered for the shear modes, $u_0^\mu = \gamma\left(1, V, 0, 0\right)^T$ and $k^\mu = \left(\omega, k, 0, 0\right)^T$, the dispersion relation (2.77) assumes the form

$$\left(\omega - kV\right)^2 - i\tau_{\text{eff}}\gamma\left(\omega - kV\right)\left(\omega V - k\right)^2 - c_s^2\left(\omega V - k\right)^2 = 0 \ . \tag{2.82}$$

In this case, we observe that the term $\Omega\kappa^2$ in Eq. (2.77) will make the equation **cubic** in ω and, as happened with the shear modes, a new solution will emerge. It is easy to see that such a new solution will be an **unstable non-hydrodynamic mode**. For this purpose, one can just look for the solutions of the equation at $k = 0$,

$$\omega\left(k=0\right) = \begin{cases} -i\dfrac{1-c_s^2V^2}{\tau_{\text{eff}}\gamma V^2} \ , \\ \pm 0 \ . \end{cases} \tag{2.83}$$

The last two solutions are simply zero and can be identified with the usual sound modes that already appeared when the unperturbed global-equilibrium state was at rest. The additional solution does not vanish at $k = 0$, i.e., it is a non-hydrodynamic mode, and has a negative imaginary part, i.e., it is an unstable exponentially growing mode. Once more, we see that **relativistic Navier-Stokes theory is unstable** when perturbed around global equilibrium: a fundamental problem that simply cannot be ignored. In the next subsection, we briefly discuss the origins behind this issue.

2.4.3 Causality and stability of Navier-Stokes theory

In the preceding subsections it was shown that relativistic Navier-Stokes theory is unstable, since perturbations on a moving fluid in global equilibrium grow exponentially, within microscopic time scales of the order of $\tau_\eta \sim \tau_{\text{eff}} \sim \eta/(\varepsilon_0 + P_0)$. This analysis was performed fixing the direction of the background velocity to be in the x-direction and only considering perturbations traveling in the same direction, i.e.,

$$\begin{aligned} u_0^\mu &= \gamma\left(1, V, 0, 0\right)^T \ , \\ k^\mu &= \left(\omega, k, 0, 0\right)^T \ . \end{aligned} \tag{2.84}$$

In this scenario, the covariant frequency, Ω, and wavenumber, κ, of the oscillations satisfy

$$\Omega = \gamma\left(\omega - Vk\right) \ , \tag{2.85}$$

$$\kappa^2 = \gamma^2\left(\omega V - k\right)^2 \ , \tag{2.86}$$

which is equivalent to a 1+1-dimensional Lorentz boost with velocity V of a 4-vector made of ω and k. Therefore, for all practical purposes, the dispersion relation for modes obtained for perturbations of a moving fluid can be obtained by applying a Lorentz transformation to the same dispersion relation obtained for perturbations on a fluid at rest. This can help us understand the connection between the acausal nature of the theory and its perturbative instability when considering moving background fluids. We shall make this argument also noting a connection between the modes of Navier-Stokes theory, at asymptotically large values of wavenumber, and the dispersion relation of the diffusion equation.

We already demonstrated in the previous subsections that, for perturbations of a static fluid, both the transverse and longitudinal modes of Navier-Stokes theory behave in the same way when $k \to \infty$, $\omega\left(k\right) \sim i\tau_{\text{eff}}k^2/2 \sim i\tau_\eta k^2$, being purely non-propagating modes that are quadratic in k. This behavior is identical to the one found in the diffusion equation (1.113), in which case there is a single non-propagating mode with dispersion relation $\omega(k) = iDk^2$, with D being the diffusion coefficient. It is well known that the diffusion equation is acausal and, consequently, we conjecture that a k^2-dependence of any non-propagating mode can also be considered a sign of acausality.

Now we show that the parabolic nature of diffusion-like dispersion relations (given by their quadratic dependence on k) will lead to additional modes once a Lorentz transformation is performed. Let us consider the Lorentz boost of a 4-vector with time-component ω and spatial component k in boost direction,

$$\begin{pmatrix} \omega \\ k \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma V \\ -\gamma V & \gamma \end{pmatrix} \begin{pmatrix} \omega' \\ k' \end{pmatrix} = \begin{pmatrix} \gamma\omega' - \gamma V k' \\ -\gamma V \omega' + \gamma k' \end{pmatrix} . \tag{2.87}$$

Then, we substitute the boosted variables into the diffusion dispersion relation, which changes in the following way

$$\omega = iDk^2 \quad \longrightarrow \quad \omega' - Vk' = i\gamma D \left(k' - V\omega'\right)^2 . \tag{2.88}$$

It is straightforward to see that, in the limit of vanishing k', there is always a **non-hydrodynamic mode** given by

$$\omega'(k'=0) = -\frac{i}{\gamma D V^2} , \tag{2.89}$$

which has a **negative imaginary part**, i.e., it is **exponentially growing and thus unstable**. Note that all unstable modes obtained so far in Navier-Stokes theory have the structure above, i.e., at $k = 0$ they always look like boosted modes of the diffusion equation. In this sense, it is not too surprising that a theory with parabolic, diffusion-type modes in a non-moving background will end up featuring unstable modes in a moving background. Naturally, **non-relativistic theories** will **never** display this property since the modes transform following **Galileo's transformation**, which **never mixes frequency and wavenumber**.

It is important to note how the structure of Lorentz boosts ends up connecting the **infinite-wavenumber behavior** of a mode in a **static background** (usually considered to be irrelevant for macroscopic dynamics) with the **vanishing-wavenumber** behavior of the same mode in a **moving background**. This is partially why the **acausality** of a mode, which is an **asymptotic feature**, can affect its **stability,** which is a feature influencing the **small-wavenumber behavior** of the fluctuation. This is certainly an interesting aspect of relativistic theories.

Finally, we shall see in the next section that a transient theory of fluid dynamics does not display any non-propagating modes that match those of the diffusion equation. Also, we shall see that intrinsically unstable modes do not exist – the modes can be rendered stable as long as the transport coefficients satisfy a set of conditions that will be derived below.

2.5 Transient theory of fluid dynamics

We now analyze the modes of the complete theory, which includes the effect of the relaxation times. For the sake of simplicity, we shall perform this analysis neglecting contributions from net-charge fluctuations and in the conformal limit (vanishing bulk-viscous pressure, $\delta\tilde{\Pi} = 0$, and vanishing bulk viscosity, $\zeta = 0$). In this case, Eqs. (2.37), (2.38), and (2.42) for the longitudinal fluctuations simplify to

$$\Omega\delta\tilde{\varepsilon} - \kappa\left(\varepsilon_0 + P_0\right)\delta\tilde{u}_\parallel = 0\ , \tag{2.90}$$

$$\left(\varepsilon_0 + P_0\right)\Omega\delta\tilde{u}_\parallel - \kappa\left(c_s^2\delta\tilde{\varepsilon} + \delta\tilde{\pi}_\parallel\right) = 0\ , \tag{2.91}$$

$$\left(1 + i\tau_\pi\Omega\right)\delta\tilde{\pi}_\parallel - \frac{4}{3}i\eta\kappa\delta\tilde{u}_\parallel = 0\ . \tag{2.92}$$

The remaining equations for the shear modes do not change under these assumptions and are still given by Eqs. (2.43), (2.44), and (2.46). As already discussed above, the solution of Eq. (2.46) is already determined by Eq. (2.92) and does not need to be addressed further.

2.5.1 Transverse modes in the rest frame

The equations of motion (2.43), (2.44) for the transverse (i.e., shear) modes can be cast into the following matrix form

$$\begin{pmatrix} \Omega & -\kappa \\ -i\tau_\eta\kappa & 1 + i\tau_\pi\Omega \end{pmatrix} \begin{pmatrix} \delta\tilde{u}_\perp^\mu \\ \dfrac{\delta\tilde{\pi}_\perp^\mu}{\varepsilon_0 + P_0} \end{pmatrix} = 0\ , \tag{2.93}$$

leading to the following the dispersion relation,

$$\Omega\left(1 + i\tau_\pi\Omega\right) - i\tau_\eta\kappa^2 = 0\ . \tag{2.94}$$

Naturally, this equation will have two solutions in both the moving background and the one at rest, since its highest power in Ω is the same as its highest power in κ, i.e., κ^2.

When the background fluid is at rest, i.e., when we set $u_0^\mu = (1, 0, 0, 0)^T$, the dispersion relation becomes

$$\omega\left(1 + i\tau_\pi\omega\right) - i\tau_\eta k^2 = 0\ , \tag{2.95}$$

with two distinct solutions

$$\omega(k) = \frac{i}{2\tau_\pi} \pm k\sqrt{\frac{\tau_\eta}{\tau_\pi}}\sqrt{1 - \left(\frac{\kappa_c^{\text{shear}}}{k}\right)^2}\ , \tag{2.96}$$

where similarly to Navier-Stokes theory we defined a critical wavenumber

$$\kappa_c^{\text{shear}} \equiv \frac{1}{\sqrt{4\tau_\eta\tau_\pi}}\ . \tag{2.97}$$

We see that, due to the nonzero relaxation time τ_π, the transverse shear modes become propagating for $k > \kappa_c^{\text{shear}}$, with propagation speed $\sqrt{\tau_\eta/\tau_\pi}\sqrt{1 - (\kappa_c^{\text{shear}}/k)^2}$. They also have a nonvanishing imaginary part, describing damping on a time scale $\sim 2\tau_\pi$.

At small wavenumbers, $k \to 0$, the shear modes become

$$\omega\left(k \to 0\right) = \begin{cases} i\tau_\eta k^2 + \mathcal{O}(k^4)\ , \\ \dfrac{i}{\tau_\pi} + \mathcal{O}(k^2)\ . \end{cases} \tag{2.98}$$

We see that, for transient theories of fluid dynamics, we already have a **non-hydrodynamic mode**, even for perturbations on a background that is **at rest**. But such a mode is **stable** and **decays exponentially to zero within a time scale** τ_π – the new transport coefficient introduced when constructing transient theories of fluid dynamics. The other mode is hydrodynamical and, at small wavenumbers, is identical to the one obtained from relativistic Navier-Stokes theory. Therefore, at small wavenumbers, the transient theory is very similar to Navier-Stokes theory, the only difference being the appearance of a **damped** – and therefore **stable** – non-hydrodynamic mode $\sim i/\tau_\pi$.

At large values of the wavenumber, $k \to \infty$, the solution becomes

$$\omega\,(k \to \infty) = \pm\sqrt{\frac{\tau_\eta}{\tau_\pi}}k + \mathcal{O}(1)\ . \tag{2.99}$$

We see that this mode is **linear** in the wavenumber and, contrary to Navier-Stokes theory, the order of the dispersion relation in powers of frequency will **not be modified by a Lorentz boost**. As discussed, this asymptotic behavior of the mode may provide a **causal and stable mode**.

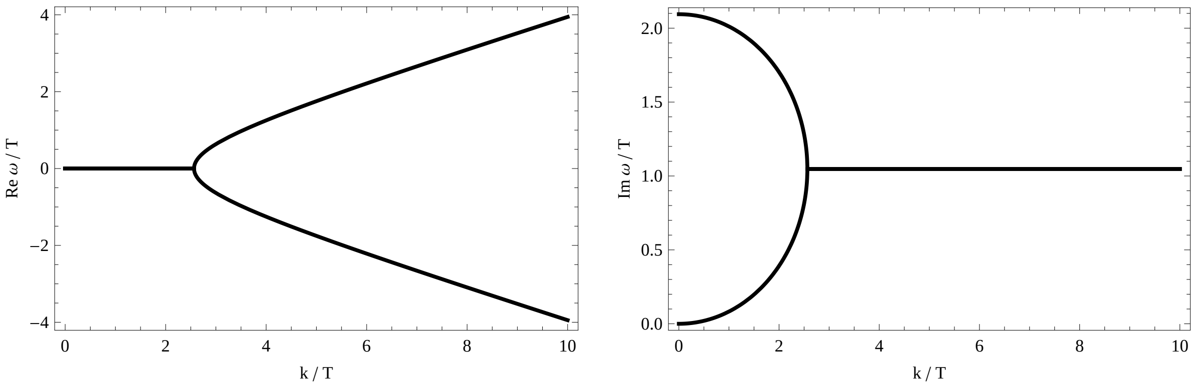

Figure 2.2: The real parts (left panel) and the imaginary parts (right panel) of the dispersion relations for the transverse modes in a static background. We fixed the relaxation time as $\tau_\pi = 6\tau_\eta$ and $\eta/s_0 = 1/(4\pi)$. Figure taken from Ref. [5].

In Fig. 2.2 we show the real (left panel) and imaginary (right panel) parts of the transverse modes in units of temperature. In this plot, we parametrize the relaxation time as $\tau_\pi = 6\tau_\eta$ [7] and fix the shear viscosity to be $\eta/s_0 = 1/(4\pi)$, a value usually associated to strongly coupled conformal fluids [25].

The dispersion relations for the shear modes resulting from Eq. (2.96) change their behavior from non-propagating to propagating at the critical wavenumber (2.97), as shown in Fig. 2.2. In this sense, it may be instructive to address the issue of causality by looking at the **group velocity** of such propagating modes. For wavenumbers larger than κ_c^{shear}, the (modulus of the) group velocity of the propagating shear mode is

$$v_g(k) = \frac{\partial\,\mathrm{Re}\,\omega(k)}{\partial k} = v_{g,\text{shear}}^{\text{as}}\,\frac{k/\kappa_c^{\text{shear}}}{\sqrt{(k/\kappa_c^{\text{shear}})^2 - 1}}\ , \tag{2.100}$$

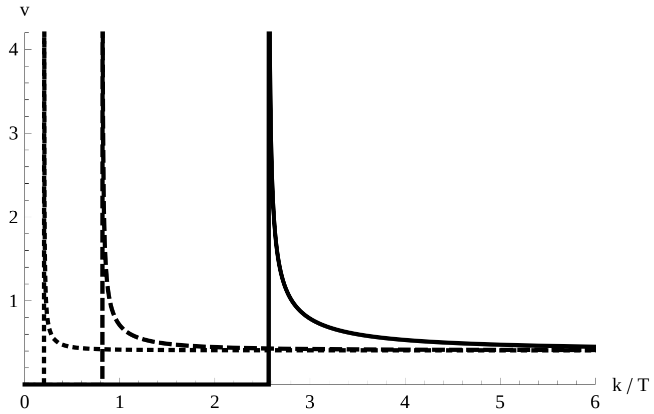

Figure 2.3: The group velocity for the transverse mode for $\tau_\pi = 6\tau_\eta$, $c_s^2 = 1/3$, and $\eta/s_0 = 1/(4\pi)$ (full line), 1/4 (dashed line), and 1 (dotted line). Figure taken from Ref. [5].

where we defined the **asymptotic group velocity** of this mode,

$$v_{g,\text{shear}}^{\text{as}} \equiv \sqrt{\frac{\tau_\eta}{\tau_\pi}} , \tag{2.101}$$

as the asymptotic value of v_g in the large-wavenumber limit. The group velocity as a function of the wavenumber is shown in Fig. 2.3.

We can see from Eq. (2.100) (and also from Fig. 2.3) that the group velocity diverges near the critical wavenumber κ_c and approaches its asymptotic value ($k \to \infty$) from above. For certain choices of shear relaxation time and viscosity, it may thus happen that the group velocity becomes **superluminal**. In Sec. 2.5.4 we shall show that this apparent violation of causality of the group velocity does **not** cause the theory **as a whole to become acausal**. The necessary condition required in order to insure that the theory is causal is actually that the **asymptotic group velocity** (2.101) **does not exceed the speed of light**,

$$\lim_{k\to\infty} \frac{\partial \text{Re}\ \omega(k)}{\partial k} = v_{g,\text{shear}}^{\text{as}} \leq 1 . \tag{2.102}$$

This condition, which we shall refer to as **asymptotic causality condition**, is not automatically satisfied by the linearized Israel-Stewart theory. It actually must be **imposed additionally**, leading to the following constraint that must be satisfied by the shear relaxation time (if the effects of bulk-viscous pressure and net-charge diffusion were included, also additional constraints for the relaxation times related to these quantities would have been obtained [4, 19]),

$$\frac{\tau_\eta}{\tau_\pi} \leq 1 \quad \Longleftrightarrow \quad \tau_\pi \geq \frac{\eta}{\varepsilon_0 + P_0} . \tag{2.103}$$

That is, the relaxation time must be larger than the time scale τ_η.

When analyzing the longitudinal (sound) modes, we shall find that the causality condition (2.103) is actually not sufficient to guarantee the stability and causality of the sound modes. Since all modes must be causal and stable, the validity of Eq. (2.103) does not guarantee the causality of the linearized theory as a whole.

2.5.2 Longitudinal modes in the rest frame

The equations of motion (2.90) – (2.92) for the longitudinal modes can be cast into the following matrix form

$$\begin{pmatrix} \Omega & -\kappa & 0 \\ -c_s^2\kappa & \Omega & -\kappa \\ 0 & -i\tau_{\rm eff}\kappa & 1+i\tau_\pi\Omega \end{pmatrix} \begin{pmatrix} \dfrac{\delta\tilde{\varepsilon}}{\varepsilon_0+P_0} \\ \delta\tilde{u}_\parallel \\ \dfrac{\delta\tilde{\pi}_\parallel}{\varepsilon_0+P_0} \end{pmatrix} = 0 \,, \tag{2.104}$$

with the dispersion relations being given by

$$\left(\Omega^2 - c_s^2\kappa^2\right)(1+i\tau_\pi\Omega) - i\tau_{\rm eff}\Omega\kappa^2 = 0 \,. \tag{2.105}$$

Naturally, this equation will have three solutions in both the moving background and the static one, since its highest power in frequency, Ω^3, is the same as its highest power in both κ and Ω, which is $\Omega\kappa^2$.

Exercise 2.8: Prove Eq. (2.105) by computing the determinant of the coefficient matrix in Eq. (2.104).

When the unperturbed fluid is at rest, the dispersion relation becomes

$$\left(\omega^2 - c_s^2 k^2\right)(1+i\tau_\pi\omega) - i\tau_{\rm eff}\omega k^2 = 0 \,. \tag{2.106}$$

The longitudinal modes are the solutions of a cubic equation, which cannot be expressed in a simple analytical form. In the following, we initially restrict our discussion to the limiting form of these modes for $k \to 0$ and $k \to \infty$, respectively. The analytic solution in the limit of small wavenumber k is

$$\omega(k) = \begin{cases} \dfrac{i}{\tau_\pi} \,, \\ \pm c_s\, k + i\,\dfrac{\tau_{\rm eff}}{2}\,k^2 \,. \end{cases} \tag{2.107}$$

The first solution is obtained by setting $k \equiv 0$ in Eq. (2.106), while the second is obtained by assuming that $\omega \sim k \ll \tau_\pi^{-1}$, thus mapping Eq. (2.106) onto Eq. (2.78).

On the other hand, for large wavenumber one obtains from Eq. (2.106)

$$\omega(k) = \begin{cases} \dfrac{i}{\tau_\pi}\left[1+\dfrac{\tau_{\rm eff}}{\tau_\pi c_s^2}\right]^{-1} \,, \\ \pm c_s\, k\sqrt{1+\dfrac{\tau_{\rm eff}}{\tau_\pi c_s^2}} + \dfrac{i}{2\tau_\pi}\left[1+\dfrac{\tau_\pi c_s^2}{\tau_{\rm eff}}\right]^{-1} \,. \end{cases} \tag{2.108}$$

This corresponds to one non-propagating mode and two propagating (sound) modes. As observed for the transverse modes, at small wavenumber the longitudinal propagating

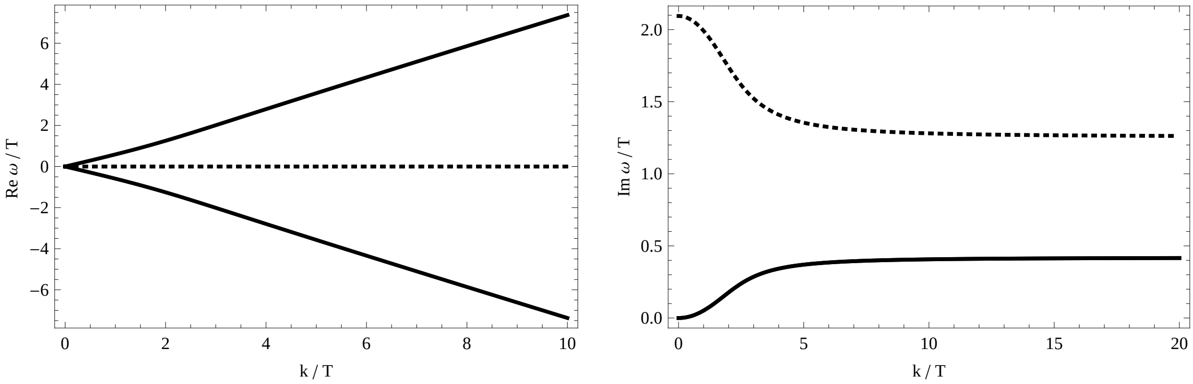

Figure 2.4: The real parts (left panel) and the imaginary parts (right panel) of the dispersion relations for the sound modes (full lines) and the non-propagating mode (dashed line). The parameters are $\eta/s_0 = 1/(4\pi)$, $\tau_\pi = 6\tau_\eta$, $c_s^2 = 1/3$. Figure taken from Ref. [5].

modes are exactly the same as the ones obtained from Navier-Stokes theory. Also, we note that all imaginary parts are positive and therefore all longitudinal modes are stable (as it was the case for the transverse modes).

Exercise 2.9: Prove Eq. (2.108).

(Hint: For the first solution, assume $\omega^2 \ll c_s^2 k^2$ in the first term of Eq. (2.106). The second solution can be obtained via perturbation theory. To this end, first consider Eq. (2.106) to leading order in ω and k. Show that the solution of the leading-order equation gives the first term in Eq. (2.108). Then consider corrections to this solution. Linearize Eq. (2.106) in these corrections and determine them in the limit of large wavenumber. This gives the second term in Eq. (2.108).)

The non-propagating mode does not have a real part, $\mathrm{Re}\,\omega = 0$, and, hence, we cannot discuss the causality of this mode using the group velocity. This is similar to what happened in Navier-Stokes theory, where all modes became purely imaginary above a critical wavenumber. For this reason, we analyze this mode by comparing it to the mode of the diffusion equation (1.113) – the same procedure adopted with Navier-Stokes theory. As already stated, the diffusion equation is known to be acausal and has a single non-propagating mode with dispersion relation $\omega = iDk^2$. We have already demonstrated that such a k^2–dependence in any non-propagating mode (even if in the limit of infinite wavenumber) will lead to unstable non-hydrodynamic modes once boosted to a moving frame. However, the non-propagating mode (2.108) is **independent** of k in the asymptotic limit (cf. Fig. 2.4) and should not lead to acausal signal propagation. Furthermore, this lack of dependence on the wavenumber leads to a dispersion relation that is well behaved under Lorentz transformations, i.e., no additional mode will appear when k^μ is boosted.

The dispersion relations resulting from Eq. (2.106) are shown in Fig. 2.4, and the

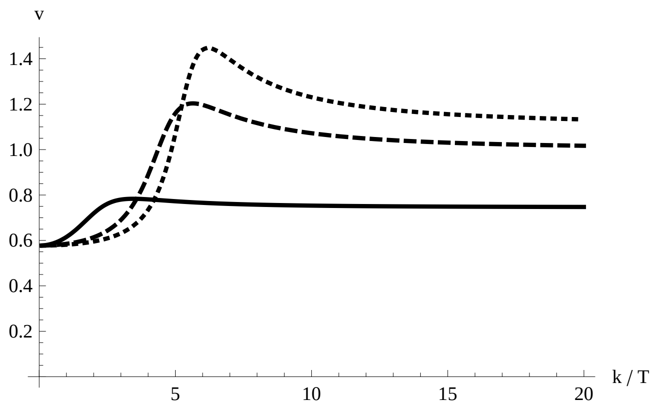

Figure 2.5: The group velocity of longitudinal modes for $\eta/s_0 = 1/(4\pi)$, $c_s^2 = 1/3$, and $\tau_\pi = 6\tau_\eta$ (full line), $\tau_\pi = 2\tau_\eta$ (dashed line), as well as $\tau_\pi = 3\tau_\eta/2$ (dotted line). Figure taken from Ref. [5].

corresponding group velocity in Fig. 2.5. The group velocity has a maximum for a finite value of k/T and approaches its asymptotic value ($k \to \infty$) from above. But it does not diverge for any finite value of k, as happened with the group velocity for the shear modes. We see that for small values of the ratio τ_π/τ_η the group velocity can become superluminal in some wavenumber domains. Nevertheless, as will be discussed in Sec. 2.5.4, this is not necessarily a problem since only the asymptotic value of the group velocity determines whether the mode as a whole violates causality or not. However, if $\tau_\pi/\tau_{\rm eff}$ is sufficiently small, even the asymptotic group velocity can become larger than the velocity of light. As a matter of fact, the **asymptotic value of the group velocity for the longitudinal modes** is

$$v_{g,\rm sound}^{\rm as} = \lim_{k\to\infty} \frac{\partial {\rm Re}\,\omega}{\partial k} = c_s \sqrt{1 + \frac{\tau_{\rm eff}}{\tau_\pi c_s^2}} \,. \tag{2.109}$$

Consequently, for the asymptotic group velocity of sound waves to be less than the speed of light, τ_π and $\tau_{\rm eff}$ should satisfy the following **asymptotic causality condition** [14]:

$$\frac{\tau_{\rm eff}}{\tau_\pi} \leq 1 - c_s^2 \,. \tag{2.110}$$

This is similar to the causality condition for the group velocity in the case of bulk viscosity, found originally in Ref. [4]. For conformal fluids, where $c_s^2 = 1/3$, the condition (2.110) simplifies to

$$\frac{\tau_{\rm eff}}{\tau_\pi} \leq \frac{2}{3} \,. \tag{2.111}$$

Since for conformal fluids $\zeta = 0$, we also have $\tau_{\rm eff} = 4\tau_\eta/3$, and thus

$$\frac{\tau_\eta}{\tau_\pi} \leq \frac{1}{2} \,, \tag{2.112}$$

which is more restrictive than the asymptotic causality condition (2.103) for the shear modes.

2.5.3 Stability for a moving background

So far, we have studied the dispersion relations of transient fluid dynamics for perturbations on a fluid at rest. In this scheme, it was demonstrated that such theories are **stable**, and also **causal**, if the relaxation times satisfy a certain **asymptotic causality condition**. Now we must check if this situation changes when we consider perturbations in a moving (or Lorentz-boosted) background – in Navier-Stokes theory, this completely changed the behavior with the appearance of unstable non-hydrodynamic modes.

We consider the same moving background used when analyzing the longitudinal and transverse modes of Navier-Stokes theory, which we parametrized as $u_0^\mu = \gamma\,(1, V, 0, 0)^T$ and $k^\mu = (\omega, k, 0, 0)^T$. This corresponds to fluctuations traveling parallel to the background velocity. The dispersion relation for the transverse modes, Eq. (2.94), transforms to

$$(\omega - Vk)\left[1 + i\gamma\tau_\pi\,(\omega - Vk)\right] - i\tau_\eta\gamma\,(\omega V - k)^2 = 0\,. \tag{2.113}$$

where we used Eqs. (2.85) and (2.86). This is a quadratic equation and, consequently, it has two solutions, each with a twofold degeneracy. These are

$$\omega_\pm(k) = \frac{1}{2(\tau_\pi - V^2\tau_\eta)\gamma}\left[i - 2\gamma(\tau_\eta - \tau_\pi)kV \pm \sqrt{-1 - 4i\gamma^{-1}\tau_\eta Vk + 4\gamma^{-2}\tau_\eta\tau_\pi k^2}\right]\,. \tag{2.114}$$

Exercise 2.10: Prove that Eq. (2.114) is the solution of Eq. (2.113).

At $k = 0$, it is straightforward to see that the two solutions simplify to

$$\omega\,(k = 0) = \begin{cases} 0\,, \\ \dfrac{i}{\gamma(\tau_\pi - V^2\tau_\eta)}\,, \end{cases} \tag{2.115}$$

which agree with the solution (2.98) in a non-moving background ($V = 0$) and with the results obtained from Navier-Stokes theory in a moving background, Eq. (2.74), when τ_π is set to zero. The main and fundamental difference is that the non-propagating mode that appears in Navier-Stokes theory, when perturbed around a moving fluid, always has a negative imaginary part. As can be seen from the expression above, it is the introduction of a shear relaxation time that makes it possible to change the sign of the imaginary part of such a non-propagating mode. As a matter of fact, we observe that, as long as $\tau_\pi \geq \tau_\eta$, this mode is always stable: a condition that coincides with the asymptotic causality condition (2.103). So we see once more, in a more practical setting, the connection between causality and stability.

Similarly, the dispersion relation for the longitudinal modes, Eq. (2.105), transforms to

$$\left[(\omega - Vk)^2 - c_s^2\,(\omega V - k)^2\right]\left[1 + i\gamma\tau_\pi\,(\omega - Vk)\right] - i\gamma\tau_{\text{eff}}\,(\omega - Vk)\,(\omega V - k)^2 = 0\,. \tag{2.116}$$

This equation is cubic and has three solutions, which are too long to write down in this context. Once more, we restrict ourselves to study the solution at $k = 0$, where the instability of the non-propagating modes was already obvious in Navier-Stokes theory (for a moving background). In this case,

$$\omega\,(k=0) = \begin{cases} \pm 0\,, \\ \dfrac{i\,(1-c_s^2V^2)}{\gamma\,[\tau_\pi\,(1-c_s^2V^2)-\tau_{\text{eff}}V^2]}\,. \end{cases} \tag{2.117}$$

Exercise 2.11: Prove that Eq. (2.117) is a solution for Eq. (2.116) for $k = 0$.

This is very similar to what we found for the shear modes. We have one non-vanishing solution, corresponding to the mode that was unstable in Navier-Stokes theory, see Eq. (2.83), which agrees with the above when setting $\tau_\pi =$. We see once more that a **non-vanishing shear relaxation time** τ_π can render this mode **stable**. For any value of $0 \leq V < 1$, if $\tau_{\text{eff}} \leq \tau_\pi(1-c_s^2)$, the imaginary part of this mode will always be positive and, thus, it will be linearly stable. This condition corresponds to the asymptotic causality condition (2.110) derived for the longitudinal modes for a background fluid at rest, confirming that, **once causality is assured**, the boosted fluctuations will be **stable**.

In Fig. 2.6, the dependence of the group velocity on the wavenumber is shown for various values of the boost velocity V. The left panel shows the behavior of one of the shear modes and the right panel one of the sound modes. The parameter set used here is $\eta/s_0 = 1/\,(4\pi)\,,\ \tau_\pi = 6\tau_\eta,\ c_s^2 = 1/3$, which satisfies the asymptotic causality condition. We observe that the divergence of the group velocity of the shear mode in the rest frame is tampered by the movement of the unperturbed fluid, resulting in a peak of finite height. However, the group velocity may still exceed the speed of light in a certain range of wavenumbers. Nevertheless, we note that as we increase the velocity of the unperturbed fluid, the peak height gradually diminishes, until the group velocity remains below the speed of light for all wavenumbers. However, the same does not occur with the longitudinal group velocity, which we found to be superluminal in some ranges of wavenumber even when the velocity of the unperturbed fluid becomes close to the speed of light.

Although the group velocity of the shear or the sound mode may exceed the speed of light, the theory is still stable as long as the asymptotic causality condition is fulfilled. This is demonstrated in the left panel of Fig. 2.7, where the imaginary parts of the modes are shown for the parameter set $\eta/s_0 = 1/\,(4\pi)\,,\ \tau_\pi = 6\tau_\eta,\ c_s^2 = 1/3$. We observe that all imaginary parts are positive, indicating the stability of the theory.

In contrast to the case where the background fluid is at rest, where the theory was found to be stable even for parameters which violate the asymptotic causality condition (2.110), this is no longer the case for perturbations performed on a moving fluid. In the right panel of Fig. 2.7, the imaginary parts of the modes are calculated with the parameter set $\eta/s_0 = 1/\,(4\pi)$, $c_s^2 = 1/3$ and $\tau_\pi = \tau_\eta$ – the latter violating the causality and stability conditions derived in this section. Now one observes the appearance of negative imaginary parts, indicating that the theory becomes unstable.

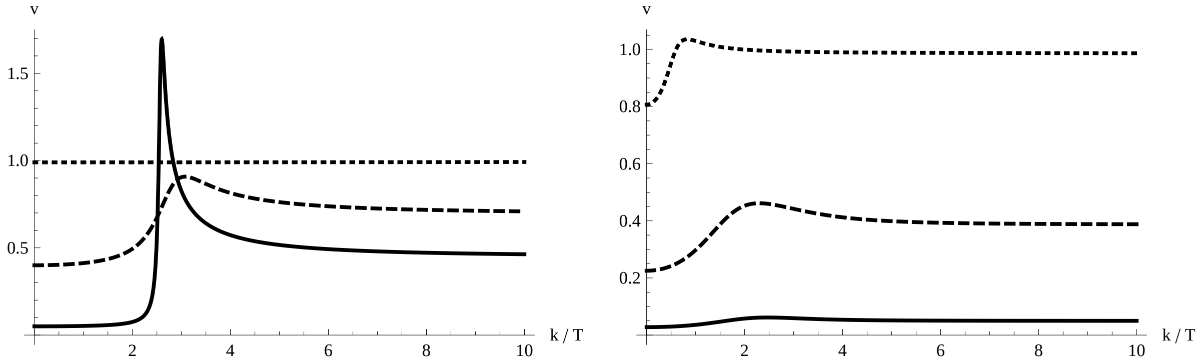

Figure 2.6: The group velocity calculated for one of the shear modes (left panel) and one of the sound modes (right panel). We set $\eta/s_0 = 1/(4\pi), \tau_\pi = 6\tau_\eta, c_s^2 = 1/3$. The solid line is for a boost velocity $V = 0.05$, the dashed line for $V = 0.4$ and the dotted line for $V = 0.99$, respectively. Figure taken from Ref. [5].

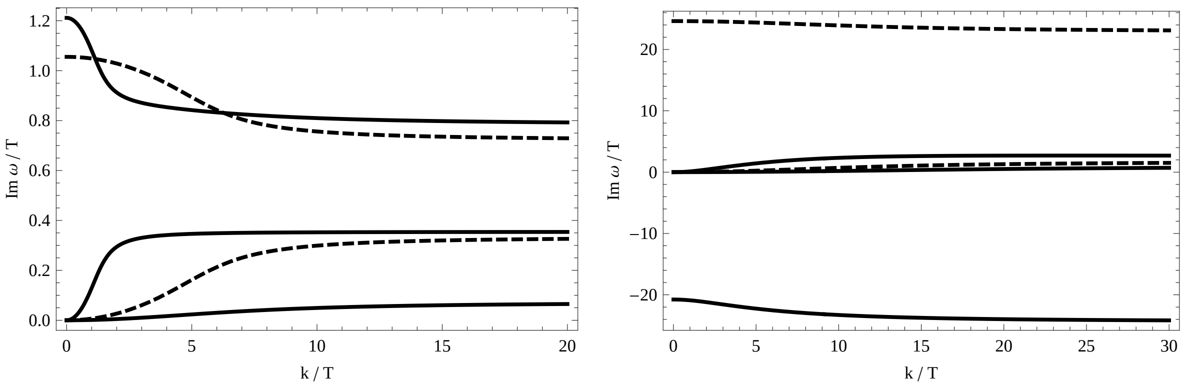

Figure 2.7: The imaginary parts of the dispersion relations for a boost in x-direction with velocity $V = 0.9$. The left panel shows the results for the parameter set $\eta/s_0 = 1/(4\pi)$, $\tau_\pi = 6\tau_\eta$, $c_s^2 = \frac{1}{3}$, which fulfills the asymptotic causality condition, while the right panel is for $\eta/s_0 = 1/(4\pi)$, $\tau_\pi = \tau_\eta$, $c_s^2 = \frac{1}{3}$, which violates this condition. The dashed lines are for the shear modes, while the solid lines are for the sound modes. Figure taken from Ref. [5].

2.5.4 Causality of wave propagation

In the preceding discussion we have seen that transient theories of fluid dynamics can be designed to be stable as long as the transverse and longitudinal modes fulfill asymptotic causality conditions. In this section, we shall show that the **causality** of the theory **as a whole** is guaranteed if the asymptotic stability condition is fulfilled. The group velocity may become superluminal, or even diverge, as long as this apparent violation of causality is restricted to a finite range of wavenumbers. The argument leading to this conclusion is analogous to that of **Sommerfeld and Brillouin in classical electrodynamics** [26, 27]. For instance, in the case of anomalous dispersion the group velocity may become superluminal, but the causality of the theory as a whole is not affected.

The change in a fluid-dynamical variable induced by a general perturbation is given by

$$\delta X(x,t) = \sum_j \int d\omega \, \widetilde{\delta X}_j(\omega) \, e^{i\omega t - ik_j(\omega)x} \, , \tag{2.118}$$

where $\delta X(x,t)$ stands for $\delta\varepsilon$, δu^μ, or $\delta\pi^{\mu\nu}$. The index j denotes the different modes, i.e., the shear modes, the sound modes etc. The function $k_j(\omega)$ is the inverted dispersion relation $\omega_j(k)$ of the respective mode. The Fourier components are given by

$$\sum_j \widetilde{\delta X}_j(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \delta X(0,t) \, e^{-i\omega t} \, . \tag{2.119}$$

We assume that the incident wave has a well-defined front that reaches $x = 0$ not before $t = 0$. Thus $\delta X(0,t) = 0$ for $t < 0$. This condition on $\delta X(0,t)$ ensures that $\sum_j \widetilde{\delta X}_j(\omega)$ is **analytic in the lower half of the complex ω–plane** [26]. On the other hand, in Secs. 2.5.1, 2.5.3 we have found that the group velocity of the shear modes **diverges** for certain values of k. These divergences correspond to **singularities in the complex ω–plane**. However, if the asymptotic causality condition is **fulfilled**, the imaginary part of the dispersion relation is always **positive**, i.e., the singularities only appear in the **upper half of the complex ω–plane**. In this case, the system is also **stable**. On the other hand, if the asymptotic causality condition is **violated**, the singularities may appear also in the **lower half-plane**, i.e., for **negative imaginary part** of the dispersion relation, and the system is **unstable**.

We shall now demonstrate that the divergences in the group velocity do not violate causality as long as the asymptotic causality condition is satisfied, i.e., as long as the asymptotic group velocity remains subluminal. To this end, we compute Eq. (2.118) by contour integration in the complex ω–plane. To close the contour, we have to know the asymptotic behavior of the dispersion relations. In our calculation, we found that the real part of the dispersion relation at large k is proportional to k (see Eqs. (2.99) and (2.108)), with a coefficient which is the large-k limit of the group velocity, i.e., v_{gj}^{as},

$$\lim_{k\to\infty} \mathrm{Re}\, \omega_j(k) = v_{gj}^{\mathrm{as}} \, k \, . \tag{2.120}$$

Then, in the large-k limit, the exponential becomes

$$\exp[i\omega t - ik_j(\omega)x] \to \exp\left[-i \, \frac{\omega}{v_{gj}^{\mathrm{as}}} \, (x - v_{gj}^{\mathrm{as}} \, t)\right] \, . \tag{2.121}$$

In the case $x > v_{gj}^{\mathrm{as}}\, t$, we have to close the integral contour in the **lower half-plane**. If the asymptotic causality condition is **fulfilled**, there are **no singularities in the lower half-plane**, and Eq. (2.118) vanishes. On the other hand, the contour should be closed in the **upper half-plane** if $x \leq v_{gj}^{\mathrm{as}}\, t$. Then, because of the singularities, Eq. (2.118) may have a nonzero value. However, as long as we choose a parameter set for which the asymptotic group velocity v_{gj}^{as} is smaller than the speed of light, i.e., for which the asymptotic causality condition is fulfilled, **the signal propagation does not violate causality**, since the locations x where the disturbance has travelled lie within the cone given by v_{gj}^{as}, which, in turn, lies within the lightcone, q.e.d.

To conclude this section, we have shown that the **asymptotic causality condition** not only implies **stability in a general (Lorentz-boosted) frame**, but also **causality** of the theory **as a whole**.

2.6 Summary

In this chapter, we discussed the properties of linearized fluid-dynamical equations around global equilibrium. The main goal was to determine whether hydrodynamic fluctuations around global equilibrium are stable and causal – two fundamental properties that any physical description should satisfy. We investigated these issues for ideal fluids and the two formulations of relativistic dissipative fluid dynamics considered so far in this book: Navier-Stokes theory and transient fluid dynamics (usually represented in the form of Israel-Stewart theory).

First, we studied the linear properties of ideal fluids and recovered all its basic and well-known features. In this case, transverse modes do not appear and the longitudinal modes are purely propagating, describing the propagation of sound waves. The velocity of sound of relativistic fluids was then obtained, and given by the (square root of the) derivative of the pressure with respect to the energy density, at fixed entropy per particle (or per net-charge). All modes in ideal fluid dynamics are stable and the theory is causal, as long as the velocity of sound $c_s \leq 1$ – a property that is satisfied by all known microscopic theories.

Next, we discussed the linear properties of relativistic Navier-Stokes theory. When investigating perturbations performed on a fluid that is initially at rest, the linear properties of relativistic Navier-Stokes theory essentially reduce to those of its non-relativistic version. In this case, transverse modes do appear and describe the diffusion of the velocity field due to viscosity. The longitudinal modes still display the propagation of sound waves, which now exhibit diffusion-type damping due to the shear and bulk viscosity of the fluid. Nevertheless, novel striking features of a relativistic version of Navier-Stokes theory appear when considering perturbations on a moving fluid. In this case, we demonstrated that unstable non-hydrodynamic modes, which simply do not exist in the non-relativistic Navier-Stokes theory, appear in both the transverse and longitudinal degrees of freedom. The existence of such unstable modes renders the general global-equilibrium solution of relativistic Navier-Stokes theory linearly unstable, posing a fundamental problem to the application of this theory to describe any relativistic fluid existing in Nature. It is this fundamental problem that the transient theories of fluid dynamics, constructed in the previous chapter, aim to correct.

We then investigated the transverse and longitudinal modes of (second-order) transient fluid dynamics and demonstrated that such theories can be constructed to be causal and stable, at least when perturbed around global equilibrium. We argued (and later proceeded to demonstrate) that as long as the **asymptotic value of the group velocity remains subluminal**, $v_g^{\rm as} \leq 1$, the theory is causal. We have shown that the condition $v_g^{\rm as} \leq 1$is equivalent to the requirement that the relaxation time scale τ_π must not be smaller than the ratio $\tau_{\rm eff}/(1-c_s^2)$, where $\tau_{\rm eff} \sim \eta/(\varepsilon_0 + P_0) \sim \beta_0 \eta/s_0$ Thus, second-order transient theories of fluid dynamics are not *per se* stable and causal; they **may become unstable and acausal if this condition is violated**. This is an important conclusion for practitioners of fluid dynamics, who frequently consider τ_π and the shear viscosity-to-entropy density ratio η/s_0 to be independent from each other. We have demonstrated that this is not the case if one wants the theory to remain causal. These findings also illuminate, from a different perspective, why Navier-Stokes theory violates causality, because there $\tau_\pi \to 0$, while η remains nonzero. Therefore, transient theories of fluid dynamics, such as Israel-Stewart theory, can actually be successfully applied to describe the dynamics of relativistic fluids.

We finally recounted a time-honored argument from electrodynamics, proving that causality of a theory is guaranteed if the large-wavenumber limit of the group velocity remains subluminal. Thus, a divergence of the group velocity at some finite wavenumber, which actually occurs for some modes of transient fluid dynamics, does not necessarily imply that the theory as a whole becomes acausal.

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3 Fluid Dynamics from Kinetic Theory: Traditional Approaches

05/17/2022

In Chapter 1 we have investigated how the equations of relativistic dissipative fluid dynamics can be derived phenomenologically, and in Chapter 2 we have discussed their basic linear properties around equilibrium. In the remainder of this lecture series, the derivation of relativistic dissipative fluid dynamics from a microscopic theory, in this case the **relativistic Boltzmann equation**, will be carried out in great detail. In this chapter, we start addressing this topic by discussing the two most widespread methods usually employed to derive relativistic fluid dynamics from the Boltzmann equation: the **Chapman-Enskog expansion** [1] and the **method of moments as proposed by Israel and Stewart** [2, 3, 4].

This chapter is organized as follows: In Sec. 3.1 we briefly introduce the Boltzmann equation and discuss how the fluid-dynamical degrees of freedom are defined in this setting. In Sec. 3.2 we introduce Chapman-Enskog theory and derive the fluid-dynamical equations following this procedure. In Sec. 3.3 we discuss Israel's and Stewart's original derivation of transient fluid dynamics and the differences to Chapman and Enskog's approach.

3.1 Matching fluid-dynamical with kinetic degrees of freedom

We start with the **relativistic Boltzmann equation** for the **single-particle distribution function** $f_{\mathbf{k}}$,

$$k^\mu \partial_\mu f_{\mathbf{k}} = C[f] \ . \tag{3.1}$$

Here, the 4-momentum of a particle is $k^\mu = (k^0, \mathbf{k})^T$, with $k^0 = \sqrt{\mathbf{k}^2 + m^2}$ being the on-shell energy and m its mass. For the **collision term**, we consider only **binary elastic collisions**,

$$C[f] = \frac{1}{\nu} \int dK' dP dP' \, W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right) , \tag{3.2}$$

where ν is a symmetry factor ($= 2$ for identical particles), $W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'}$ is the **Lorentz-invariant transition rate**, and $dK \equiv d_{\rm dof} d^3\mathbf{k} / \left[(2\pi)^3 k^0\right]$ is the **Lorentz-invariant momentum-space volume**, with $d_{\rm dof}$ being the number of internal degrees of freedom. We introduced the notation

$$\tilde{f}_{\mathbf{k}} \equiv 1 - a f_{\mathbf{k}} \ , \tag{3.3}$$

where $a = 1$ ($a = -1$) for fermions (bosons) and $a = 0$ for a classical gas.

In kinetic theory, the conserved **particle current** N^μ and the **energy-momentum tensor** $T^{\mu\nu}$ are expressed as **moments of the single-particle distribution function** $f_{\mathbf{k}}$,

$$N^\mu = \langle k^\mu \rangle \ , \tag{3.4}$$
$$T^{\mu\nu} = \langle k^\mu k^\nu \rangle \ , \tag{3.5}$$

where the angular brackets are defined as

$$\langle \cdots \rangle \equiv \int dK \ (\cdots) \, f_{\mathbf{k}} \ , \tag{3.6}$$

i.e., a momentum-space average with the distribution function $f_{\mathbf{k}}$ as weight factor.

3.1.1 Macroscopic conservation laws

The **macroscopic conservation laws** are expressed in terms of **continuity equations**. They can be obtained using basic properties of the collision operator, as will be derived in this subsection. To this end, it is convenient to consider the following set of integrals of the collision operator,

$$\int dK \, G_{\mathbf{k}} C\,[f] = \frac{1}{\nu} \int dK dK' dP dP' \, W_{\mathbf{kk'}\to\mathbf{pp'}} G_{\mathbf{k}} \left(f_{\mathbf{p}} f_{\mathbf{p'}} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k'}} - f_{\mathbf{k}} f_{\mathbf{k'}} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p'}} \right) \ , \tag{3.7}$$

where $G_{\mathbf{k}}$ is an arbitrary function of the 3-momentum $\mathbf{k}$. The function $G_{\mathbf{k}}$ can be a Lorentz-tensor of arbitrary rank.

We now use a set of properties satisfied by the transition rate, $W_{\mathbf{kk'}\to\mathbf{pp'}}$. First, we note that $W_{\mathbf{kk'}\to\mathbf{pp'}}$ is **invariant under the exchange** $\mathbf{k} \leftrightarrow \mathbf{k'}$ or $\mathbf{p} \leftrightarrow \mathbf{p'}$, i.e.,

$$W_{\mathbf{kk'}\to\mathbf{pp'}} = W_{\mathbf{k'k}\to\mathbf{pp'}} = W_{\mathbf{k'k}\to\mathbf{p'p}} \ . \tag{3.8}$$

In other words, the transition probability per unit of time from an initial state (particles before the collision) to a final state (particles after the collision) cannot depend on which particle carries the incoming momentum $\mathbf{k}$ or $\mathbf{k'}$ or the outgoing momentum $\mathbf{p}$ or $\mathbf{p'}$. This property allows us to rewrite the collision integral (3.7) in the following form

$$\int dK \, G_{\mathbf{k}} C\,[f] = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{kk'}\to\mathbf{pp'}} \frac{G_{\mathbf{k}} + G_{\mathbf{k'}}}{2} \left(f_{\mathbf{p}} f_{\mathbf{p'}} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k'}} - f_{\mathbf{k}} f_{\mathbf{k'}} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p'}} \right) \ . \tag{3.9}$$

Next, we note that **time-reversal symmetry** further imposes that the transition rate is invariant under the exchange $\mathbf{kk'} \leftrightarrow \mathbf{pp'}$, i.e.,

$$W_{\mathbf{kk'}\to\mathbf{pp'}} = W_{\mathbf{pp'}\to\mathbf{kk'}} \ . \tag{3.10}$$

That is, collisions that change the momenta of the particles in the direction $(\mathbf{k}, \mathbf{k'}) \to (\mathbf{p}, \mathbf{p'})$ are as likely to happen as those that change the momenta in the direction $(\mathbf{p}, \mathbf{p'}) \to (\mathbf{k}, \mathbf{k'})$. If we impose this fundamental symmetry, we derive the following relation,

$$\int dK \, G_{\mathbf{k}} C\,[f] = \frac{1}{4\nu} \int dK dK' dP dP' W_{\mathbf{kk'}\to\mathbf{pp'}} \left(G_{\mathbf{k}} + G_{\mathbf{k'}} - G_{\mathbf{p}} - G_{\mathbf{p'}} \right) \times \left(f_{\mathbf{p}} f_{\mathbf{p'}} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k'}} - f_{\mathbf{k}} f_{\mathbf{k'}} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p'}} \right) \ . \tag{3.11}$$

Expression (3.11) for the collision integral (3.7) is very useful since it clearly shows that it **vanishes** if $G_{\mathbf{k}}$ is a quantity that is **conserved in microscopic collisions** between the particles. Such quantities are well-known in physics and are the **4-momentum**, k^μ, and the **charge** or, since we only consider binary elastic collisions, the **particle number**. Thus, taking $G_{\mathbf{k}} = 1$ or $G_{\mathbf{k}} = k^\mu$, we obtain

$$\int dK\, C[f] \;=\; 0\,, \tag{3.12}$$

$$\int dK\, k^\mu C[f] \;=\; 0\,. \tag{3.13}$$

Since there are no other quantities which are conserved in particle collisions, no other integral of the collision term vanishes – all other integrals will lead to finite contributions.

If we integrate the Boltzmann equation over 4-momentum K and use property (3.12) of the collision operator, we arrive at the continuity equation related to **particle-number conservation** (or net-charge conservation, if we consider processes that change the particle number),

$$\int dK\, k^\mu \partial_\mu f_{\mathbf{k}} = \int dK\, C[f] = 0 \quad \Longleftrightarrow \quad \partial_\mu \langle k^\mu \rangle = \partial_\mu N^\mu = 0\,. \tag{3.14}$$

Similarly, if we multiply the Boltzmann equation by k^ν, integrate over K, and use property (3.13) of the collision operator, we arrive at the continuity equation expressing **energy-momentum conservation**,

$$\int dK\, k^\mu k^\nu \partial_\mu f_{\mathbf{k}} = \int dK\, k^\nu C[f] = 0 \quad \Longleftrightarrow \quad \partial_\mu \langle k^\mu k^\nu \rangle = \partial_\mu T^{\mu\nu} = 0\,. \tag{3.15}$$

As already mentioned, any other integral of the collision operator will not vanish and hence, no additional conservation laws can be derived from the Boltzmann equation. We note, for spin-zero particles, angular-momentum conservation is a consequence of energy-momentum conservation, cf. Exercise 1.5, as long as the energy-momentum tensor is symmetric – a property that is fulfilled by definition, cf. Eq. (3.5).

3.1.2 Fluid-dynamical variables and matching conditions

The particle current N^μ and the energy-momentum tensor $T^{\mu\nu}$ can be **tensor-decomposed** with respect to the fluid 4-velocity u^μ. We introduce u^μ as the **time-like, normalized** **($u_\mu u^\mu = 1$) eigenvector** of the energy-momentum tensor,

$$T^{\mu\nu} u_\nu = \varepsilon u^\mu\,, \tag{3.16}$$

where the **eigenvalue** ε is the **energy density**. This choice of fluid velocity is traditionally referred to as the **Landau frame** [5], see Sec. 1.3.3. Next, we divide the momentum of the particles k^μ into two parts: one **parallel** and one **orthogonal** to u^μ,

$$k^\mu = E_{\mathbf{k}} u^\mu + k^{\langle\mu\rangle}\,, \tag{3.17}$$

where we defined the scalar

$$E_{\mathbf{k}} \equiv k^{\mu} u_{\mu} \;, \tag{3.18}$$

which is identical to the energy of the particle in the rest frame of the fluid, $E_{\mathbf{k}} = k^{\mu} u_{\mathrm{LR},\mu} = k^0 = \sqrt{\mathbf{k}^2 + m^2}$, and used the notation $A^{\langle\mu\rangle} = \Delta^{\mu}_{\nu} A^{\nu}$, with $\Delta^{\mu\nu}$ the projection operator onto the 3-space orthogonal to u^{μ} given by Eq. (1.25).

Then, the **tensor decomposition** of N^{μ} and $T^{\mu\nu}$ reads

$$N^{\mu} = n u^{\mu} + n^{\mu} \;, \tag{3.19}$$

$$T^{\mu\nu} = \varepsilon\, u^{\mu} u^{\nu} - \Delta^{\mu\nu} \left(P_0 + \Pi\right) + \pi^{\mu\nu} \;, \tag{3.20}$$

where the **particle density** n, the **particle-diffusion current** n^{μ}, the **energy density** ε, the **shear-stress tensor** $\pi^{\mu\nu}$, and the sum of **thermodynamic pressure** P_0 and **bulk-viscous pressure** Π are defined by

$$n \equiv \langle E_{\mathbf{k}} \rangle \;,\; n^{\mu} \equiv \left\langle k^{\langle\mu\rangle} \right\rangle \;,\; \varepsilon \equiv \left\langle E_{\mathbf{k}}^2 \right\rangle \;,\; \pi^{\mu\nu} \equiv \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle \;,\; P_0 + \Pi \equiv -\frac{1}{3} \left\langle \Delta^{\mu\nu} k_{\mu} k_{\nu} \right\rangle \;, \tag{3.21}$$

where $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$, with $\Delta^{\mu\nu}_{\alpha\beta}$ as defined in Eq. (1.60).

Next, we introduce the **local-equilibrium distribution function** as

$$f_{0\mathbf{k}} = \left[\exp\left(\beta_0\, E_{\mathbf{k}} - \alpha_0\right) + a\right]^{-1} \;, \tag{3.22}$$

where $\beta_0 \equiv 1/T$ is the **inverse temperature** and $\alpha_0 = \mu/T$ is the **thermal potential**, i.e., the ratio of the **chemical potential** μ to **temperature**, respectively. The values of α_0 and β_0 are defined by the **matching conditions**, as explained in Sec. 1.3.1,

$$n \equiv n_0 = \langle E_{\mathbf{k}} \rangle_0 \;, \quad \varepsilon \equiv \varepsilon_0 = \left\langle E_{\mathbf{k}}^2 \right\rangle_0 \;, \tag{3.23}$$

where we use the following notation for the momentum-space average with the local-equilibrium distribution function as weight factor

$$\langle \cdots \rangle_0 \equiv \int dK \,(\cdots)\, f_{0\mathbf{k}} \;. \tag{3.24}$$

Then, the separation between thermodynamic pressure and bulk-viscous pressure is achieved by

$$P_0 = -\frac{1}{3} \left\langle \Delta^{\mu\nu} k_{\mu} k_{\nu} \right\rangle_0 \tag{3.25}$$

and

$$\Pi = -\frac{1}{3} \left\langle \Delta^{\mu\nu} k_{\mu} k_{\nu} \right\rangle_{\delta} \;, \tag{3.26}$$

where

$$\langle \cdots \rangle_{\delta} = \langle \cdots \rangle - \langle \cdots \rangle_0 \;. \tag{3.27}$$

In equilibrium, the bulk-viscous pressure vanishes by definition, $\Pi_0 \equiv 0$. The particle or charge diffusion current n^{μ} and the shear-stress tensor $\pi^{\mu\nu}$ vanish as well, $n_0^{\mu} \equiv \langle k^{\langle\mu\rangle} \rangle_0 = 0$, $\pi_0^{\mu\nu} = \langle k^{\langle\mu} k^{\nu\rangle} \rangle_0 = 0$. This is obvious from the symmetries of the equilibrium distribution $f_{0\mathbf{k}}$. This, in turn, implies that

$$n^{\mu} = \left\langle k^{\langle\mu\rangle} \right\rangle_{\delta} \;, \qquad \pi^{\mu\nu} = \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle_{\delta} \;. \tag{3.28}$$

As shown in Sec. 1.3.3, the conservation laws provide equations of motion only for n, ε, and u^μ, and hence one still needs to derive the equations of motion for the dissipative corrections Π, n^μ, and $\pi^{\mu\nu}$. In kinetic theory, this task can be performed **rigorously**. Nevertheless, there are several methods that can be used to determine the required equations and transport coefficients. In the following, we review the two most traditional methods: **Chapman-Enskog theory** [1] and the **method of moments proposed by Israel and Stewart** [2, 3, 4].

3.2 Chapman-Enskog theory

The **Chapman-Enskog expansion** [1] is the most traditional formalism used to derive fluid dynamics from the Boltzmann equation. It was originally developed for non-relativistic systems, but Israel proved that it could be used with almost no modifications to describe relativistic systems as well [6, 7].

The Chapman-Enskog formalism corresponds to the **microscopic implementation of the gradient expansion**, already discussed in Chapter 1. It assumes that the single-particle distribution function depends only on the five **primary fluid-dynamical variables**, i.e., temperature, chemical potential, and the three independent components of the fluid-velocity field, as well as their gradients. The corrections to the local distribution function are then systematically arranged in terms of an **expansion in powers of the Knudsen number**. As is well known and will be shown in this section, the **first-order truncation** of the expansion leads to **Navier-Stokes theory**. Keeping second and higher-order terms one obtains the Burnett and super-Burnett equations, respectively [8]. In principle, one can construct the solution to an arbitrarily high order in Knudsen number.

As first pointed out by Grad, the Chapman-Enskog expansion is an **asymptotic series** [9]. Also, in the relativistic case, the Chapman-Enskog expansion leads to unstable equations of motion [10] and, therefore, has very little use in the description of realistic systems. Despite this major drawback, the Chapman-Enskog expansion is an important development in kinetic theory and its results are useful to understand the asymptotic behavior of the Boltzmann equation.

The first step is to rewrite the Boltzmann equation (3.1) by decomposing the 4-derivative ∂_μ into its time-like and space-like parts,

$$\partial_\mu = u_\mu u^\nu \partial_\nu + \Delta_\mu^\nu \partial_\nu \equiv u_\mu D + \nabla_\mu \; , \tag{3.29}$$

where we defined the **comoving derivative**

$$D \equiv u^\nu \partial_\nu \; , \tag{3.30}$$

which is equal to the ordinary time derivative in the local rest frame of the fluid, $D_{LR} \equiv \partial/\partial t$, and the **3-space gradient**

$$\nabla_\mu \equiv \Delta_\mu^\nu \partial_\nu \; , \tag{3.31}$$

which is equal to the ordinary spatial gradient in the local rest frame of the fluid, $\nabla_{LR,\mu} \equiv (0, \boldsymbol{\nabla})$. The Boltzmann equation (3.1) then reads

$$D f_{\mathbf{k}} + \frac{1}{E_{\mathbf{k}}} k^\mu \nabla_\mu f_{\mathbf{k}} = \frac{1}{E_{\mathbf{k}}} C[f] \; . \tag{3.32}$$

In the local rest frame of the system, the gradient ∇_μ determines an inverse **macroscopic length scale**, L^{-1}, over which the single-particle distribution function (and its momentum integrals) varies in space. Similarly, the covariant derivative D defines an inverse **macroscopic time scale**, $\bar{\tau}^{-1}$. It is convenient to redefine these derivatives as $\nabla_\mu \equiv L^{-1}\hat{\nabla}_\mu$ and $D \equiv \bar{\tau}^{-1}\hat{D}$, where $\hat{\nabla}_\mu$ and $\hat{D}$ are unitless derivatives of order one. Then, by multiplying the Boltzmann equation by the **mean free path** λ of the particles, we obtain the **dimensionless equation of motion**

$$\overline{\mathrm{Kn}}\ \hat{D} f_{\mathbf{k}} + \frac{\mathrm{Kn}}{E_{\mathbf{k}}} k^\mu \hat{\nabla}_\mu f_{\mathbf{k}} = \frac{\lambda}{E_{\mathbf{k}}} C[f]\ , \tag{3.33}$$

where Kn$= \lambda/L$ is the standard Knudsen number. For the sake of completeness, we also introduced another type of Knudsen number, $\overline{\mathrm{Kn}} = \lambda/\bar{\tau}$, which characterizes the **macroscopic time variations relative to the mean free path**.

In Chapman-Enskog theory a solution of the Boltzmann equation is obtained by expanding the single-particle distribution function in powers of Kn,

$$f_{\mathbf{k}} = f^{(0)}_{\mathbf{k}} + \mathrm{Kn}\ f^{(1)}_{\mathbf{k}} + \mathrm{Kn}^2\ f^{(2)}_{\mathbf{k}} + \ldots\ . \tag{3.34}$$

If the Knudsen number is small, it should be possible to truncate this expansion and find an approximate expression for $f_{\mathbf{k}}$. This solution is found using perturbation theory, by substituting Eq. (3.34) into Eq. (3.33) and solving it order by order in Knudsen number.

First, we substitute expression (3.34) into the collision term (3.2), obtaining

$$C[f] = \mathcal{C}^{(0)} + \mathrm{Kn}\ \mathcal{C}^{(1)} + \mathrm{Kn}^2\ \mathcal{C}^{(2)} + \ldots\ , \tag{3.35}$$

where the first two terms of the expansion are

$$\mathcal{C}^{(0)} \equiv \frac{1}{\nu}\int dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} \left(f^{(0)}_{\mathbf{p}} f^{(0)}_{\mathbf{p}'} \tilde{f}^{(0)}_{\mathbf{k}} \tilde{f}^{(0)}_{\mathbf{k}'} - f^{(0)}_{\mathbf{k}} f^{(0)}_{\mathbf{k}'} \tilde{f}^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}'} \right)\ , \tag{3.36}$$

$$\begin{aligned}\mathcal{C}^{(1)} \equiv \frac{1}{\nu}\int dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} \Bigg[& f^{(0)}_{\mathbf{p}} f^{(0)}_{\mathbf{p}'} \tilde{f}^{(0)}_{\mathbf{k}} \tilde{f}^{(0)}_{\mathbf{k}'} \left(\frac{f^{(1)}_{\mathbf{p}}}{f^{(0)}_{\mathbf{p}}} + \frac{f^{(1)}_{\mathbf{p}'}}{f^{(0)}_{\mathbf{p}'}} - a\frac{f^{(1)}_{\mathbf{k}}}{\tilde{f}^{(0)}_{\mathbf{k}}} - a\frac{f^{(1)}_{\mathbf{k}'}}{\tilde{f}^{(0)}_{\mathbf{k}'}} \right) \\ & - f^{(0)}_{\mathbf{k}} f^{(0)}_{\mathbf{k}'} \tilde{f}^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}'} \left(\frac{f^{(1)}_{\mathbf{k}}}{f^{(0)}_{\mathbf{k}}} + \frac{f^{(1)}_{\mathbf{k}'}}{f^{(0)}_{\mathbf{k}'}} - a\frac{f^{(1)}_{\mathbf{p}}}{\tilde{f}^{(0)}_{\mathbf{p}}} - a\frac{f^{(1)}_{\mathbf{p}'}}{\tilde{f}^{(0)}_{\mathbf{p}'}} \right) \Bigg]\ .\end{aligned} \tag{3.37}$$

Exercise 3.1: Prove Eq. (3.37).

Note that we introduced **two types of Knudsen numbers**: one related to **temporal variations** of integrals of $f_{\mathbf{k}}$, $\overline{\mathrm{Kn}}$, and another related to **spatial variations** of integrals of $f_{\mathbf{k}}$, Kn. In general, these two quantities do not need to be equal or even related. On the other hand, in the fluid-dynamical limit, it is reasonable to expect that these two quantities are related or, at least, of the same order of magnitude. In the Chapman-Enskog expansion, one goes a step further and assumes that Kn and $\overline{\mathrm{Kn}}$ are **exactly the**

same, $\mathrm{Kn} = \overline{\mathrm{Kn}}$. Only with this assumption it becomes possible to solve the Boltzmann equation perturbatively in powers of Kn, as was initially proposed. We shall see later that this assumption will ensure that time-like gradients can always be replaced by space-like gradients and, consequently, one can always arrange the solution of the single-particle distribution function in **powers of space-like gradients**.

Therefore, using $\mathrm{Kn} = \overline{\mathrm{Kn}}$, and inserting the expansions (3.34) and (3.35) into the Boltzmann equation (3.33), one obtains by comparing **order by order in Kn** the following solution,

$$0 = \mathcal{C}^{(0)} , \tag{3.38}$$

$$\left[\hat{D} f_{\mathbf{k}}\right]^{(1)} + \frac{1}{E_{\mathbf{k}}} k^{\mu} \hat{\nabla}_{\mu} f_{\mathbf{k}}^{(0)} = \frac{\lambda}{E_{\mathbf{k}}} \mathcal{C}^{(1)} , \tag{3.39}$$

$$\left[\hat{D} f_{\mathbf{k}}\right]^{(2)} + \frac{1}{E_{\mathbf{k}}} k^{\mu} \hat{\nabla}_{\mu} f_{\mathbf{k}}^{(1)} = \frac{\lambda}{E_{\mathbf{k}}} \mathcal{C}^{(2)} . \tag{3.40}$$

Equation (3.38) can be used to solve for $f_{\mathbf{k}}^{(0)}$. Once $f_{\mathbf{k}}^{(0)}$ is known, Eq. (3.39) can be used to solve for $f_{\mathbf{k}}^{(1)}$ and so on. In principle, one could go on indefinitely and construct the solution $f_{\mathbf{k}}$ to any order in Knudsen number. However, after solving for the correction of first order in Knudsen number, the calculations start to become cumbersome.

Note that in Eqs. (3.38) – (3.40), we wrote $\left[\hat{D} f_{\mathbf{k}}\right]^{(n)}$ to indicate terms of order Kn^n from the time derivative. This needs to be distinguished from $\hat{D} f_{\mathbf{k}}^{(n)}$ because, as we shall see below, the latter can have contributions of all orders in Kn, i.e.,

$$\hat{D} f_{\mathbf{k}}^{(n)} = \left[\hat{D} f_{\mathbf{k}}^{(n)}\right]^{(1)} + \mathrm{Kn} \left[\hat{D} f_{\mathbf{k}}^{(n)}\right]^{(2)} + \dots , \tag{3.41}$$

where $\left[\hat{D} f_{\mathbf{k}}^{(n)}\right]^{(m)}$ is the coefficient of order Kn^{n+m} in the Boltzmann equation (3.33). Therefore,

$$\left[\hat{D} f_{\mathbf{k}}\right]^{(0)} = 0 , \tag{3.42}$$

$$\left[\hat{D} f_{\mathbf{k}}\right]^{(1)} = \left[\hat{D} f_{\mathbf{k}}^{(0)}\right]^{(1)} , \tag{3.43}$$

$$\left[\hat{D} f_{\mathbf{k}}\right]^{(2)} = \left[\hat{D} f_{\mathbf{k}}^{(0)}\right]^{(2)} + \left[\hat{D} f_{\mathbf{k}}^{(1)}\right]^{(2)} , \tag{3.44}$$

and so on.

Let us now clarify why time derivatives contain also higher orders in the Knudsen number. To this end, we rewrite the conservation laws Eqs. (1.77) – (1.79), under the assumption $\mathrm{Kn} = \overline{\mathrm{Kn}}$ as

$$\hat{D}\varepsilon_0 + (\varepsilon_0 + P_0 + \Pi)\,\hat{\theta} - \pi^{\alpha\beta}\hat{\sigma}_{\alpha\beta} = 0 , \tag{3.45}$$

$$(\varepsilon_0 + P_0 + \Pi)\,\hat{D}u^{\mu} - \hat{\nabla}^{\mu}(P_0 + \Pi) - \pi^{\mu\beta}\hat{D}u_{\beta} + \Delta^{\mu}_{\alpha}\hat{\nabla}_{\beta}\pi^{\alpha\beta} = 0 , \tag{3.46}$$

$$\hat{D}n_0 + n_0\hat{\theta} - n^{\mu}\hat{D}u_{\mu} + \hat{\nabla}_{\mu}n^{\mu} = 0 . \tag{3.47}$$

Here, we introduced $\theta \equiv L^{-1}\hat{\theta}$ and $\sigma^{\mu\nu} \equiv L^{-1}\hat{\sigma}^{\mu\nu}$, with the expansion scalar θ and the shear tensor $\sigma^{\mu\nu}$ defined in Eqs. (1.33) and (1.80). In order to obtain Eqs. (3.46), (3.47) from Eqs. (1.78), (1.79) we also rewrote $\partial_\mu n^\mu$ and $\Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta}$ using Eq. (3.29) and the orthogonality relations $n^\mu u_\mu = 0$, $u_\beta \pi^{\alpha\beta} = 0$.

Let us now define the thermodynamic functions

$$I_{nq} \equiv \frac{(-1)^q}{(2q+1)!!} \int dK\, f_{0\mathbf{k}} E_{\mathbf{k}}^{n-2q} \left(m^2 - E_{\mathbf{k}}^2\right)^q \equiv \frac{(-1)^q}{(2q+1)!!} \left\langle (k^\mu u_\mu)^{n-2q} (\Delta^{\mu\nu} k_\mu k_\nu)^q \right\rangle_0 \,, \tag{3.48}$$

$$J_{nq} \equiv \frac{(-1)^q}{(2q+1)!!} \int dK\, f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{n-2q} \left(m^2 - E_{\mathbf{k}}^2\right)^q \equiv \frac{(-1)^q}{(2q+1)!!} \left\langle \tilde{f}_{0\mathbf{k}} (k^\mu u_\mu)^{n-2q} (\Delta^{\mu\nu} k_\mu k_\nu)^q \right\rangle_0 \,. \tag{3.49}$$

From Eqs. (3.23) – (3.25) we observe that

$$n_0 = I_{10} \,, \quad \varepsilon_0 = I_{20} \,, \quad P_0 = I_{21} \,. \tag{3.50}$$

From the definition of $f_{0\mathbf{k}}$, Eq. (3.22), we derive the identity

$$dI_{nq} = J_{nq}\, d\alpha_0 - J_{n+1,q}\, d\beta_0 \,. \tag{3.51}$$

Using Eq. (3.22) in the definitions (3.48), (3.49) and integrating by parts also yields the identity

$$\beta_0 J_{nq} = I_{n-1,q-1} + (n-2q)\, I_{n-1,q} \,, \tag{3.52}$$

from which we deduce

$$I_{10} = \beta_0 J_{21} \,, \quad I_{20} + I_{21} = \beta_0 J_{31} \,. \tag{3.53}$$

From the definition (3.49) one also readily proves the identity

$$J_{nq} \equiv (2q+3) J_{n,q+1} + m^2 J_{n-2,q} \,, \tag{3.54}$$

which also holds for the integrals I_{nq}.

Exercise 3.2: Prove Eqs. (3.51), (3.52), and (3.54).

For further use, we also define the thermodynamic functions

$$G_{nm} = J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0} \,, \quad D_{nq} = J_{n+1,q} J_{n-1,q} - J_{nq}^2 \,. \tag{3.55}$$

From Eqs. (3.50) and (3.51) we then derive the following thermodynamic relations,

$$d\alpha_0 = -\frac{J_{20}}{D_{20}} d\varepsilon_0 + \frac{J_{30}}{D_{20}} dn_0 \,, \tag{3.56}$$

$$d\beta_0 = -\frac{J_{10}}{D_{20}} d\varepsilon_0 + \frac{J_{20}}{D_{20}} dn_0 \,. \tag{3.57}$$

Exercise 3.3: Prove Eqs. (3.56) and (3.57).

Rewriting the total derivative into a (dimensionless time-like derivative) and using the hydrodynamic equations of motion (3.45) and (3.47), we finally obtain equations of motion for α_0 and β_0,

$$\hat{D}\alpha_0 = \frac{1}{D_{20}} \left\{ \left[(\varepsilon_0 + P_0) J_{20} - n_0 J_{30} \right] \hat{\theta} + J_{20} \left(\Pi\hat{\theta} - \pi^{\alpha\beta}\hat{\sigma}_{\alpha\beta} \right) - J_{30} \left(\hat{\nabla}_\mu n^\mu - n^\mu \hat{D} u_\mu \right) \right\} , \tag{3.58}$$

$$\hat{D}\beta_0 = \frac{1}{D_{20}} \left\{ \left[(\varepsilon_0 + P_0) J_{10} - n_0 J_{20} \right] \hat{\theta} + J_{10} \left(\Pi\hat{\theta} - \pi^{\alpha\beta}\hat{\sigma}_{\alpha\beta} \right) - J_{20} \left(\hat{\nabla}_\mu n^\mu - n^\mu \hat{D} u_\mu \right) \right\} . \tag{3.59}$$

Exercise 3.4: Prove Eqs. (3.58) and (3.59).

We see that the time-like derivatives are proportional to terms of first order in gradients (the terms $\sim \hat{\theta}$) or, equivalently, proportional to one power of the Knudsen number, respectively, but that also terms enter which are proportional to **gradients times dissipative quantities** (Π, n^μ, $\pi^{\mu\nu}$), which (according to Navier-Stokes theory) are itself of at least first order in the Knudsen number. Therefore, time derivatives of a quantity of given order in Kn in general involve also **higher orders in Kn**, as written formally in Eq. (3.41) for $f_{\mathbf{k}}^{(n)}$.

3.2.1 Solving the Chapman-Enskog expansion: zeroth- and first-order solutions

The solution of Eq. (3.38) is well known, and is given by the **local-equilibrium single-particle distribution function** (3.22),

$$f_{\mathbf{k}}^{(0)} = f_{0\mathbf{k}} = \left[\exp\left(\beta_0 E_{\mathbf{k}} - \alpha_0 \right) + a \right]^{-1} , \tag{3.60}$$

where $\alpha_0(X)$, $\beta_0(X)$, and $u^\mu(X)$ are functions of space and time (global equilibrium corresponds to the particular case where there is no space-time dependence). Therefore, the **zeroth-order truncation of the Chapman-Enskog expansion** leads to the equations of **ideal fluid dynamics**, with all conserved currents being given by their respective **equilibrium values**. In this case, we have already demonstrated in Chapter 1 that the continuity equations describing (net-)particle and energy-momentum conservation are sufficient to describe the dynamics of the fluid, since the pressure is determined by an **equation of state** (here, the equation of state is the one of a dilute gas, given by $P_0 = I_{21}(\alpha_0, \beta_0)$).

However, ideal fluid dynamics does **not** arise from a solution of the Boltzmann equation since $f_{0\mathbf{k}}$ alone does **not** satisfy this equation: the **collision term vanishes**, Eq. (3.38), but the **left-hand side of the Boltzmann equation** obviously does **not vanish** if α_0, β_0, and u^μ are functions of space and time. On the other hand, the above result implies that **ideal fluid dynamics** can be interpreted as the **zeroth-order truncation of an expansion in powers of Knudsen number**. In this sense, it should be understood solely as an **approximate solution**, that will never be exactly realized but, in practice, may lead to a reliable description of several physical systems.

The local-equilibrium variables α_0 and β_0 are defined by the matching conditions (3.23), while the fluid 4-velocity is defined by the choice of frame (in our case the Landau frame, Eq. (3.16)). In order to satisfy these constraints, for all $n \geq 1$ the corrections $f_{\mathbf{k}}^{(n)}$ must satisfy

$$\int dK\, E_{\mathbf{k}}\, f_{\mathbf{k}}^{(n)} = 0\,, \tag{3.61}$$

$$\int dK\, E_{\mathbf{k}}^2\, f_{\mathbf{k}}^{(n)} = 0\,, \tag{3.62}$$

$$\int dK\, E_{\mathbf{k}} k^{\langle\mu\rangle}\, f_{\mathbf{k}}^{(n)} = 0\,. \tag{3.63}$$

The last condition can be obtained by projecting Eq. (3.16) onto Δ_μ^α and ensures that there is no flow of energy relative to u^μ. These conditions guarantee that, **to any order of approximation**, the solution depends solely on α_0, β_0, and u^μ, and their gradients. Also, they remove the freedom that we could add the solution of the homogeneous Boltzmann equation to the solution of the perturbative series (3.34), i.e., a global-equilibrium distribution function with another temperature, chemical potential, and velocity.

In the following, we shall construct the solution of the Chapman-Enskog expansion to first order in Knudsen number, i.e., the solution to Eq. (3.39). Using Eqs. (3.43) and (3.60), as well as simplifying Eq. (3.37) using Eq. (3.60), we must solve

$$\left[\hat{D} f_{0\mathbf{k}}\right]^{(1)} + \frac{1}{E_{\mathbf{k}}} k^\mu \hat{\nabla}_\mu f_{0\mathbf{k}} = -\lambda \hat{C} f_{\mathbf{k}}^{(1)}\,, \tag{3.64}$$

where we defined the **linear collision operator** acting on $f_{\mathbf{k}}^{(1)}$ as

$$\hat{C} f_{\mathbf{k}}^{(1)} \equiv \frac{1}{\nu E_{\mathbf{k}}} \int dK' dP dP'\, W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \times \left(\frac{f_{\mathbf{k}}^{(1)}}{f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}} + \frac{f_{\mathbf{k}'}^{(1)}}{f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{k}'}} - \frac{f_{\mathbf{p}}^{(1)}}{f_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}}} - \frac{f_{\mathbf{p}'}^{(1)}}{f_{0\mathbf{p}'}\tilde{f}_{0\mathbf{p}'}} \right)\,. \tag{3.65}$$

Exercise 3.5: Prove Eq. (3.65), i.e., that $\hat{C} f_{\mathbf{k}}^{(1)} \equiv -\mathcal{C}^{(1)}/E_{\mathbf{k}}$, with $\mathcal{C}^{(1)}$ given by Eq. (3.37).

(Hint: Using Eq. (3.22) and the fact that energy-momentum is conserved in binary collisions, first prove that $f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \equiv f_{0\mathbf{p}} f_{0\mathbf{p}'} \tilde{f}_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}'}$.)

The time-like and space-like derivatives of $f_{0\mathbf{k}}$ appearing on the left-hand side of Eq. (3.64) are straightforward to calculate using Eq. (3.60),

$$\hat{D} f_{0\mathbf{k}} = -f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \left(E_{\mathbf{k}} \hat{D}\beta_0 + \beta_0 k^{\langle\nu\rangle} \hat{D} u_\nu - \hat{D}\alpha_0 \right) , \quad (3.66)$$

$$\hat{\nabla}_\mu f_{0\mathbf{k}} = -f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \left(E_{\mathbf{k}} \hat{\nabla}_\mu \beta_0 + \beta_0 k^{\langle\nu\rangle} \hat{\nabla}_\mu u_\nu - \hat{\nabla}_\mu \alpha_0 \right) , \quad (3.67)$$

where we used that $u^\nu \hat{D} u_\nu = u^\nu \hat{\nabla}_\mu u_\nu = 0$.

All time-like derivatives of α_0, β_0, and u^μ that appear in Eq. (3.66) can be replaced by space-like gradients using the conservation equations (3.45) – (3.47), as well as Eqs. (3.58), (3.59), respectively. However, note that the bulk-viscous pressure, particle-diffusion 4-current, and shear-stress tensor do not have zeroth-order contributions in the Knudsen number, because they vanish in equilibrium,

$$(\Pi)^{(0)} = -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_0 - P_0 = 0 , \quad (3.68)$$

$$(n^\mu)^{(0)} = \left\langle k^{\langle\mu\rangle} \right\rangle_0 = 0 , \quad (3.69)$$

$$(\pi^{\mu\nu})^{(0)} = \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle_0 = 0 , \quad (3.70)$$

i.e., the dissipative currents are **at least of first order in Knudsen number**. Therefore, substituting these results into Eqs. (3.46), (3.58), and (3.59), one obtains

$$\hat{D}\alpha_0 = \frac{(\varepsilon_0 + P_0) J_{20} - n_0 J_{30}}{D_{20}} \hat{\theta} + \mathcal{O}\left(\mathrm{Kn}^2\right) , \quad (3.71)$$

$$\hat{D}\beta_0 = \frac{(\varepsilon_0 + P_0) J_{10} - n_0 J_{20}}{D_{20}} \hat{\theta} + \mathcal{O}\left(\mathrm{Kn}^2\right) , \quad (3.72)$$

$$\hat{D}u^\mu = \frac{1}{\varepsilon_0 + P_0} \hat{\nabla}^\mu P_0 + \mathcal{O}\left(\mathrm{Kn}^2\right) . \quad (3.73)$$

Therefore, to first order in Knudsen number, the comoving time derivatives of α_0, β_0, and u^μ are linearly proportional to their spatial gradients. This whole procedure illustrates the subtleties involved in replacing time derivatives by spatial ones in Chapman-Enskog theory. To first order, the procedure is fairly simple, but it can get rather complicated when one goes to higher orders, since it is extremely difficult to establish *a priori* what is the general structure of the contributions of higher order in Knudsen number.

Then, Eq. (3.64) can be reduced to

$$f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \left(A_{\mathbf{k}} \hat{\theta} + B_{\mathbf{k}} k^{\langle\mu\rangle} \hat{\nabla}_\mu \alpha_0 + \frac{\beta_0}{E_{\mathbf{k}}} k^{\langle\mu} k^{\nu\rangle} \hat{\sigma}_{\mu\nu} \right) = \lambda \hat{C} f^{(1)}_{\mathbf{k}} , \quad (3.74)$$

where we made use of the thermodynamic relation (1.103) and the fact that $k^{\langle\mu\rangle} k^{\langle\nu\rangle} = k^{\langle\mu} k^{\nu\rangle} + \Delta^{\mu\nu} \left(\Delta^{\alpha\beta} k_\alpha k_\beta \right) /3$. We furthermore introduced the scalar functions

$$A_{\mathbf{k}} \equiv \frac{(\varepsilon_0 + P_0) J_{10} - n_0 J_{20}}{D_{20}} E_{\mathbf{k}} - \frac{(\varepsilon_0 + P_0) J_{20} - n_0 J_{30}}{D_{20}} + \frac{\beta_0}{3E_{\mathbf{k}}} \Delta^{\alpha\beta} k_\alpha k_\beta , \quad (3.75)$$

$$B_{\mathbf{k}} \equiv h_0^{-1} - \frac{1}{E_{\mathbf{k}}} . \quad (3.76)$$

Exercise 3.6: Prove Eq. (3.74).

We note that the linear operator $\hat{C}$ defined in Eq. (3.65) has **five degenerate eigenfunctions** (1, $E_{\mathbf{k}}$, and $k^{\langle\mu\rangle}$, multiplied by $f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}$), with **zero eigenvalues**

$$\hat{C}(f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\ 1) = 0\ , \quad \hat{C}(f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\ E_{\mathbf{k}}) = 0\ , \quad \hat{C}(f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\ k^{\langle\mu\rangle}) = 0\ . \tag{3.77}$$

These correspond to quantities which are **conserved** in microscopic collisions.

Nevertheless, solving Eq. (3.74) is still a complicated task because one needs to invert the collision operator $\hat{C}$ in order to obtain $f_{\mathbf{k}}^{(1)}$. Still, since the equation is **linear** in $f_{\mathbf{k}}^{(1)}$ and the left-hand side is **linear** in the gradients $\hat{\theta}$, $\hat{\nabla}_\mu\alpha_0$, and $\hat{\sigma}_{\mu\nu}$, one already knows that the general solution for $f_{\mathbf{k}}^{(1)}$ must have the following form,

$$\frac{f_{\mathbf{k}}^{(1)}}{f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}} = \varphi_{\mathbf{k}}^{\mathrm{s}}\,\hat{\theta} + \varphi_{\mathbf{k}}^{\mathrm{v}}\,\beta_0 k^{\langle\mu\rangle}\hat{\nabla}_\mu\alpha_0 + \varphi_{\mathbf{k}}^{\mathrm{t}}\,\beta_0^2 k^{\langle\mu}k^{\nu\rangle}\hat{\sigma}_{\mu\nu} + \varphi_{\mathbf{k}}^{\mathrm{hom}}\ . \tag{3.78}$$

The functions $\varphi_{\mathbf{k}}^{i}$, i =s,v,t, are dimensionless and depend on momentum only through $E_{\mathbf{k}}$, i.e., $\varphi_{\mathbf{k}}^{i} = \varphi_{\mathbf{k}}^{i}(E_{\mathbf{k}})$. The function $\varphi_{\mathbf{k}}^{\mathrm{hom}}$ is the homogeneous solution,

$$\hat{C}\varphi_{\mathbf{k}}^{\mathrm{hom}} = 0\ , \tag{3.79}$$

which is constructed as a linear combination of the eigenfunctions 1, $E_{\mathbf{k}}$, and $k^{\langle\mu\rangle}$,

$$\varphi_{\mathbf{k}}^{\mathrm{hom}} = a_0 + a_1 E_{\mathbf{k}} + a_{2\mu}k^{\langle\mu\rangle}\ . \tag{3.80}$$

The coefficients a_0, a_1, and $a_{2\mu}$ must be determined using the matching conditions (3.61) – (3.63) for $n = 1$.

For the following, it is advantageous to define

$$\begin{aligned}
\alpha_r^{\mathrm{s}} &\equiv \int dK\, E_{\mathbf{k}}^r A_{\mathbf{k}} f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \\
&= \frac{(\varepsilon_0 + P_0)\,J_{10} - n_0 J_{20}}{D_{20}}\,J_{r+1,0} - \frac{(\varepsilon_0 + P_0)\,J_{20} - n_0 J_{30}}{D_{20}}\,J_{r,0} - \beta_0\,J_{r+1,1}\ , && (3.81)\\
\alpha_r^{\mathrm{v}} &\equiv \frac{1}{3}\int dK\, E_{\mathbf{k}}^r B_{\mathbf{k}}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right) f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} = J_{r+1,1} - h_0^{-1}J_{r+2,1}\ , && (3.82)\\
\alpha_r^{\mathrm{t}} &\equiv \frac{2}{15}\beta_0\int dK\, E_{\mathbf{k}}^{r-1}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^2 f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} = 2\beta_0 J_{r+3,2}\ , && (3.83)
\end{aligned}$$

where we used Eqs. (3.49), (3.75), and (3.76).

Exercise 3.7: Show that, in the massless limit, $\alpha_r^{\rm s}$ vanishes for any value of r.
(Hint: Note that, for $m = 0$, $I_{n0} = 3I_{n1}$, $J_{n0} = 3J_{n1}$, cf. Eqs. (3.48), (3.49). For the proof, use the relations (1.103), (3.51), and (3.53).)

Then, inserting the Ansatz (3.78) into Eq. (3.74), multiplying by an arbitrary power of energy, $E_{\mathbf{k}}^r$, and integrating over momentum, dK, one obtains the equation satisfied by $\varphi_{\mathbf{k}}^{\rm s}$,

$$\alpha_r^{\rm s} = \frac{\lambda}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} \left(\varphi_{\mathbf{k}}^{\rm s} + \varphi_{\mathbf{k}'}^{\rm s} - \varphi_{\mathbf{p}}^{\rm s} - \varphi_{\mathbf{p}'}^{\rm s} \right) . \tag{3.84}$$

Here, we have used **orthogonality relations** on the left- and right-hand sides. On the left-hand side, we have used the relation

$$\int dK \, \mathrm{F}(E_{\mathbf{k}}) \, k^{\langle \mu_1} \cdots k^{\mu_m \rangle} k_{\langle \nu_1} \cdots k_{\nu_n \rangle} = \frac{n! \, \delta_{mn}}{(2n+1)!!} \Delta^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \int dK \, \mathrm{F}(E_{\mathbf{k}}) \, \left(\Delta^{\alpha\beta} k_\alpha k_\beta \right)^n , \tag{3.85}$$

for an arbitrary function $\mathrm{F}(E_{\mathbf{k}})$, which depends only on $E_{\mathbf{k}} = k^\mu u_\mu$. The proof of this relation can be found in App. 4.7. To obtain the right-hand side of Eq. (3.84), we have used the fact that the tensor structure of a quantity

$$\begin{aligned} (\mathcal{A}_r)^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \equiv & \frac{1}{\nu} \int dK dK' dP dP' \, W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} k^{\langle \mu_1} \cdots k^{\mu_m \rangle} \\ & \times \left(\mathrm{H}_{\mathbf{k}} \, k_{\langle \nu_1} \cdots k_{\nu_n \rangle} + \mathrm{H}_{\mathbf{k}'} \, k'_{\langle \nu_1} \cdots k'_{\nu_n \rangle} - \mathrm{H}_{\mathbf{p}} \, p_{\langle \nu_1} \cdots p_{\nu_n \rangle} - \mathrm{H}_{\mathbf{p}'} \, p'_{\langle \nu_1} \cdots p'_{\nu_n \rangle} \right) , \end{aligned} \tag{3.86}$$

where $\mathrm{H}_{\mathbf{k}}$ is an arbitrary function of $E_{\mathbf{k}}$, can only be of a form which satisfies

$$(\mathcal{A}_r)^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} = \delta_{mn} \, \mathcal{A}_r^{(n)} \, \Delta^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} , \tag{3.87}$$

with

$$\begin{aligned} \mathcal{A}_r^{(n)} = & \frac{1}{\nu \, (2n+1)} \int dK dK' dP dP' \, W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} k^{\langle \mu_1} \cdots k^{\mu_n \rangle} \\ & \times \left(\mathrm{H}_{\mathbf{k}} \, k_{\langle \mu_1} \cdots k_{\mu_n \rangle} + \mathrm{H}_{\mathbf{k}'} \, k'_{\langle \mu_1} \cdots k'_{\mu_n \rangle} - \mathrm{H}_{\mathbf{p}} \, p_{\langle \mu_1} \cdots p_{\mu_n \rangle} - \mathrm{H}_{\mathbf{p}'} \, p'_{\langle \mu_1} \cdots p'_{\mu_n \rangle} \right) , \end{aligned} \tag{3.88}$$

cf. Eqs. (4.46) and (4.48) in Chapter 4, where these relations are explicitly proven.

Similarly, multiplying Eq. (3.74) by $E_{\mathbf{k}}^r k^{\langle \nu \rangle}$ and integrating over dK, one obtains the equation for $\varphi_{\mathbf{k}}^{\rm v}$,

$$\begin{aligned} \alpha_r^{\rm v} = & \frac{\beta_0 \lambda}{3\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \\ & \times E_{\mathbf{k}}^{r-1} k_{\langle \mu \rangle} \left(\varphi_{\mathbf{k}}^{\rm v} k^{\langle \mu \rangle} + \varphi_{\mathbf{k}'}^{\rm v} k'^{\langle \mu \rangle} - \varphi_{\mathbf{p}}^{\rm v} p^{\langle \mu \rangle} - \varphi_{\mathbf{p}'}^{\rm v} p'^{\langle \mu \rangle} \right) . \end{aligned} \tag{3.89}$$

Finally, multiplying by $E_{\mathbf{k}}^r k^{\langle \alpha} k^{\beta \rangle}$ and once more integrating over dK, one obtains the equation for $\varphi_{\mathbf{k}}^{\rm t}$,

$$\begin{aligned} \alpha_r^{\rm t} = & \frac{\beta_0^2 \lambda}{5\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \\ & \times E_{\mathbf{k}}^{r-1} k_{\langle \mu} k_{\nu \rangle} \left(\varphi_{\mathbf{k}}^{\rm t} k^{\langle \mu} k^{\nu \rangle} + \varphi_{\mathbf{k}'}^{\rm t} k'^{\langle \mu} k'^{\nu \rangle} - \varphi_{\mathbf{p}}^{\rm t} p^{\langle \mu} p^{\nu \rangle} - \varphi_{\mathbf{p}'}^{\rm t} p'^{\langle \mu} p'^{\nu \rangle} \right) . \end{aligned} \tag{3.90}$$

Exercise 3.8: Verify Eqs. (3.84), (3.89), and (3.90) by an explicit calculation.

The next step is to expand the functions $\varphi^i_{\mathbf{k}}$ in terms of a **complete basis formed of** $E_{\mathbf{k}}$. The most common approach is to simply take a power series,

$$\varphi^i_{\mathbf{k}} = \sum_{n=0}^{N^i} \epsilon^i_n E^n_{\mathbf{k}} \,, \tag{3.91}$$

where for practical purposes the series is truncated at some power $N^i < \infty$. Note that for the scalar contribution ($i = \mathrm{s}$), the terms $n = 0, 1$ of the expansion can be incorporated into the homogeneous solution, while for the vector contribution ($i = \mathrm{v}$) the same occurs for the $n = 0$ term of the expansion. Overall, this reduces the problem of solving for the remaining coefficients ϵ^i_n. Inserting the expansion (3.91) into Eqs. (3.84), (3.89), and (3.90) leads to the following linear algebraic relations for the ϵ^i_n,

$$\alpha^i_r = \lambda \sum_{n=0}^{N^i} \mathcal{A}^i_{rn} \epsilon^i_n \,, \quad i = \mathrm{s}, \mathrm{v}, \mathrm{t} \,. \tag{3.92}$$

The dimension of the matrices $\mathcal{A}^i_{rn}$ is determined by N^i; the more terms one includes in the expansion (3.91), the higher the dimension of the matrix becomes. In practice, one can never include an infinite number of terms in the series, but one can check its convergence and truncate at that N^i for which $\varphi^i_{\mathbf{k}}$ reaches the required accuracy (up to a given value of momentum).

The matrices $\mathcal{A}^i_{rn}$ are defined as

$$\begin{aligned}
\mathcal{A}^{\mathrm{s}}_{rn} &= \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{kk'} \to \mathbf{pp'}} f_{0\mathbf{k}} f_{0\mathbf{k'}} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p'}} E^{r-1}_{\mathbf{k}} \left(E^n_{\mathbf{k}} + E^n_{\mathbf{k'}} - E^n_{\mathbf{p}} - E^n_{\mathbf{p'}} \right) \,, \\
\mathcal{A}^{\mathrm{v}}_{rn} &= \frac{\beta_0}{3\nu} \int dK dK' dP dP' W_{\mathbf{kk'} \to \mathbf{pp'}} f_{0\mathbf{k}} f_{0\mathbf{k'}} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p'}} \\
&\quad \times E^{r-1}_{\mathbf{k}} k_{\langle \mu \rangle} \left(E^n_{\mathbf{k}} k^{\langle \mu \rangle} + E^n_{\mathbf{k'}} k'^{\langle \mu \rangle} - E^n_{\mathbf{p}} p^{\langle \mu \rangle} - E^n_{\mathbf{p'}} p'^{\langle \mu \rangle} \right) \,, \\
\mathcal{A}^{\mathrm{t}}_{rn} &= \frac{\beta_0^2}{5\nu} \int dK dK' dP dP' W_{\mathbf{kk'} \to \mathbf{pp'}} f_{0\mathbf{k}} f_{0\mathbf{k'}} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p'}} \\
&\quad \times E^{r-1}_{\mathbf{k}} k_{\langle \mu} k_{\nu \rangle} \left(E^n_{\mathbf{k}} k^{\langle \mu} k^{\nu \rangle} + E^n_{\mathbf{k'}} k'^{\langle \mu} k'^{\nu \rangle} - E^n_{\mathbf{p}} p^{\langle \mu} p^{\nu \rangle} - E^n_{\mathbf{p'}} p'^{\langle \mu} p'^{\nu \rangle} \right) \,.
\end{aligned} \tag{3.93}$$

3.2.2 Minimal truncation scheme

The conservation of the number of particles (in binary collisions), energy, and momentum make the following components of the collision matrices $\mathcal{A}^{\mathrm{s}}$ and $\mathcal{A}^{\mathrm{v}}$ vanish: $\mathcal{A}^{\mathrm{s}}_{r0}$, $\mathcal{A}^{\mathrm{s}}_{r1}$, $\mathcal{A}^{\mathrm{s}}_{1n}$, $\mathcal{A}^{\mathrm{s}}_{2n}$, $\mathcal{A}^{\mathrm{v}}_{r0}$, and $\mathcal{A}^{\mathrm{v}}_{1n}$. Hence, these matrices have some rows and columns with all their entries being zero. Furthermore, from the definitions (3.81), (3.82), the identities (3.50), (3.51), and (3.53), as well as the thermodynamical relation (1.103) one can prove that the components α_1^{s}, α_2^{s}, and α_1^{v} are also zero and, consequently, Eq. (3.92) becomes a trivial identity for $r = 1, 2$ ($i = \mathrm{s}$) and $r = 1$ ($i = \mathrm{v}$). When inverting Eq. (3.92), these trivial lines and columns must be removed, making it possible to solve for all, except three, expansion coefficients ϵ_n^i. The three coefficients that are undetermined in Eq. (3.92) must be obtained from the matching conditions – they actually correspond to terms that can be incorporated into the homogeneous solution specified in Eq. (3.80). For this reason, when truncating the expansion (3.91), one must have at least $N^{\mathrm{s}} \geq 2$ and $N^{\mathrm{v}} \geq 1$.

Exercise 3.9:

(i) Using particle-number and energy-momentum conservation in binary elastic collisions it is fairly easy to see that $\mathcal{A}_{r0}^{\mathrm{s}}$, $\mathcal{A}_{r1}^{\mathrm{s}}$, and $\mathcal{A}_{r0}^{\mathrm{v}}$ vanish identically. Using the symmetries of $W_{\mathbf{kk}'\to\mathbf{pp}'}$ and the properties of the equilibrium distribution (3.22), show that also $\mathcal{A}_{1n}^{\mathrm{s}}$, $\mathcal{A}_{2n}^{\mathrm{s}}$, and $\mathcal{A}_{1n}^{\mathrm{v}}$ vanish.

(ii) Following the hints in the text, show that α_1^{s}, α_2^{s}, and α_1^{v} are zero.

In order to better understand this, let us assume the simplest possible truncation scheme for Eq. (3.91), i.e., $N^{\mathrm{s}} = 2$, $N^{\mathrm{v}} = 1$, and $N^{\mathrm{t}} = 0$. Then, the non-trivial lines of Eq. (3.92) (obtained by removing all contributions related to the homogeneous solution) lead to the following equations for ϵ_2^{s}, ϵ_1^{v}, and ϵ_0^{t} ,

$$\epsilon_2^{\mathrm{s}} = \frac{\alpha_0^{\mathrm{s}}}{\lambda \mathcal{A}_{02}^{\mathrm{s}}} \ , \quad \epsilon_1^{\mathrm{v}} = \frac{\alpha_0^{\mathrm{v}}}{\lambda \mathcal{A}_{01}^{\mathrm{v}}} \ , \quad \epsilon_0^{\mathrm{t}} = \frac{\alpha_0^{\mathrm{t}}}{\lambda \mathcal{A}_{00}^{\mathrm{t}}} \ , \tag{3.94}$$

leaving ϵ_0^{s}, ϵ_1^{s}, and ϵ_0^{v} still undetermined. On the other hand, the matching conditions (3.61) – (3.63), together with the orthogonality relation (3.85), provide the following additional constraints that must be satisfied by $\varphi_{\mathbf{k}}^{\mathrm{s}}$ and $\varphi_{\mathbf{k}}^{\mathrm{v}}$

$$\int dK \; E_{\mathbf{k}} \varphi_{\mathbf{k}}^{\mathrm{s}} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \;=\; 0 \, , \tag{3.95}$$

$$\int dK \; E_{\mathbf{k}}^2 \varphi_{\mathbf{k}}^{\mathrm{s}} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \;=\; 0 \, , \tag{3.96}$$

$$\int dK \; E_{\mathbf{k}} \left(\Delta_{\mu\nu} k^\mu k^\nu\right) \varphi_{\mathbf{k}}^{\mathrm{v}} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \;=\; 0 \, . \tag{3.97}$$

These equations relate the so far undetermined coefficients to ϵ_2^{s} and ϵ_1^{v},

$$\begin{pmatrix} J_{10} & J_{20} \\ J_{20} & J_{30} \end{pmatrix} \begin{pmatrix} \epsilon_0^{\mathrm{s}} \\ \epsilon_1^{\mathrm{s}} \end{pmatrix} = -\begin{pmatrix} J_{30} \\ J_{40} \end{pmatrix} \epsilon_2^{\mathrm{s}} \, , \tag{3.98}$$

$$J_{31} \epsilon_0^{\mathrm{v}} = -J_{41} \epsilon_1^{\mathrm{v}} \, . \tag{3.99}$$

With Eq. (3.94), the solution is

$$\epsilon_0^{\rm s} = \frac{D_{30}}{D_{20}} \frac{\alpha_0^{\rm s}}{\lambda \mathcal{A}_{02}^{\rm s}} , \tag{3.100}$$

$$\epsilon_1^{\rm s} = \frac{G_{23}}{D_{20}} \frac{\alpha_0^{\rm s}}{\lambda \mathcal{A}_{02}^{\rm s}} , \tag{3.101}$$

$$\epsilon_0^{\rm v} = -\frac{J_{41}}{J_{31}} \frac{\alpha_0^{\rm v}}{\lambda \mathcal{A}_{01}^{\rm v}} , \tag{3.102}$$

where we used the thermodynamic functions (3.55).

Exercise 3.10: Prove Eqs. (3.95) – (3.102).

With all the coefficients of the truncated expansion solved for, we can write down the first-order solutions to the bulk-viscous pressure,

$$\begin{aligned} (\Pi)^{(1)} &= -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle^{(1)} = -\frac{1}{3} \int dK \, (\Delta^{\mu\nu} k_\mu k_\nu) \, f_{\mathbf{k}}^{(1)} \\ &= -\frac{1}{3} \int dK \, (\Delta^{\mu\nu} k_\mu k_\nu) \, \varphi_{\mathbf{k}}^{\rm s} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \, \hat{\theta} \\ &= \left(\frac{J_{21} D_{30} + J_{31} G_{23}}{D_{20}} + J_{41} \right) \frac{\alpha_0^{\rm s}}{\lambda \mathcal{A}_{02}^{\rm s}} \, \hat{\theta} , \end{aligned} \tag{3.103}$$

to the diffusion 4-current,

$$\begin{aligned} (n^\mu)^{(1)} &= \left\langle k^{\langle \mu \rangle} \right\rangle^{(1)} = \int dK \, k^{\langle \mu \rangle} f_{\mathbf{k}}^{(1)} \\ &= \beta_0 \int dK \, k^{\langle \mu \rangle} k^{\langle \nu \rangle} \varphi_{\mathbf{k}}^{\rm v} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \, \hat{\nabla}_\nu \alpha_0 \\ &= \frac{D_{31}}{J_{31}} \frac{\beta_0 \alpha_0^{\rm v}}{\lambda \mathcal{A}_{01}^{\rm v}} \, \hat{\nabla}^\mu \alpha_0 , \end{aligned} \tag{3.104}$$

and to the shear-stress tensor,

$$\begin{aligned} (\pi^{\mu\nu})^{(1)} &= \left\langle k^{\langle \mu} k^{\nu \rangle} \right\rangle^{(1)} = \int dK \, k^{\langle \mu} k^{\nu \rangle} \, f_{\mathbf{k}}^{(1)} \\ &= \beta_0^2 \int dK \, k^{\langle \mu} k^{\nu \rangle} k^{\langle \alpha} k^{\beta \rangle} \varphi_{\mathbf{k}}^{\rm t} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \, \hat{\sigma}_{\alpha\beta} = 2 J_{42} \frac{\beta_0^2 \alpha_0^{\rm t}}{\lambda \mathcal{A}_{00}^{\rm t}} \, \hat{\sigma}^{\mu\nu} . \end{aligned} \tag{3.105}$$

Exercise 3.11: Using the orthogonality condition (3.85), prove Eqs. (3.103) – (3.105).

Therefore, as already mentioned at the beginning of this section, in the **Chapman-Enskog expansion relativistic Navier-Stokes theory appears as the first-order**

truncation of an expansion in powers of gradients of temperature, chemical potential, and velocity. With $\mathrm{Kn} = \lambda/L$, $\hat{\theta} = L\theta$, $\hat{\nabla}^\mu = L\nabla^\mu$, and $\hat{\sigma}^{\mu\nu} = L\sigma^{\mu\nu}$, we then identify by comparison with Eqs. (1.85), (1.86), and (1.87) the microscopic formulas for bulk-viscosity, particle-diffusion, and shear-viscosity coefficients as

$$\zeta = -\left(\frac{J_{21}D_{30} + J_{31}G_{23}}{D_{20}} + J_{41}\right)\frac{\alpha_0^{\mathrm{s}}}{\mathcal{A}_{02}^{\mathrm{s}}} , \tag{3.106}$$

$$\varkappa = \frac{D_{31}}{J_{31}}\frac{\beta_0\alpha_0^{\mathrm{v}}}{\mathcal{A}_{01}^{\mathrm{v}}} , \tag{3.107}$$

$$\eta = J_{42}\frac{\beta_0^2\alpha_0^{\mathrm{t}}}{\mathcal{A}_{00}^{\mathrm{t}}} . \tag{3.108}$$

We thus succeeded in our goal to derive **expressions for the first-order transport coefficients** via an explicit calculation using an underlying **microscopic theory**.

3.3 Israel-Stewart theory

05/19/2022

In this section we explain **Israel's and Stewart's 14-moment approximation** as it was originally proposed in Ref. [4]. This method is **conceptually different** from the Chapman-Enskog theory and is not constructed from an expansion in a small parameter, such as the Knudsen number. The main assumption made in this approach is that, since a fluid-dynamical description requires only the conserved currents N^μ and $T^{\mu\nu}$ to specify the state of the fluid, the single-particle distribution function should also be well described by these fields. In practice, this is accomplished by expanding $f_{\mathbf{k}}$ in terms of its moments and truncate it in such a way that it **only** depends on N^μ and $T^{\mu\nu}$. We shall describe the details of this approach next.

3.3.1 14-moment approximation

The starting point is the Israel-Stewart Ansatz for the non-equilibrium single-particle distribution function

$$f_{\mathbf{k}} = \left[\exp\left(-y_{\mathbf{k}}\right) + a\right]^{-1} , \tag{3.109}$$

where $a = 1$ ($a = -1$) for fermions (bosons) and $a = 0$ for a classical gas. In the traditional Israel-Stewart approach, the parameter $y_{\mathbf{k}}$ is expanded in momentum space around its local-equilibrium value, $y_{0\mathbf{k}} = \alpha_0 - \beta_0 u_\mu k^\mu$, in terms of a series of (reducible) **Lorentz-tensors** formed from the particle 4-momentum k^μ, i.e., 1, k^μ, $k^\mu k^\nu$, ... Therefore,

$$\delta y_{\mathbf{k}} \equiv y_{\mathbf{k}} - y_{0\mathbf{k}} = \epsilon + k^\mu \epsilon_\mu + k^\mu k^\nu \epsilon_{\mu\nu} + k^\mu k^\nu k^\lambda \epsilon_{\mu\nu\lambda} + \cdots . \tag{3.110}$$

For small momenta, the non-equilibrium single-particle distribution function can be further expanded around the local-equilibrium state

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\delta y_{\mathbf{k}} + \mathcal{O}\left(\delta y_{\mathbf{k}}^2\right) . \tag{3.111}$$

In kinetic theory, the particle 4-current and the energy-momentum tensor are given by Eqs. (3.4), (3.5). Substituting Eqs. (3.110), (3.111) into these equations, we express the conserved currents in terms of the expansion coefficients $\epsilon_{\mu_1 \cdots \mu_m}$,

$$N^\mu = I_0^\mu + \epsilon J_0^\mu + J_0^{\mu\nu}\epsilon_\nu + J_0^{\mu\nu\lambda}\epsilon_{\nu\lambda} + J_0^{\mu\nu\lambda\rho}\epsilon_{\nu\lambda\rho} + \cdots \ , \tag{3.112}$$
$$T^{\mu\nu} = I_0^{\mu\nu} + \epsilon J_0^{\mu\nu} + J_0^{\mu\nu\lambda}\epsilon_\lambda + J_0^{\mu\nu\lambda\rho}\epsilon_{\lambda\rho} + J_0^{\mu\nu\lambda\rho\sigma}\epsilon_{\lambda\rho\sigma} + \cdots \ , \tag{3.113}$$

where we introduced the tensors,

$$I_0^{\alpha_1 \ldots \alpha_n} \equiv \int dK \, k^{\alpha_1} \ldots k^{\alpha_n} \, f_{0\mathbf{k}} \ , \tag{3.114}$$
$$J_0^{\alpha_1 \ldots \alpha_n} \equiv \int dK \, k^{\alpha_1} \ldots k^{\alpha_n} \, f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \ . \tag{3.115}$$

The terms that are not multiplied by the expansion coefficients ϵ, ϵ^μ, and $\epsilon^{\mu\nu}$ are identified as the equilibrium currents (1.21) and (1.23),

$$N^\mu_{\text{ideal}} = I_0^\mu \ , \tag{3.116}$$
$$T^{\mu\nu}_{\text{ideal}} = I_0^{\mu\nu} \ . \tag{3.117}$$

Naturally, the remaining terms originate from the **dissipative corrections** to $f_{\mathbf{k}}$. The tensors (3.114), (3.115) depend only on the thermal potential, α_0, the inverse temperature, β_0, and the 4-velocity, u^μ. Therefore, their tensor structure must be constructed solely from combinations of u^μ and the metric tensor, $g^{\mu\nu}$. For the sake of convenience, we tensor decompose the moments (3.114), (3.115) in terms of the fluid velocity u^μ and the projection operator $\Delta^{\mu\nu}$. One then obtains

$$\begin{aligned}
I_0^\mu &= I_{10} u^\mu \ , \\
I_0^{\mu\nu} &= I_{20} u^\mu u^\nu - I_{21} \Delta^{\mu\nu} \ , \\
J_0^\mu &= J_{10} u^\mu \ , \\
J_0^{\mu\nu} &= J_{20} u^\mu u^\nu - J_{21} \Delta^{\mu\nu} \ , \\
J_0^{\mu\nu\lambda} &= J_{30} u^\mu u^\nu u^\lambda - 3 J_{31} u^{(\mu} \Delta^{\nu\lambda)} \ , \\
J_0^{\mu\nu\lambda\rho} &= J_{40} u^\mu u^\nu u^\lambda u^\rho - 6 J_{41} u^{(\mu} u^\nu \Delta^{\lambda\rho)} + 3 J_{42} \Delta^{(\mu\nu} \Delta^{\lambda\rho)} \ ,
\end{aligned} \tag{3.118}$$

where the thermodynamic integrals I_{nq} and J_{nq} were defined in Eqs. (3.48) and (3.49). Due to Eq. (3.50), Eqs. (3.116) and (3.117) reduce to what was derived in Chapter 1, Eqs. (1.21) and (1.23). The parentheses around the Lorentz indices indicate symmetrization of the tensor with respect to all indices (where only independent terms are counted), whereas the prefactor counts the number of (independent) terms resulting from this symmetrization.

Exercise 3.12: Prove Eqs. (3.118).

Obtaining the expansion coefficients $\epsilon^{\mu_1\cdots\mu_m}$ is not a trivial task. In the Chapman-Enskog expansion, these were expressed in terms of gradients of α_0, β_0, and u^μ using perturbation theory, by assuming an expansion in powers of Knudsen number. Following the developments made by Grad in the non-relativistic regime, Israel and Stewart proposed a different approach. Instead of a gradient expansion, they suggested an *ad hoc* **truncation of the expansion (3.110) at second order in momentum**, i.e., one only keeps the tensors 1, k^μ, and $k^\mu k^\nu$ in the expansion,

$$\delta y_{\mathbf{k}} \approx \epsilon + k^\mu \epsilon_\mu + k^\mu k^\nu \epsilon_{\mu\nu} \; . \tag{3.119}$$

Without loss of generality, $\epsilon_{\mu\nu}$ can be assumed to be symmetric and traceless, i.e., $\epsilon^{\mu\nu} = \epsilon^{\nu\mu}$ and $\epsilon^\mu_\mu = 0$ (the symmetry under $\mu \leftrightarrow \nu$ is obvious, since any antisymmetric part would vanish in the contraction with $k^\mu k^\nu$, while the trace of $\epsilon_{\mu\nu}$ can always be absorbed in the scalar coefficient ϵ). This leaves us with 14 unknown degrees of freedom in the expansion coefficients ϵ, ϵ_μ, and $\epsilon_{\mu\nu}$. We note that the equilibrium variables introduced in $y_{0\mathbf{k}}$, i.e., α_0, β_0, and u^μ, are defined by the matching conditions (3.23) and the definition of the local rest frame, e.g. the Landau choice (3.16).

3.3.2 Matching procedure

The 14 degrees of freedom of the **truncated** expansion can be uniquely related to the 14 components of the particle 4-current, N^μ, and the energy-momentum tensor, $T^{\mu\nu}$, the so-called **matching procedure**. This procedure will generate a single-particle distribution function that is **completely determined by the components of the conserved currents**. Israel and Stewart expected this to be a good approximation in the fluid-dynamical regime, where N^μ and $T^{\mu\nu}$ are considered to be sufficient to describe the state of the system.

The expansion coefficients can be solved using the constraints already derived in Eqs. (3.16), (3.21), and (3.23), that is

$$\Delta_{\mu\nu} N^\nu = n_\mu \; , \tag{3.120}$$

$$\Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} = \pi^{\mu\nu} \; , \tag{3.121}$$

$$-\frac{1}{3}\Delta_{\mu\nu}\left(T^{\mu\nu} - T^{\mu\nu}_{\text{ideal}}\right) = \Pi \; , \tag{3.122}$$

$$u_\mu\left(N^\mu - N^\mu_{\text{ideal}}\right) = 0 \; , \tag{3.123}$$

$$u_\nu\left(T^{\mu\nu} - T^{\mu\nu}_{\text{ideal}}\right) = 0 \; . \tag{3.124}$$

By solving this set of 14 linear equations, the expansion coefficients ϵ, ϵ_μ, and $\epsilon_{\mu\nu}$ can be expressed in terms of the 14 (independent) variables α_0, β_0, Π, u^μ, n^μ, and $\pi^{\mu\nu}$. The relations (3.120) – (3.122) define the dissipative currents while the restrictions (3.123), (3.124) come from the matching conditions and the Landau choice for the local rest frame. If we were using the Eckart frame, for example, we would have to use, instead of Eqs. (3.123) and (3.124),

$$N^\mu - N^\mu_{\text{ideal}} = 0 \; , \tag{3.125}$$

$$u_\mu u_\nu\left(T^{\mu\nu} - T^{\mu\nu}_{\text{ideal}}\right) = 0 \; . \tag{3.126}$$

Combining Eqs. (3.116), (3.117), and (3.118), and truncating the expansion at terms quadratic in 4-momentum, the conditions (3.120), (3.121), and (3.122) imply that

$$n^\mu = -J_{21}\Delta^{\mu\nu}\epsilon_\nu - 2J_{31}\Delta^{\mu\nu}u^\lambda\epsilon_{\nu\lambda} \,, \tag{3.127}$$

$$\pi^{\mu\nu} = 2J_{42}\Delta^{\mu\nu}_{\lambda\rho}\epsilon^{\lambda\rho} \,, \tag{3.128}$$

$$\Pi = J_{21}\epsilon + J_{31}u^\lambda\epsilon_\lambda + \left(J_{41} + \frac{5}{3}J_{42}\right)u^\lambda u^\rho\epsilon_{\lambda\rho} \,, \tag{3.129}$$

where in the last equation we have used the fact that $\epsilon_{\lambda\rho}$ is traceless.

Exercise 3.13: Prove Eqs. (3.127) – (3.129).

The expressions (3.127) – (3.129) motivate the following Ansatz for the expansion coefficients,

$$\begin{aligned} \epsilon &= E_0\Pi \,, \\ \epsilon_\lambda &= D_0\Pi u_\lambda + D_1 n_\lambda \,, \\ \epsilon_{\lambda\rho} &= B_0\left(\Delta_{\lambda\rho} - 3u_\lambda u_\rho\right)\Pi + B_1 u_{(\lambda}n_{\rho)} + B_2\pi_{\lambda\rho} \,. \end{aligned} \tag{3.130}$$

Then, in order to determine the single-particle distribution function (3.111), we have to compute the coefficients E_0, D_0, D_1, B_0, B_1, and B_2. This can be done by substituting the Ansatz (3.130) into Eqs. (3.123), (3.124), (3.127), (3.128), and (3.129), leading to

$$J_{21}D_1 + J_{31}B_1 = -1 \,, \tag{3.131}$$

$$2J_{42}B_2 = 1 \,, \tag{3.132}$$

$$J_{21}E_0 + J_{31}D_0 - (3J_{41} + 5J_{42})B_0 = 1 \,, \tag{3.133}$$

$$J_{10}E_0 + J_{20}D_0 - 3(J_{30} + J_{31})B_0 = 0 \,, \tag{3.134}$$

$$J_{31}D_1 + J_{41}B_1 = 0 \,, \tag{3.135}$$

$$J_{20}E_0 + J_{30}D_0 - 3(J_{40} + J_{41})B_0 = 0 \,. \tag{3.136}$$

Equations (3.131), (3.132), and (3.133) come directly from (3.127), (3.128), and (3.129), respectively. Equations (3.134), (3.135), and (3.136) are consequences of Eqs. (3.123) and (3.124). The solution of this set of equations is

$$\begin{aligned} \frac{E_0}{3B_0} &= m^2 + 4\,\frac{J_{31}J_{30} - J_{41}J_{20}}{D_{20}} \equiv -C_1 \\ \frac{D_0}{3B_0} &= -4\,\frac{J_{31}J_{20} - J_{41}J_{10}}{D_{20}} \equiv -C_2 \,, \\ B_0 &= -\frac{1}{3C_1J_{21} + 3C_2J_{31} + 3J_{41} + 5J_{42}} \,, \\ B_1 &= \frac{J_{31}}{D_{31}} \,, \\ D_1 &= -\frac{J_{41}}{D_{31}} \,, \\ B_2 &= \frac{1}{2J_{42}} \,. \end{aligned} \tag{3.137}$$

In the first two equations we have used the identity (3.54).

Exercise 3.14: Prove Eqs. (3.131) – (3.137).

3.3.3 Moment equations

Now that the momentum distribution function is known, we can calculate the equations of motion satisfied by the dissipative currents. Israel and Stewart derived the equations of motion for Π, n^μ, and $\pi^{\mu\nu}$ from the **second moment of the Boltzmann equation** [2, 3, 4, 11]

$$\partial_\mu \left\langle k^\mu k^\nu k^\lambda \right\rangle = \int dK k^\nu k^\lambda C\left[f\right] \ . \tag{3.138}$$

In this case, the equations for Π, n^μ, and $\pi^{\mu\nu}$ are obtained from the projections $u_\nu u_\lambda \partial_\mu \left\langle k^\mu k^\nu k^\lambda \right\rangle$, $\Delta^\alpha_\lambda u_\nu \partial_\mu \left\langle k^\mu k^\nu k^\lambda \right\rangle$, and $\Delta^{\alpha\beta}_{\nu\lambda} \partial_\mu \left\langle k^\mu k^\nu k^\lambda \right\rangle$, respectively, together with the 14-moment approximation, Eqs. (3.111), (3.119), and (3.130). These equations determine the time evolution of Π, n^μ, and $\pi^{\mu\nu}$ through their (projected) comoving derivatives, $D\Pi$, $Dn^{\langle\mu\rangle} \equiv \Delta^\mu_\nu Dn^\nu$, and $D\pi^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta}$, respectively.

However, extracting the equations of motion from the second moment of the Boltzmann equation is just a choice. The 14-moment approximation itself does not specify which moment of the Boltzmann equation should be chosen to close the conservation laws – once the 14-moment approximation is employed **any** moment of the Boltzmann equation will lead to a closed set of equations for the dissipative currents [12, 13]. This happens because the distribution function itself is already a unique function of the fluid-dynamical variables. In general, the form of the equations of motion will always be the same, but the **transport coefficients** appearing in the final equations will **depend on the choice of the moment** [13]. This happens because the 14-moment approximation is not a truncation in Knudsen number and there is no moment of the Boltzmann equation that carries the complete contribution to each term.

In the following, we will slightly deviate from Israel and Stewart and derive the equations satisfied by Π, n^μ, and $\pi^{\mu\nu}$ following Ref. [12], where it was shown that using the second moment of the Boltzmann equation to obtain the equations of motion for the dissipative currents introduces an unnecessary ambiguity in the derivation of fluid dynamics. This happens because the exact equations of motion for Π, n^μ, and $\pi^{\mu\nu}$ are known and can be derived **directly** from the Boltzmann equation. Therefore, instead of choosing an **arbitrary** moment of the Boltzmann equation to derive such equations, one should just use the **exact** equations of motion for the dissipative currents.

Following Ref. [12], the comoving derivatives of Π, n^μ, and $\pi^{\mu\nu}$ are calculated exactly using

$$D\Pi = -\frac{1}{3} m^2 \int dK \, D\delta f_{\mathbf{k}} \ , \tag{3.139}$$

$$Dn^{\langle\mu\rangle} = \int dK \, k^{\langle\mu\rangle} \, D\delta f_{\mathbf{k}} \ , \tag{3.140}$$

$$D\pi^{\langle\mu\nu\rangle} = \int dK \, k^{\langle\mu} k^{\nu\rangle} \, D\delta f_{\mathbf{k}} \ . \tag{3.141}$$

Note that in the first equation, we used the matching condition $\varepsilon = \varepsilon_0$, such that $\langle E_{\mathbf{k}}^2 \rangle_\delta = 0$. Then, replacing $D\delta f_{\mathbf{k}}$ by using the Boltzmann equation (3.1) in the form

$$D\delta f_{\mathbf{k}} = -Df_{0\mathbf{k}} - \frac{1}{E_{\mathbf{k}}} k^{\langle\mu\rangle} \nabla_\mu f_{\mathbf{k}} + \frac{1}{E_{\mathbf{k}}} C[f] \,, \tag{3.142}$$

we obtain the **exact** equations

$$\begin{aligned}
D\Pi + C &= -\frac{m^2}{3}\alpha_0^{\mathrm{s}}\theta - \left(\frac{2}{3} - \frac{m^2}{3}\frac{G_{20}}{D_{20}}\right)\Pi\theta - \frac{m^2}{3}\frac{G_{20}}{D_{20}}\pi^{\mu\nu}\sigma_{\mu\nu} - \frac{m^2}{3}\frac{G_{30}}{D_{20}}\partial_\mu n^\mu \\
&\quad + \frac{m^4}{9}\left\langle E_{\mathbf{k}}^{-2}\right\rangle_\delta \theta + \frac{m^2}{3}\left\langle E_{\mathbf{k}}^{-2} k^{\langle\mu} k^{\nu\rangle}\right\rangle_\delta \sigma_{\mu\nu} + \frac{m^2}{3}\nabla_\mu \left\langle E_{\mathbf{k}}^{-1} k^{\langle\mu\rangle}\right\rangle_\delta \,, \qquad (3.143)\\
Dn^{\langle\mu\rangle} - C^\mu &= \alpha_0^{\mathrm{v}}\nabla^\mu\alpha_0 + n^\nu\omega^\mu_{\ \nu} - n^\mu\theta - \frac{3}{5}n^\nu\sigma^\mu_\nu \\
&\quad + \frac{\beta_0 J_{21}}{\varepsilon_0 + P_0}\left(\Pi Du^\mu - \nabla^\mu\Pi - \pi^{\mu\nu}Du_\nu + \Delta^\mu_\nu\nabla_\lambda\pi^{\lambda\nu}\right) \\
&\quad - \frac{m^2}{3}\left\langle E_{\mathbf{k}}^{-2}k^{\langle\mu\rangle}\right\rangle_\delta \theta - \Delta^\mu_\lambda\nabla_\nu\left\langle E_{\mathbf{k}}^{-1}k^{\langle\lambda}k^{\nu\rangle}\right\rangle_\delta - \frac{2m^2}{5}\left\langle E_{\mathbf{k}}^{-2}k^{\langle\nu\rangle}\right\rangle_\delta \sigma^\mu_\nu \\
&\quad - \frac{m^2}{3}\nabla^\mu\left\langle E_{\mathbf{k}}^{-1}\right\rangle_\delta - \left\langle E_{\mathbf{k}}^{-2}k^{\langle\mu}k^\nu k^{\lambda\rangle}\right\rangle_\delta \sigma_{\lambda\nu} \,, \qquad (3.144)\\
D\pi^{\langle\mu\nu\rangle} - C^{\mu\nu} &= \alpha_0^{\mathrm{t}}\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + 2\pi^{\lambda\langle\mu}\omega^{\nu\rangle}_{\ \lambda} + \frac{6}{5}\Pi\sigma^{\mu\nu} \\
&\quad - \frac{4m^2}{7}\Delta^{\mu\nu}_{\alpha\beta}\left\langle E_{\mathbf{k}}^{-2}k^{\langle\lambda}k^{\alpha\rangle}\right\rangle_\delta \sigma^\beta_\lambda - \frac{2m^4}{15}\left\langle E_{\mathbf{k}}^{-2}\right\rangle_\delta \sigma^{\mu\nu} \\
&\quad - \frac{2m^2}{5}\Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha\left\langle E_{\mathbf{k}}^{-1}k^{\langle\beta\rangle}\right\rangle_\delta - \frac{m^2}{3}\left\langle E_{\mathbf{k}}^{-2}k^{\langle\mu}k^{\nu\rangle}\right\rangle_\delta \theta \\
&\quad - \left\langle E_{\mathbf{k}}^{-2}k^{\langle\mu}k^\nu k^\lambda k^{\rho\rangle}\right\rangle_\delta \sigma_{\lambda\rho} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\left\langle E_{\mathbf{k}}^{-1}k^{\langle\alpha}k^\beta k^{\lambda\rangle}\right\rangle_\delta \,. \qquad (3.145)
\end{aligned}$$

The functions α_0^{i} were defined previously in the context of the Chapman-Enskog expansion in Eqs. (3.81) – (3.83). We also used the equations of motion (3.46), (3.58), and (3.59) (in their unscaled versions), as well as the thermodynamic identity (1.103). We also used the relations

$$\begin{aligned}
k^{\langle\mu\rangle}k^{\langle\nu\rangle} &= k^{\langle\mu}k^{\nu\rangle} + \frac{1}{3}\Delta^{\mu\nu}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right) \,, \qquad (3.146)\\
k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\lambda\rangle} &= k^{\langle\mu}k^\nu k^{\lambda\rangle} + \frac{1}{5}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)\left(\Delta^{\mu\nu}k^{\langle\lambda\rangle} + \Delta^{\mu\lambda}k^{\langle\nu\rangle} + \Delta^{\nu\lambda}k^{\langle\mu\rangle}\right) \,, \qquad (3.147)\\
k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\lambda\rangle}k^{\langle\rho\rangle} &= k^{\langle\mu}k^\nu k^\lambda k^{\rho\rangle} + \frac{1}{7}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)\left(\Delta^{\mu\nu}k^{\langle\lambda}k^{\rho\rangle} + \Delta^{\mu\lambda}k^{\langle\nu}k^{\rho\rangle} + \Delta^{\mu\rho}k^{\langle\nu}k^{\lambda\rangle}\right.\\
&\quad \left. + \Delta^{\nu\lambda}k^{\langle\mu}k^{\rho\rangle} + \Delta^{\nu\rho}k^{\langle\mu}k^{\lambda\rangle} + \Delta^{\lambda\rho}k^{\langle\mu}k^{\nu\rangle}\right) \\
&\quad + \frac{1}{15}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^2\left(\Delta^{\mu\nu}\Delta^{\lambda\rho} + \Delta^{\mu\lambda}\Delta^{\nu\rho} + \Delta^{\mu\rho}\Delta^{\nu\lambda}\right) \,, \qquad (3.148)
\end{aligned}$$

where $k^{\langle\mu}k^\nu k^{\lambda\rangle} = \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}k^\alpha k^\beta k^\gamma$, $k^{\langle\mu}k^\nu k^\lambda k^{\rho\rangle} = \Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}k^\alpha k^\beta k^\gamma k^\delta$, with the projection operators $\Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}$ and $\Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}$ defined in App. 4.7. Finally, we also defined the collision integrals,

$$C = \frac{m^2}{3} \int dK \, E_{\mathbf{k}}^{-1} C[f] \ , \tag{3.149}$$

$$C^{\mu} = \int dK \, E_{\mathbf{k}}^{-1} k^{\langle\mu\rangle} C[f] \ , \tag{3.150}$$

$$C^{\mu\nu} = \int dK \, E_{\mathbf{k}}^{-1} k^{\langle\mu} k^{\nu\rangle} C[f] \ . \tag{3.151}$$

Exercise 3.15: Prove Eqs. (3.143) – (3.145).

3.3.4 Calculation of the collision integrals

In Israel-Stewart theory, the distribution function was expanded in powers of $\delta y_{\mathbf{k}}$, retaining only the **first-order correction**, cf. Eq. (3.111). For the sake of consistency, the **same approximation** has to be made in the calculation of the collision terms (3.149) – (3.151). In the end, this is equivalent to using the **linearized collision operator** in the integrals above. Up to terms of order $\mathcal{O}(\delta y_{\mathbf{k}}^2)$, the collision terms reduce to

$$C = \frac{m^2}{3\nu} \int dK dK' dP dP' \, \frac{1}{E_{\mathbf{k}}} W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \left(y_{\mathbf{p}} + y_{\mathbf{p}'} - y_{\mathbf{k}} - y_{\mathbf{k}'} \right) \ , \tag{3.152}$$

$$C^{\mu} = \frac{1}{\nu} \int dK dK' dP dP' \, \frac{k^{\langle\mu\rangle}}{E_{\mathbf{k}}} W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \left(y_{\mathbf{p}} + y_{\mathbf{p}'} - y_{\mathbf{k}} - y_{\mathbf{k}'} \right) \ , \tag{3.153}$$

$$C^{\mu\nu} = \frac{1}{\nu} \int dK dK' dP dP' \, \frac{k^{\langle\mu} k^{\nu\rangle}}{E_{\mathbf{k}}} W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \left(y_{\mathbf{p}} + y_{\mathbf{p}'} - y_{\mathbf{k}} - y_{\mathbf{k}'} \right) \ . \tag{3.154}$$

Here, we have written $y_{\mathbf{k}}$ instead of $\delta y_{\mathbf{k}}$, since $y_{0\mathbf{k}} = \alpha_0 - \beta_0 E_{\mathbf{k}}$ consists of two collision invariants (1 and $E_{\mathbf{k}}$), and consequently $y_{0\mathbf{p}} + y_{0\mathbf{p}'} - y_{0\mathbf{k}} - y_{0\mathbf{k}'} = 0$.

Exercise 3.16: Prove Eqs. (3.152) – (3.154).

Inserting the moment expansion (3.119) of $\delta y_{\mathbf{k}}$ into the collision integrals, they further simplify to

$$C = -\frac{4}{9} m^2 \mathcal{A}^{\mathrm{s}}_{02} u_\alpha u_\beta \epsilon^{\alpha\beta} \ , \tag{3.155}$$

$$C^{\mu} = -2 \frac{\mathcal{A}^{\mathrm{v}}_{01}}{\beta_0} u_\alpha \Delta^{\mu}_{\beta} \epsilon^{\alpha\beta} \ , \tag{3.156}$$

$$C^{\mu\nu} = -\frac{\mathcal{A}^{\mathrm{t}}_{00}}{\beta_0^2} \Delta^{\mu\nu}_{\alpha\beta} \epsilon^{\alpha\beta} \ , \tag{3.157}$$

where we have used the definition (3.93) of the matrix elements $\mathcal{A}^{i}_{mn}$, $i =$s,v,t. We note that the only term from the moment expansion that contributes to the collision integral is $k_\alpha k_\beta \epsilon^{\alpha\beta}$. The remaining two terms, ϵ and $\epsilon_\mu k^\mu$, are proportional to collision invariants, and consequently, make the collision integrals vanish. Furthermore, we have made use of the orthogonality relation (3.87) and of the tracelessness of $\epsilon^{\alpha\beta}$.

Exercise 3.17: Prove Eqs. (3.155) – (3.157).

Then, using Eq. (3.130), which expresses the expansion coefficient $\epsilon^{\mu\nu}$ in terms of Π, n^μ, and $\pi^{\mu\nu}$, we obtain the final expression for the coefficients C, C^μ, and $C^{\mu\nu}$ in the 14-moment approximation

$$C = \frac{4}{3} m^2 B_0 \mathcal{A}^{\rm s}_{02} \Pi \,, \tag{3.158}$$

$$C^\mu = -B_1 \frac{\mathcal{A}^{\rm v}_{01}}{\beta_0} n^\mu \,, \tag{3.159}$$

$$C^{\mu\nu} = -B_2 \frac{\mathcal{A}^{\rm t}_{00}}{\beta_0^2} \pi^{\mu\nu} \,. \tag{3.160}$$

The thermodynamic functions B_0, B_1, and B_2 were calculated in Eq. (3.137) while the matrices $\mathcal{A}^i_{nm}$ are defined in Eq. (3.93). The dependence of the collision integrals on the particle cross sections come exclusively from $\mathcal{A}^i_{nm}$.

Exercise 3.18: Prove Eqs. (3.158) – (3.160).

In addition, the following terms appearing in the exact equations of motion can be computed using Eqs. (3.85), (3.119), and (3.130) as

$$\left\langle E_{\mathbf{k}}^{-1} \right\rangle_\delta = \left[J_{-1,0} E_0 + J_{00} D_0 - 3 \left(J_{11} + J_{10} \right) B_0 \right] \Pi \equiv \gamma_1^\Pi \Pi \,, \tag{3.161}$$

$$\left\langle E_{\mathbf{k}}^{-2} \right\rangle_\delta = \left[J_{-2,0} E_0 + J_{-1,0} D_0 - 3 \left(J_{01} + J_{00} \right) B_0 \right] \Pi \equiv \gamma_2^\Pi \Pi \,, \tag{3.162}$$

$$\left\langle E_{\mathbf{k}}^{-1} k^{\langle\mu\rangle} \right\rangle_\delta = - \left(J_{11} D_1 + J_{21} B_1 \right) n^\mu \equiv \gamma_1^n n^\mu \,, \tag{3.163}$$

$$\left\langle E_{\mathbf{k}}^{-2} k^{\langle\mu\rangle} \right\rangle_\delta = - \left(J_{01} D_1 + J_{11} B_1 \right) n^\mu \equiv \gamma_2^n n^\mu \,, \tag{3.164}$$

$$\left\langle E_{\mathbf{k}}^{-1} k^{\langle\mu} k^{\nu\rangle} \right\rangle_\delta = \frac{J_{32}}{J_{42}} \pi^{\mu\nu} \equiv \gamma_1^\pi \pi^{\mu\nu} \,, \tag{3.165}$$

$$\left\langle E_{\mathbf{k}}^{-2} k^{\langle\mu} k^{\nu\rangle} \right\rangle_\delta = \frac{J_{22}}{J_{42}} \pi^{\mu\nu} \equiv \gamma_2^\pi \pi^{\mu\nu} \,, \tag{3.166}$$

$$\left\langle E_{\mathbf{k}}^{-2} k^{\langle\mu} k^{\nu} k^{\lambda\rangle} \right\rangle_\delta = 0 \,, \quad \left\langle E_{\mathbf{k}}^{-2} k^{\langle\mu} k^{\nu} k^{\lambda} k^{\rho\rangle} \right\rangle_\delta = 0 \,, \tag{3.167}$$

where the coefficients γ_1^Π, γ_2^Π, γ_1^n, γ_2^n, γ_1^π, and γ_2^π are defined by the right-hand sides of these equations.

Exercise 3.19: Prove Eqs. (3.161) – (3.167).

3.3.5 Hydrodynamic equations of motion

Implementing the simplifications (3.158) – (3.167) on account of the 14-moment approximation, the system of equations (3.143) – (3.145) will be closed in terms of the fluid-dynamical variables. Then, we obtain the following equations of motion,

$$\tau_\Pi D\Pi = -\Pi - \zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} - \ell_{\Pi n}\nabla_\mu n^\mu - \tau_{\Pi n}n^\mu Du_\mu - \lambda_{\Pi n}n^\mu\nabla_\mu\alpha_0 \,, \tag{3.168}$$

$$\begin{aligned}\tau_n Dn^{\langle\mu\rangle} &= -n^\mu + \varkappa\nabla^\mu\alpha_0 - \tau_n n_\nu\omega^{\nu\mu} - \delta_{nn}n^\mu\theta - \lambda_{nn}n^\nu\sigma^\mu_\nu + \ell_{n\pi}\Delta^{\mu\nu}\nabla_\lambda\pi^\lambda_\nu - \tau_{n\pi}\pi^\mu_\nu Du^\nu \\ &\quad - \lambda_{n\pi}\pi^{\mu\nu}\nabla_\nu\alpha_0 - \ell_{n\Pi}\nabla^\mu\Pi + \tau_{n\Pi}\Pi Du^\mu + \lambda_{n\Pi}\Pi\nabla^\mu\alpha_0 \,, \end{aligned} \tag{3.169}$$

$$\begin{aligned}\tau_\pi D\pi^{\langle\mu\nu\rangle} &= -\pi^{\mu\nu} + 2\eta\sigma^{\mu\nu} + 2\tau_\pi\pi^{\langle\mu}_\alpha\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\langle\mu}_\alpha\sigma^{\nu\rangle\alpha} \\ &\quad - \tau_{\pi n}n^{\langle\mu}Du^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \,, \end{aligned} \tag{3.170}$$

where we neglected terms of third order in dissipative currents or gradients. In total, these equations contain 25 transport coefficients. The coefficients of **bulk viscosity**, **particle diffusion**, and **shear viscosity** are identified as

$$\zeta = \frac{\alpha_0^{\mathrm{s}}}{4B_0\mathcal{A}^{\mathrm{s}}_{02}} \,, \qquad \varkappa = \frac{\beta_0\alpha_0^{\mathrm{v}}}{B_1\mathcal{A}^{\mathrm{v}}_{01}} \,, \qquad \eta = \frac{\beta_0^2\alpha_0^{\mathrm{t}}}{2B_2\mathcal{A}^{\mathrm{t}}_{00}} \,. \tag{3.171}$$

Note that these expressions for ζ, $\varkappa$, and η are the **same** as those obtained in Chapman-Enskog theory, Eqs. (3.106) – (3.108), when the simplest truncation scheme possible is employed, $N^{\mathrm{s}} = 2$, $N^{\mathrm{v}} = 1$, and $N^{\mathrm{t}} = 0$. (The proof that the bulk-viscosity coefficient agrees with Eq. (3.106) utilizes the relation (3.54).)

The **relaxation times**, which have no analogue in Navier-Stokes theory, are given by

$$\tau_\Pi = \frac{3}{4m^2B_0\mathcal{A}^{\mathrm{s}}_{02}} \,, \quad \tau_n = \frac{\beta_0}{B_1\mathcal{A}^{\mathrm{v}}_{01}} \,, \quad \tau_\pi = \frac{\beta_0^2}{B_2\mathcal{A}^{\mathrm{t}}_{00}} \,. \tag{3.172}$$

Note that the ratios of the relaxation times to their corresponding viscosity and diffusion coefficients are actually thermodynamic functions and independent of the collision term,

$$\frac{\tau_\Pi}{\zeta} = \frac{3}{m^2\alpha_0^{\mathrm{s}}} \,, \quad \frac{\tau_n}{\varkappa} = \frac{1}{\alpha_0^{\mathrm{v}}} \,, \quad \frac{\tau_\pi}{\eta} = \frac{2}{\alpha_0^{\mathrm{t}}} \,. \tag{3.173}$$

In particular, in the classical and massless limit $\tau_\pi/\eta = 1/(\beta_0 I_{32}) = 5/(\varepsilon_0 + P_0)$. This value respects the asymptotic causality condition discussed in Chapter 2.

The remaining transport coefficients related to the bulk-viscous pressure are

$$\begin{aligned}
\frac{\delta_{\Pi\Pi}}{\tau_\Pi} &= \frac{2}{3} - \frac{m^2}{3}\frac{G_{20}}{D_{20}} - \frac{m^4}{9}\gamma_2^\Pi \,, & \frac{\lambda_{\Pi\pi}}{\tau_\Pi} &= \frac{m^2}{3}\left(\gamma_2^\pi - \frac{G_{20}}{D_{20}}\right) \,, \\
\frac{\lambda_{\Pi n}}{\tau_\Pi} &= -\frac{m^2}{3}\left(\frac{\partial\gamma_1^n}{\partial\alpha_0} + h_0^{-1}\frac{\partial\gamma_1^n}{\partial\beta_0}\right) \,, & \frac{\ell_{\Pi n}}{\tau_\Pi} &= \frac{m^2}{3}\left(\frac{G_{30}}{D_{20}} - \gamma_1^n\right) \,, \\
\frac{\tau_{\Pi n}}{\tau_\Pi} &= -\frac{m^2}{3}\left(\frac{G_{30}}{D_{20}} - \beta_0\frac{\partial\gamma_1^n}{\partial\beta_0}\right) \,,
\end{aligned} \tag{3.174}$$

while those related to the particle-diffusion current are

$$\begin{aligned}
\frac{\delta_{nn}}{\tau_n} &= 1 + \frac{m^2}{3}\gamma_2^n , & \frac{\lambda_{nn}}{\tau_n} &= \frac{3}{5} + \frac{2}{5} m^2 \gamma_2^n , \\
\frac{\ell_{n\pi}}{\tau_n} &= \frac{\beta_0 J_{21}}{\varepsilon_0 + P_0} - \gamma_1^\pi , & \frac{\tau_{n\pi}}{\tau_n} &= \beta_0 \frac{J_{21}}{\varepsilon_0 + P_0} - \beta_0 \frac{\partial \gamma_1^\pi}{\partial \beta_0} , \\
\frac{\lambda_{n\pi}}{\tau_n} &= \frac{\partial \gamma_1^\pi}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_1^\pi}{\partial \beta_0} , & \frac{\ell_{n\Pi}}{\tau_n} &= \frac{\beta_0 J_{21}}{\varepsilon_0 + P_0} + \frac{m^2}{3}\gamma_1^\Pi , \\
\frac{\tau_{n\Pi}}{\tau_n} &= \frac{\beta_0 J_{21}}{\varepsilon_0 + P_0} + \frac{m^2}{3}\beta_0 \frac{\partial \gamma_1^\Pi}{\partial \beta_0} , & \frac{\lambda_{n\Pi}}{\tau_n} &= -\frac{m^2}{3}\left(\frac{\partial \gamma_1^\Pi}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_1^\Pi}{\partial \beta_0} \right) ,
\end{aligned} \tag{3.175}$$

and, finally, those related to the shear-stress tensor are

$$\begin{aligned}
\frac{\delta_{\pi\pi}}{\tau_\pi} &= \frac{4}{3} + \frac{m^2}{3}\gamma_2^\pi , & \frac{\tau_{\pi\pi}}{\tau_\pi} &= \frac{10}{7} + \frac{4}{7} m^2 \gamma_2^\pi , \\
\frac{\tau_{\pi n}}{\tau_\pi} &= -\frac{2}{5} m^2 \beta_0 \frac{\partial \gamma_1^n}{\partial \beta_0} , & \frac{\ell_{\pi n}}{\tau_\pi} &= -\frac{2}{5} m^2 \gamma_1^n , \\
\frac{\lambda_{\pi n}}{\tau_\pi} &= -\frac{2}{5} m^2 \left(\frac{\partial \gamma_1^n}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_1^n}{\partial \beta_0} \right) , & \frac{\lambda_{\pi\Pi}}{\tau_\pi} &= \frac{6}{5} - \frac{2}{15} m^4 \gamma_2^\Pi .
\end{aligned} \tag{3.176}$$

Exercise 3.20: Prove Eqs. (3.168) – (3.176).

We note that equations (3.168) – (3.170) are not identical to the original equations obtained by Israel and Stewart in Ref. [4]. The following terms did not appear in the original Israel-Stewart equations: in the equation for the bulk-viscous pressure the terms proportional to $\Pi\theta$, $\pi^{\mu\nu}\sigma_{\mu\nu}$, and $n_\mu \nabla^\mu \alpha_0$; in the equation for the diffusion 4-current the terms proportional to $n^\mu\theta$, $n^\nu\sigma^\mu_\nu$, $\pi^{\mu\nu}\nabla_\nu\alpha_0$, and $\Pi\nabla^\mu\alpha_0$; and in the equation for the shear-stress tensor the terms proportional to $\pi^{\mu\nu}\theta$, $\pi^{\langle\mu}_\alpha \sigma^{\nu\rangle\alpha}$, $n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0$, and $\Pi\sigma^{\mu\nu}$. These missing terms made some believe that the formalism proposed by Israel and Stewart to derive fluid dynamics necessarily led to incomplete equations of motion. As we demonstrated in this chapter, this is certainly not the case. The issue was that such terms were originally dropped in their original work because they were considered to be unimportant for applications to cosmology, where the expansion rate of the fluid is usually quite small and the shear tensor is zero. Therefore, the disappearance of such terms is only a reflection of the power-counting scheme originally adopted by Israel-Stewart, it is not a permanent feature of the formalism.

However, for the purposes of describing the quark-gluon plasma produced in heavy-ion collisions, such a power-counting scheme is simply not sufficient, and the terms that were originally dropped can be of relevance to the description of such a rapidly expanding system. The first to note that Israel-Stewart theory was not incomplete were the authors of Refs. [14, 15, 16], who actually wrote down the complete equations that follow from Israel-Stewart's derivation procedure, i.e., Eqs. (3.168) – (3.170).

Furthermore, we note that the transport coefficients derived in this section are slightly different from those derived by Israel and Stewart in Ref. [4]. The reason behind this is well known: we did not use the second moment of the Boltzmann equation to derive the equations of motion for the dissipative currents using the 14-moment approximation – we instead used the exact equation of motion for the dissipative currents. The moment of the Boltzmann equation employed to derive the equations of motion does not change the form of the equations of motion, but does affect the microscopic expressions obtained for the transport coefficients. We remark that this is basically a relativistic effect and, in the non-relativistic (or low-temperature) limit, the set of transport coefficients will have the same values regardless of the moment employed.

In contrast to Chapman-Enskog theory, Israel's and Stewart's framework allows for a rather simple derivation of fluid dynamics. This derivation leads to a fairly accurate theory of fluid dynamics for dilute gases, which takes into account the transient dynamics of the dissipative currents and includes several higher-order terms. Following the arguments constructed in Chapter 2, linearized Israel-Stewart theory can be shown to be stable under perturbations and also to respect causality, depending on the values taken for η/τ_π, $\varkappa/\tau_n$, and ζ/τ_Π. The microscopic expressions derived in this section for the viscosity coefficients and relaxation times are **consistent with the causality and stability conditions** obtained in Chapter 2. Naturally, improvements to this formalism are still required, as will be discussed in detail in the forthcoming chapters.

3.4 Summary

In this chapter we have discussed in detail the **derivation of relativistic dissipative fluid dynamics from kinetic theory** following the two most traditional approaches: the **Chapman-Enskog expansion** and the **14-moment approximation**, as originally proposed by Israel and Stewart.

In Sec. 3.1 we showed how the fluid-dynamical degrees of freedom can be matched to moments of the single-particle momentum distribution function. This is the first step required to derive fluid dynamics from the Boltzmann equation and this section's results will also be employed in the following chapter.

In Sec. 3.2 we discussed the Chapman-Enskog expansion. In this scheme, an asymptotic solution of the Boltzmann is constructed using perturbation theory, by expanding the single-particle distribution function in **powers of the Knudsen number**. This results in a **gradient expansion**, in which all moments of the distribution function depend **solely** on the five **primary fluid-dynamical variables and their gradients**. The corrections to the local-equilibrium distribution function are then systematically arranged in terms of an expansion in powers of the Knudsen number. We showed in this section that the **zeroth-order truncation** of this expansion leads to **ideal fluid dynamics** and the **first-order truncation** to **Navier-Stokes theory**. We further obtained the **microscopic expressions for the viscosity and diffusion coefficients**. Keeping second- and higher-order terms one would obtain the Burnett and super-Burnett equations, but these higher-order solutions were not explicitly calculated here.

In Sec. 3.3 we discussed **Israel's and Stewart's derivation of relativistic fluid**

dynamics. This procedure is based on a truncation of the moment expansion of the single-particle distribution function. There, the distribution function is expressed solely in terms of **14 degrees of freedom**, which can be matched to the **14 independent components of the conserved currents**. This truncated version of the single-particle distribution function, the so-called **14-moment approximation**, is substituted into the **exact** equations of motion for the dissipative currents, leading to a **closed set of dynamical equations** for these fields. The novel equations of motion obtained satisfy the causality condition obtained in Chapter 2 and, therefore, are stable. In the next chapters, we shall see how the method of moments can be improved, in order to derive even more precise fluid-dynamical equations of motion.

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4 Method of Moments

The two widespread methods to derive relativistic fluid dynamics from the Boltzmann equation, the Chapman-Enskog expansion [1] and the 14-moment approximation [2, 3, 4], both described in the previous chapter, have **flaws**. The relativistic **Chapman-Enskog expansion** is essentially based on the gradient expansion and as such does not feature transient effects, which allow to restore causality and stability. It thus leads to fluid-dynamical equations of motion that are **acausal and intrinsically unstable** [5] and, consequently, it should not be applied to derive the equations of relativistic fluid dynamics from kinetic theory. The **method of moments** [2, 3, 4, 6] is in principle free of such problems and leads to transient fluid-dynamical equations that can be constructed to be **causal and stable** [7, 8, 9, 10, 11], as shown in Chapter 2. However, in the form presented in the previous chapter it still features an **ambiguity** as to how to close the system of fluid-dynamical equations of motion. In this chapter we will discuss the necessary prerequisites to resolve this problem.

The method of moments is generally considered the method of choice for deriving the fluid-dynamical equations of motion from the Boltzmann equation. This approach was first formulated consistently by H. Grad [12] for non-relativistic systems and consists of expanding the non-equilibrium correction to the single-particle distribution function in terms of a **complete set of Hermite polynomials** [13]. The generalization of Grad's method of moments to relativistic systems is, nevertheless, not trivial and has been pursued by several authors [14, 15, 16, 17, 18, 19]. The main challenge is to find a suitable **set of orthogonal polynomials** which could **replace** the Hermite polynomials in a **relativistic formulation** [3, 6]. This issue was circumvented by Israel and Stewart by simply expanding the non-equilibrium correction to the single-particle distribution function in a Taylor series in 4-momentum. The main **disadvantage** of this approach is that, since the expansion is not realized in terms of an orthogonal basis, the expansion coefficients cannot be determined in terms of moments of the single-particle distribution function. Israel and Stewart then introduced another approximation, the so-called **14-moment approximation** [4], already discussed in Sec. 3.3. In this case, the Taylor expansion in momentum is truncated at second order, leaving only 14 expansion coefficients to be determined. Israel and Stewart then introduced a set of 14 constraints which allowed to express the expansion coefficients in terms of the conserved currents, N^{μ} and $T^{\mu\nu}$ – the so-called **matching procedure**.

A **more reliable relativistic formulation of the method of moments**, which expands the single-particle distribution function in momentum using an **orthogonal and complete basis**, was formulated in Ref. [20]. The main goal of this chapter is to describe this expansion and to derive the equations of motion satisfied by the corresponding expansion coefficients, the so-called **irreducible moments**, which turn out to be **moments of the deviation of the single-particle distribution function from a chosen ref-**

erence state. As we shall discuss in Chapter 5, such a generalization of the method of moments will be essential in understanding how relativistic fluid dynamics actually emerges from the relativistic Boltzmann equation, what is its domain of applicability, and how the equations of motion can be systematically improved. In this chapter, the reference state is assumed to be in local thermodynamical equilibrium, described by the single-particle distribution function (3.60). A generalization to a different reference state (featuring an anisotropy in momentum space) was explored in Ref. [21].

This chapter is organized as follows. In Sec. 4.1 we demonstrate how to expand the single-particle distribution function in terms of a **complete, orthogonal** basis in momentum space. In contrast to Israel's and Stewart's non-orthogonal basis 1, k^μ, $k^\mu k^\nu, \ldots$, our approach uses **irreducible tensors** in 4-momentum k^μ, and is thus **orthogonal**. The coefficients of the irreducible tensors in the expansion of the single-particle distribution function are **orthogonal polynomials in the rest-frame energy**, multiplied by the above mentioned irreducible moments. Section 4.2 derives an infinite set of equations for these moments, which is still completely equivalent to the Boltzmann equation. Section 4.3 explicitly calculates the moments of the collision term, which appear in the equations of motion for the irreducible moments. In Sec. 4.4 we summarize this chapter. Several mathematical details are delegated to a set of appendices.

4.1 Moment expansion

In principle, the momentum dependence of the **non-equilibrium component** of $f_{\mathbf{k}}$, $\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}}$, should be obtained by **solving the relativistic Boltzmann equation**, which in general poses a very complicated task. In this context, the **method of moments** can be a convenient tool, since it allows us to obtain an **approximate expression** for $\delta f_{\mathbf{k}}$, which is able to capture some features of solutions of the Boltzmann equation. In this section we explain and develop the moment expansion of the single-particle distribution function.

It is convenient to factorize the local-equilibrium distribution function $f_{0\mathbf{k}}$ from $f_{\mathbf{k}}$ in the following way

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} \equiv f_{0\mathbf{k}} \left(1 + \mathcal{G}_{\mathbf{k}} \phi_{\mathbf{k}}\right) \ . \tag{4.1}$$

Above, we introduced $\mathcal{G}_{\mathbf{k}}$ as an arbitrary function of $E_{\mathbf{k}} \equiv u_\mu k^\mu$, and $\phi_{\mathbf{k}}$ as an arbitrary function of the space-time coordinates x^μ and the on-shell 4-momentum $k^\mu = (k_0, \mathbf{k})^T$, $k_0 = \sqrt{\mathbf{k}^2 + m^2}$.

The next step is to expand $\phi_{\mathbf{k}}$ in terms of a **complete basis of tensors formed of** k^μ. One choice is to follow the approach proposed by Israel and Stewart [4] and expand $\phi_{\mathbf{k}}$ using the following basis,

$$1\,,\ k^\mu\,,\ k^\mu k^\nu\,,\ k^\mu k^\nu k^\lambda\,, \ldots\ . \tag{4.2}$$

In this case, the formal expression for $\phi_{\mathbf{k}}$ becomes

$$\phi_{\mathbf{k}} = \epsilon + \epsilon_\mu k^\mu + \epsilon_{\mu\nu} k^\mu k^\nu + \mathcal{O}\left(k^\mu k^\nu k^\lambda\right)\ , \tag{4.3}$$

where the expansion coefficients $\epsilon_{\mu_1 \cdots \mu_m}$ carry all the space-time dependence of $\phi_{\mathbf{k}}$.

The main **disadvantage** of the moment expansion proposed by Israel and Stewart is that the expansion coefficients are simply unknown and can only be extracted approximately. This happens because the basis (4.2) is **not** orthogonal. In their original work, Israel and Stewart overcame this problem by truncating the expansion at second order in momentum and extracting the coefficients of the truncated expansion by **matching** them to certain moments of the single-particle distribution function. However, the coefficients extracted from such a matching procedure are not unique and can change according to the order in which the expansion is truncated.

This unpleasant aspect of Israel-Stewart theory can be easily avoided by using an **orthogonal basis** for the expansion. Here, we follow the approach developed in Ref. [20] and expand $\phi_{\mathbf{k}}$ using as a basis the **irreducible tensors**,

$$1\,,\; k^{\langle\mu\rangle}\,,\; k^{\langle\mu}k^{\nu\rangle}\,,\; k^{\langle\mu}k^{\nu}k^{\lambda\rangle}\,,\dots\,, \tag{4.4}$$

and **orthogonal polynomials** in $E_{\mathbf{k}}$,

$$P^{(\ell)}_{\mathbf{k}n} = \sum_{r=0}^{n} a^{(\ell)}_{nr} E^{r}_{\mathbf{k}}\,. \tag{4.5}$$

Note that, in the **local rest frame**, the irreducible tensors (4.4) are (certain) polynomials formed of powers of the components of **3-momentum k**, while the orthogonal polynomials (4.5) are polynomials in **on-shell energy** $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. The term "irreducible" refers to the fact that the tensors are irreducible with respect to the group of Lorentz transformations which leave the fluid velocity **invariant**, $\Lambda^{\mu}_{\nu}u^{\nu} \equiv u^{\mu}$, the so-called **little group** associated with u^{μ}. In the **local rest frame**, these are **spatial rotations**.

The irreducible tensors form a **complete and orthogonal set** and are defined by using the symmetrized and, for $m > 1$ traceless, projection orthogonal to u^{μ} of tensors constructed from k^{μ}. That is,

$$k^{\langle\mu_1}\cdots k^{\mu_m\rangle} \equiv \Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} k^{\nu_1}\cdots k^{\nu_m}\,, \tag{4.6}$$

where the projectors $\Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m}$ are constructed in App. 4.5, Eq. (4.49), see also Refs. [6, 22]. Naturally, these projectors are orthogonal to the fluid 4-velocity.

The **irreducible tensors (4.4)** satisfy an **orthogonality condition**,

$$\int dK\, \mathrm{F}(E_{\mathbf{k}})\, k^{\langle\mu_1}\cdots k^{\mu_m\rangle} k_{\langle\nu_1}\cdots k_{\nu_n\rangle} = \frac{m!\,\delta_{mn}}{(2m+1)!!}\Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} \int dK\, \mathrm{F}(E_{\mathbf{k}})\,\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^m\,, \tag{4.7}$$

for an arbitrary function $F(E_{\mathbf{k}})$, which depends only on $E_{\mathbf{k}} = k^{\mu}u_{\mu}$, for the proof see App. 4.7. Likewise, the **orthogonal polynomials (4.5)** fulfill an **orthogonality condition**,

$$\int dK\, \omega^{(\ell)}\, P^{(\ell)}_{\mathbf{k}m} P^{(\ell)}_{\mathbf{k}n} = \delta_{mn}\,, \tag{4.8}$$

with the measure $\omega^{(\ell)}$ being given by

$$\omega^{(\ell)} \equiv \frac{W^{(\ell)}}{(2\ell+1)!!}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^{\ell} \mathcal{G}_{\mathbf{k}} f_{0\mathbf{k}}\,, \tag{4.9}$$

cf. App. 4.8 for further details.

We note that the basis constructed from irreducible tensors and orthogonal polynomials is **completely equivalent** to the basis used by Israel and Stewart, as long as $\mathcal{G}_{\mathbf{k}} = \tilde{f}_{0\mathbf{k}}$. However, the fact that it is orthogonal will make it more convenient to use. Next, we explain how the orthogonality relations (4.7), (4.8) satisfied by the basis elements allow to calculate the expansion coefficients.

Using the basis introduced in Eqs. (4.4) and (4.5), the quantity $\phi_{\mathbf{k}}$ defined in Eq. (4.1) is expanded as

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} c_n^{\langle\mu_1\cdots\mu_\ell\rangle} P_{\mathbf{k}n}^{(\ell)} \, k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \, . \tag{4.10}$$

The expansion coefficients $c_n^{\langle\mu_1\cdots\mu_\ell\rangle}$ can be obtained by multiplying Eq. (4.10) by the corresponding basis element, $\mathcal{G}_{\mathbf{k}} f_{0\mathbf{k}} P_{\mathbf{k}n}^{(\ell)} \, k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle}$, and integrating over momentum, dK. Together with the definition (4.1) of $\delta f_{\mathbf{k}}$, the orthogonality conditions (4.7), (4.8) then imply that

$$c_n^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{W^{(\ell)}}{\ell!} \int dK \; P_{\mathbf{k}n}^{(\ell)} \, k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \, \delta f_{\mathbf{k}} \, . \tag{4.11}$$

Exercise 4.1: Prove Eq. (4.11).

Plugging Eq. (4.11) into Eq. (4.10) and the result into Eq. (4.1), using Eq. (4.5) the full non-equilibrium single-particle distribution function can be expressed in the following way,

$$f_{\mathbf{k}} = f_{0\mathbf{k}}(1 + \mathcal{G}_{\mathbf{k}}\phi_{\mathbf{k}}) \, , \tag{4.12}$$

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \, , \tag{4.13}$$

where we introduced the function

$$\mathcal{H}_{\mathbf{k}n}^{(\ell)} \equiv \frac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{\mathbf{k}m}^{(\ell)} \, , \tag{4.14}$$

and the **irreducible moments** $\rho_n^{\mu_1\cdots\mu_\ell}$ of $\delta f_{\mathbf{k}}$,

$$\rho_n^{\mu_1\cdots\mu_\ell} \equiv \int dK \, E_{\mathbf{k}}^n \, k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \, \delta f_{\mathbf{k}} \equiv \left\langle E_{\mathbf{k}}^n \, k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \right\rangle_\delta \, . \tag{4.15}$$

On the right-hand side, we made contact to the notation (3.27) introduced in Chapter 3.

Exercise 4.2: Prove Eqs. (4.12) and (4.13).

We remark that our choice of **matching conditions** (3.23) and the **Landau-frame definition** (3.16) of the fluid velocity [23] imply that the following irreducible moments **vanish**

$$\rho_1 = \rho_2 = \rho_1^{\mu} = 0 \,. \tag{4.16}$$

The vanishing of the two scalar moments, ρ_1 and ρ_2, define the **temperature** and **chemical potential** of the **fictitious local-equilibrium state** characterized by the single-particle distribution function $f_{0\mathbf{k}}$. The rank-1 moment ρ_1^{μ} corresponds to **heat flow**, which vanishes by definition in the Landau frame.

Note that, so far, the function $\mathcal{G}_{\mathbf{k}}$ was assumed to be independent of the index ℓ. This was done just for the sake of simplicity and, in principle, nothing would prevent us from introducing a function $\mathcal{G}_{\mathbf{k}}^{(\ell)}$, if this provides a more useful prefactor in Eq. (4.1). The arbitrary function $\mathcal{G}_{\mathbf{k}}^{(\ell)}$ would determine the overall behavior of the polynomial basis employed to expand $\phi_{\mathbf{k}}$. Setting $\mathcal{G}_{\mathbf{k}}^{(\ell)} = \tilde{f}_{0\mathbf{k}}$ corresponds to employing the same basis used by Israel and Stewart, which, once truncated, is a good approximation to describe the function for small values of $\beta_0 E_{\mathbf{k}}$ (as long as there are no singularities). This basis is sufficient to derive fluid dynamics and was used for this purpose in previous work [20]. On the other hand, one can also set $\mathcal{G}_{\mathbf{k}}^{(\ell)} = \tilde{f}_{0\mathbf{k}}/\left(1+\beta_0 E_{\mathbf{k}}\right)^{\ell}$, which, at large $\beta_0 E_{\mathbf{k}}$, is equivalent to the basis $1/E_{\mathbf{k}}$, $1/E_{\mathbf{k}}^2$, $1/E_{\mathbf{k}}^3, \ldots$, while, at small $\beta_0 E_{\mathbf{k}}$, is equivalent to the usual basis 1, $E_{\mathbf{k}}$, $E_{\mathbf{k}}^2$, $E_{\mathbf{k}}^3$, ... This (truncated) basis is expected to be viable when $\beta_0 E_{\mathbf{k}}$ is large and should provide a better estimate of the momentum dependence of $\phi_{\mathbf{k}}$. In summary, with a reasonable choice of $\mathcal{G}_{\mathbf{k}}^{(\ell)}$ one can describe the single-particle distribution function for several domains of $\beta_0 E_{\mathbf{k}}$.

For the particular choice $\mathcal{G}_k = \tilde{f}_{0\mathbf{k}}$ one can easily show that, substituting Eqs. (4.12) and (4.14) into Eq. (4.15) and using Eq. (4.5), the orthogonality condition (4.7), as well as the definition of the auxiliary thermodynamic integrals (4.31), all irreducible moments are linearly related to each other (for more details see, Eq. (72) of Ref. [24]),

$$\rho_i^{\mu_1\cdots\mu_\ell} \equiv (-1)^{\ell}\,\ell! \sum_{n=0}^{N_\ell} \rho_n^{\mu_1\cdots\mu_\ell} \gamma_{in}^{(\ell)} \,, \tag{4.17}$$

where

$$\gamma_{in}^{(\ell)} = \frac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} \sum_{r=0}^{m} a_{mn}^{(\ell)} a_{mr}^{(\ell)} J_{i+r+2\ell,\ell} \,. \tag{4.18}$$

Note that these relations are also valid for moments with **negative** i, hence it is possible to express the irreducible moments with negative powers of $E_{\mathbf{k}}$ in terms of the ones with positive i, for details see Eq. (65) of Ref. [20].

The moment expansion described in this section is a very powerful tool, with applications to all methods that derive fluid dynamics from the relativistic Boltzmann equation.

Exercise 4.3: Prove Eq. (4.17).

4.2 Equations of motion for the irreducible moments

The **time-evolution equations for the irreducible moments** $\rho_r^{\mu_1\cdots\mu_\ell}$ can be obtained directly from the Boltzmann equation by applying the comoving derivative to the definition (4.15), together with the symmetrized traceless projection,

$$\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} \frac{d}{d\tau} \int dK E_{\mathbf{k}}^r k^{\langle\nu_1} \cdots k^{\nu_\ell\rangle} \delta f_{\mathbf{k}} \ , \tag{4.19}$$

where $\dot{A} \equiv u^\mu \partial_\mu A \equiv DA \equiv dA/d\tau$ and $\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\ \dot{\rho}_r^{\nu_1\cdots\nu_\ell}$. Using the Boltzmann equation (3.1) in the form (3.142),

$$\delta\dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k^\nu \nabla_\nu f_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k^\nu \nabla_\nu \delta f_{\mathbf{k}} + E_{\mathbf{k}}^{-1} C\left[f\right] \ , \tag{4.20}$$

where $\nabla_\mu \equiv \Delta_\mu^\nu \partial_\nu$, and substituting this expression into Eq. (4.19), one can obtain the **exact** equations for the comoving derivatives of $\rho_r^{\mu_1\cdots\mu_\ell}$.

Using the **power-counting scheme** that will be developed in Sec. 5.1, we can show that, in order to derive the equations of motion for relativistic fluid dynamics, it is sufficient to know the time-evolution equations for the moments (4.15) up to **rank two**, i.e., for ρ_r, ρ_r^μ, and $\rho_r^{\mu\nu}$. Similar equations can also be derived for higher-rank irreducible moments, if needed. Thus, using Eqs. (4.19) and (4.20), we obtain after some lengthy calculation

$$\begin{aligned}\dot{\rho}_r - C_{r-1} &= \alpha_r^{(0)}\theta - \frac{G_{2r}}{D_{20}}\Pi\theta + \frac{G_{2r}}{D_{20}}\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{G_{3r}}{D_{20}}\partial_\mu n^\mu + (r-1)\,\rho_{r-2}^{\mu\nu}\sigma_{\mu\nu} \\ &\quad + r\rho_{r-1}^\mu \dot{u}_\mu - \nabla_\mu \rho_{r-1}^\mu - \frac{1}{3}\left[(r+2)\,\rho_r - (r-1)\,m^2\rho_{r-2}\right]\theta \ , \end{aligned} \tag{4.21}$$

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\rangle} - C_{r-1}^{\langle\mu\rangle} &= \alpha_r^{(1)} I^\mu + \rho_r^\nu \omega^\mu_{\ \nu} + \frac{1}{3}\left[(r-1)\,m^2\rho_{r-2}^\mu - (r+3)\,\rho_r^\mu\right]\theta - \Delta^\mu_\lambda \nabla_\nu \rho_{r-1}^{\lambda\nu} + r\rho_{r-1}^{\mu\nu}\dot{u}_\nu \\ &\quad + \frac{1}{5}\left[(2r-2)\,m^2\rho_{r-2}^\nu - (2r+3)\,\rho_r^\nu\right]\sigma^\mu_\nu + \frac{1}{3}\left[m^2 r\rho_{r-1} - (r+3)\,\rho_{r+1}\right]\dot{u}^\mu \\ &\quad + \frac{\beta_0 J_{r+2,1}}{\varepsilon_0 + P_0}\left(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta^\mu_\nu \partial_\lambda \pi^{\lambda\nu}\right) - \frac{1}{3}\nabla^\mu\left(m^2\rho_{r-1} - \rho_{r+1}\right) \\ &\quad + (r-1)\,\rho_{r-2}^{\mu\nu\lambda}\sigma_{\lambda\nu} \ , \end{aligned} \tag{4.22}$$

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} &= 2\alpha_r^{(2)}\sigma^{\mu\nu} - \frac{2}{7}\left[(2r+5)\,\rho_r^{\lambda\langle\mu} - 2m^2\,(r-1)\,\rho_{r-2}^{\lambda\langle\mu}\right]\sigma^{\nu\rangle}_\lambda + 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}_{\ \lambda} \\ &\quad + \frac{2}{15}\left[(r+4)\,\rho_{r+2} - (2r+3)\,m^2\rho_r + (r-1)\,m^4\rho_{r-2}\right]\sigma^{\mu\nu} \\ &\quad + \frac{2}{5}\nabla^{\langle\mu}\left(\rho_{r+1}^{\nu\rangle} - m^2\rho_{r-1}^{\nu\rangle}\right) - \frac{2}{5}\left[(r+5)\,\rho_{r+1}^{\langle\mu} - m^2 r\rho_{r-1}^{\langle\mu}\right]\dot{u}^{\nu\rangle} \\ &\quad - \frac{1}{3}\left[(r+4)\,\rho_r^{\mu\nu} - m^2\,(r-1)\,\rho_{r-2}^{\mu\nu}\right]\theta \\ &\quad + (r-1)\,\rho_{r-2}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda \ , \end{aligned} \tag{4.23}$$

where we introduced the generalized irreducible collision terms

$$C_r^{\langle\mu_1\cdots\mu_\ell\rangle} = \int dK E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} C\left[f\right] \ . \tag{4.24}$$

We further used the definitions of the shear tensor $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu} u^{\nu\rangle}$, the expansion scalar $\theta \equiv \nabla_\mu u^\mu$, and the vorticity tensor $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ and we introduced $I^\mu \equiv \nabla^\mu \alpha_0$. All derivatives of α_0 and β_0 that appeared during the derivation of the above equations were replaced using the **exact** equations obtained from the **conservation laws of particle number, energy, and momentum,**

$$\dot{\alpha}_0 = \frac{1}{D_{20}} \left\{-J_{30}\left(n_0\theta + \partial_\mu n^\mu\right) + J_{20}\left[\left(\varepsilon_0 + P_0 + \Pi\right)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right]\right\} , \tag{4.25}$$

$$\dot{\beta}_0 = \frac{1}{D_{20}} \left\{-J_{20}\left(n_0\theta + \partial_\mu n^\mu\right) + J_{10}\left[\left(\varepsilon_0 + P_0 + \Pi\right)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right]\right\} , \tag{4.26}$$

$$\dot{u}^\mu = \frac{1}{\varepsilon_0 + P_0} \left(\nabla^\mu P_0 - \Pi \dot{u}^\mu + \nabla^\mu \Pi - \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta}\right) , \tag{4.27}$$

cf. Eqs. (3.46), (3.58), (3.59). The coefficients $\alpha_r^{(0)}$, $\alpha_r^{(1)}$, and $\alpha_r^{(2)}$ are functions of temperature and chemical potential and have the general form

$$\alpha_r^{(0)} = (1-r)\, I_{r1} - I_{r0} - \frac{1}{D_{20}}\left[G_{2r}\left(\varepsilon_0 + P_0\right) - G_{3r} n_0\right] , \tag{4.28}$$

$$\alpha_r^{(1)} = J_{r+1,1} - h_0^{-1} J_{r+2,1} , \tag{4.29}$$

$$\alpha_r^{(2)} = I_{r+2,1} + (r-1)\, I_{r+2,2} , \tag{4.30}$$

where we defined the thermodynamic functions

$$I_{nq}\left(\alpha_0, \beta_0\right) = \frac{1}{(2q+1)!!} \left\langle E_{\mathbf{k}}^{n-2q}\left(-\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q \right\rangle_0 , \quad J_{nq} = \left.\frac{\partial I_{nq}}{\partial \alpha_0}\right|_{\beta_0} , \tag{4.31}$$

$$G_{nm} = J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0} , \quad D_{nq} = J_{n+1,q} J_{n-1,q} - J_{nq}^2 , \tag{4.32}$$

cf. also Eqs. (3.48), (3.49), (3.55).

Exercise 4.4: Prove Eqs. (4.21) – (4.23).
(Warning: This proof is quite lengthy. Try this only when you have sufficient patience and time.)

Using the matching conditions (4.16), the dissipative quantities appearing in the conservation laws can be (exactly) identified with the moments

$$\rho_0 = -\frac{3}{m^2}\Pi , \quad \rho_0^\mu = n^\mu , \quad \rho_0^{\mu\nu} = \pi^{\mu\nu} , \tag{4.33}$$

cf. Eqs. (3.26), (3.28) in Chapter 3. We note that the derivation of the equations of motion for the irreducible moments is independent of the form of the expansion of the single-particle distribution introduced in the previous section.

4.3 Collision term

In this section, we compute the collision integrals (4.24), expressing them in terms of the irreducible moments. This then formally closes the set of moment equations. We shall separate the collision term into two different components: one being **linear** in the non-equilibrium corrections $\phi_{\mathbf{p}}$ and the remaining being **nonlinear** in these functions. It is obvious that the first term will be linear in the irreducible moments, while the second one is nonlinear.

The first step is to express the collision term as a functional of $\phi_{\mathbf{p}}$, which was introduced in Eq. (4.1) as (we take $\mathcal{G}_{\mathbf{k}} \equiv \tilde{f}_{0\mathbf{k}}$)

$$\delta f_{\mathbf{p}} = f_{\mathbf{p}} - f_{0\mathbf{p}} = f_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}}\phi_{\mathbf{p}} \; . \tag{4.34}$$

Using this definition, it is straightforward to demonstrate that

$$f_{\mathbf{p}}f_{\mathbf{p}'} = f_{0\mathbf{p}}f_{0\mathbf{p}'}\left(1 + \tilde{f}_{0\mathbf{p}'}\phi_{\mathbf{p}'} + \tilde{f}_{0\mathbf{p}}\phi_{\mathbf{p}} + \tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\phi_{\mathbf{p}}\phi_{\mathbf{p}'}\right) \; , \tag{4.35}$$

$$\tilde{f}_{\mathbf{p}}\tilde{f}_{\mathbf{p}'} = \tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\left(1 - af_{0\mathbf{p}'}\phi_{\mathbf{p}'} - af_{0\mathbf{p}}\phi_{\mathbf{p}} + a^2 f_{0\mathbf{p}}f_{0\mathbf{p}'}\phi_{\mathbf{p}}\phi_{\mathbf{p}'}\right) \; . \tag{4.36}$$

The terms linear in $\phi_{\mathbf{k}}$ will be collected into the linear collision term and computed in the following, while the higher-order terms constitute the nonlinear collision term, respectively. The calculation of the latter can be found (in the classical (Boltzmann) limit) in Ref. [24].

Substituting Eqs. (4.35) and (4.36) into Eq. (4.24), and keeping only terms that are linear in $\phi_{\mathbf{k}}$, one can derive the **linearized collision term**, $L[\phi]$, defined as $C[f] = L[\phi] + \mathcal{O}(\phi^2)$. Explicitly, one obtains

$$L[\phi] = \frac{1}{\nu}\int dK'dPdP'\, W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\left(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'}\right) \; . \tag{4.37}$$

Exercise 4.5: Prove Eq. (4.37).
(Hint: Use $C[f_0] = 0$ and the equality $f_{0\mathbf{p}}f_{0\mathbf{p}'}\tilde{f}_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}'} = f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}$.)

Inserting Eq. (4.37) into the expression for the irreducible collision term (4.24), we obtain the **linearized collision integral**

$$L_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \int dK\, E_{\mathbf{k}}^{r-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} L[\phi]$$
$$= \frac{1}{\nu}\int dKdK'dPdP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\left(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'}\right) \; . \tag{4.38}$$

The next step is to substitute the moment expansion (4.13) for $\phi_{\mathbf{k}}$ into Eq. (4.38), expressing it as a **linear combination of the irreducible moments**,

$$L_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{m=0}^{\infty}\sum_{n=0}^{N_m}\left(\mathcal{A}_{rn}\right)^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}\rho_n^{\nu_1\cdots\nu_m} \; , \tag{4.39}$$

where we defined the tensor

$$(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m} \equiv \frac{1}{\nu}\int dK dK' dP dP'\, W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}$$
$$\times \left(\mathcal{H}^{(m)}_{n\mathbf{k}}\, k_{\langle\nu_1}\cdots k_{\nu_m\rangle} + \mathcal{H}^{(m)}_{n\mathbf{k}'}\, k'_{\langle\nu_1}\cdots k'_{\nu_m\rangle} - \mathcal{H}^{(m)}_{n\mathbf{p}}\, p_{\langle\nu_1}\cdots p_{\nu_m\rangle} - \mathcal{H}^{(m)}_{n\mathbf{p}'}\, p'_{\langle\nu_1}\cdots p'_{\nu_m\rangle}\right) . \tag{4.40}$$

Equation (4.39) looks like different tensor components of the irreducible moments contribute to a given tensor component of the collision term on the left-hand side. In fact, however, we will now show that the tensor components on both sides of this equation are the same, cf. Eq. (4.47).

The integral $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ is a tensor of rank $m+\ell$, which is symmetric under permutations of μ-type and ν-type indices, and which depends only on the equilibrium distribution function. The latter contains only one type of 4-vector, the fluid velocity u^μ. Therefore, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ must be constructed from tensor structures made of u^μ and the metric tensor $g^{\mu\nu}$. Also, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ was constructed to be orthogonal to u^μ and to satisfy the following property,

$$\Delta^{\alpha_1\cdots\alpha_\ell}_{\mu_1\cdots\mu_\ell}\Delta^{\nu_1\cdots\nu_m}_{\beta_1\cdots\beta_m}(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m} = (\mathcal{A}_{rn})^{\alpha_1\cdots\alpha_\ell}_{\beta_1\cdots\beta_m} . \tag{4.41}$$

Since $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ is orthogonal to u^μ, it can only be constructed from combinations of projection operators, $\Delta^{\mu\nu}$. This already constrains $m+\ell$ to be an even number, since it is impossible to construct odd-ranked tensors solely from projection operators. This means that both ℓ and m are either even or odd. Therefore, the following type of terms could appear in $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$:

(i) Terms where all μ-type indices pair up on projectors $\Delta^{\mu_i\mu_j}$ and all ν-type indices on projectors $\Delta_{\nu_p\nu_q}$, e.g.,

$$\Delta^{\mu_1\mu_2}\ldots\Delta^{\mu_i\mu_j}\ldots\Delta^{\mu_{\ell-1}\mu_\ell}\Delta_{\nu_1\nu_2}\cdots\Delta_{\nu_p\nu_q}\cdots\Delta_{\nu_{m-1}\nu_m} . \tag{4.42}$$

All possible permutations of the μ-type among themselves and ν-type indices among themselves are allowed.

(ii) Terms where at least one μ-type index pairs with a ν-type index on a projector, e.g.,

$$\Delta^{\mu_1}_{\nu_1}\Delta^{\mu_2\mu_3}\cdots\Delta^{\mu_i\mu_j}\cdots\Delta^{\mu_{\ell-1}\mu_\ell}\Delta_{\nu_2\nu_3}\cdots\Delta_{\nu_p\nu_q}\cdots\Delta_{\nu_{m-1}\nu_m} . \tag{4.43}$$

Again, all possible permutations of the μ-type and ν-type indices are allowed. If there is an odd number of projectors of the type $\Delta^{\mu_i}_{\nu_p}$, both ℓ and m must be odd. If there is an even number, both ℓ and m must be even, too. Without loss of generality, suppose that $\ell > m$. For $\ell+m$ to be even, ℓ must be $m+2, m+4, \ldots$. Then one could pair all ν-type indices with μ-type indices on projectors of the form $\Delta^{\mu_i}_{\nu_p}$, with some projectors left over which carry only μ-type indices, e.g., $\Delta^{\mu_j\mu_k}$.

(iii) If $\ell = m$, all μ-type indices could be paired up with ν-type indices on projectors of the form $\Delta^{\mu_i}_{\nu_p}$, with no left-over projectors like explained at the end of (ii),

$$\Delta^{\mu_1}_{\nu_1}\cdots\Delta^{\mu_\ell}_{\nu_\ell} . \tag{4.44}$$

Again, all permutations of the μ-type indices among themselves and ν-type indices among themselves are allowed.

Note that terms of the type (i) and (ii) by themselves do not satisfy the property (4.41). This happens because any term which contains at least one projector of the type $\Delta^{\mu_i\mu_j}$ or $\Delta_{\nu_p\nu_q}$ vanishes when contracted with $\Delta^{\alpha_1\cdots\alpha_\ell}_{\mu_1\cdots\mu_\ell}\Delta^{\nu_1\cdots\nu_m}_{\beta_1\cdots\beta_m}$. Therefore, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ cannot be solely constructed from terms of type (i) and (ii), because otherwise it would vanish trivially, and the property (4.41) would not be satisfied. There must at least be one term of type (iii). However, this implies that $m=\ell$. This does not mean that terms of type (i) and (ii) do not appear; they do occur, but in such a way that Eq. (4.41) is satisfied. In summary, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ has the form

$$(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m} = \delta_{\ell m}\left\{\mathcal{A}^{(\ell)}_{rn}\Delta^{(\mu_1}_{(\nu_1}\cdots\Delta^{\mu_\ell)}_{\nu_\ell)} + [\text{terms of type (i) and (ii)}]\right\} , \tag{4.45}$$

where the parentheses denote the symmetrization of all Lorentz indices. Contracting Eq. (4.45) with $\Delta^{\alpha_1\cdots\alpha_\ell}_{\mu_1\cdots\mu_\ell}\Delta^{\nu_1\cdots\nu_\ell}_{\beta_1\cdots\beta_\ell}$ and using Eq. (4.41), we prove that

$$(\mathcal{A}_{rn})^{\alpha_1\cdots\alpha_\ell}_{\beta_1\cdots\beta_m} = \delta_{\ell m}\,\mathcal{A}^{(\ell)}_{rn}\,\Delta^{\alpha_1\cdots\alpha_\ell}_{\beta_1\cdots\beta_\ell} . \tag{4.46}$$

Finally, substituting Eq. (4.46) into Eq. (4.39) we derive,

$$L^{\langle\mu_1\cdots\mu_\ell\rangle}_{r-1} = -\sum_{n=0}^{\infty}\mathcal{A}^{(\ell)}_{rn}\rho^{\mu_1\cdots\mu_\ell}_n . \tag{4.47}$$

This completes our goal to express the linear collision term as a linear combination of the irreducible moments. The coefficients $\mathcal{A}^{(\ell)}_{rn}$ can be obtained from the following projection of $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$,

$$\begin{aligned}\mathcal{A}^{(\ell)}_{rn} &= \frac{1}{\Delta^{\mu_1\cdots\mu_\ell}_{\mu_1\cdots\mu_\ell}}\Delta^{\nu_1\cdots\nu_\ell}_{\mu_1\cdots\mu_\ell}\,(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} \\ &= \frac{1}{\nu\,(2\ell+1)}\int dKdK'dPdP'\,W_{\mathbf{kk'}\to\mathbf{pp'}}f_{0\mathbf{k}}f_{0\mathbf{k'}}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p'}}E^{r-1}_{\mathbf{k}}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} \\ &\times\left(\mathcal{H}^{(\ell)}_{\mathbf{k}n}\,k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}+\mathcal{H}^{(\ell)}_{\mathbf{k'}n}\,k'_{\langle\mu_1}\cdots k'_{\mu_\ell\rangle}-\mathcal{H}^{(\ell)}_{\mathbf{p}n}\,p_{\langle\mu_1}\cdots p_{\mu_\ell\rangle}-\mathcal{H}^{(\ell)}_{\mathbf{p'}n}\,p'_{\langle\mu_1}\cdots p'_{\mu_\ell\rangle}\right) ,\end{aligned} \tag{4.48}$$

where we used that $\Delta^{\mu_1\cdots\mu_\ell}_{\mu_1\cdots\mu_\ell}=2\ell+1$. The coefficients $\mathcal{A}^{(\ell)}_{rn}$ are (invertible) matrices (the matrix structure is indicated by the row and column indices r and n) and contain all the information of the underlying microscopic theory. We explicitly compute some of them for the case of a classical gas of massless particles with constant cross section in Chapter 5.

We remark that, for $\ell=0$ the terms with $n=1,2$, and for $\ell=1$ the term with $n=1$ are zero, because the moments ρ_1, ρ_2, and ρ^μ_1 vanish due to the definition of the velocity field and the matching conditions, Eqs. (3.16) and (3.23). Therefore, in order to invert $\mathcal{A}^{(\ell)}_{rn}$, for $\ell=0$ we have to exclude the second and third rows and columns, and for $\ell=1$ the second row and column.

We note that similar matrices already appeared in the context of Chapman-Enskog theory and the method of moments, where they were referred to as $\mathcal{A}^{\text{s}}_{rn}$, $\mathcal{A}^{\text{v}}_{rn}$, and $\mathcal{A}^{\text{t}}_{rn}$, cf. Eq. (3.93). Note that, if one replaces $\mathcal{H}^{(\ell)}_{\mathbf{k}n}\to E^n_{\mathbf{k}}$ in Eq. (4.48), both definitions actually agree (up to prefactors): $\mathcal{A}^{\mathrm{s}}_{rn} \to \mathcal{A}^{(0)}_{rn}$, $\mathcal{A}^{\mathrm{v}}_{rn} \to \beta_0 \mathcal{A}^{(1)}_{rn}$, and $\mathcal{A}^{\mathrm{t}}_{rn} \to \beta_0^2 \mathcal{A}^{(2)}_{rn}$. Therefore, the difference between Chapman-Enskog theory and the method of moments is entirely due to a different basis employed in the moment expansion in each method. In Chapman-Enskog theory the expansion basis employed was $1, E_{\mathbf{k}}$, $E^2_{\mathbf{k}}, \ldots$, while in the method of moments an orthogonal basis of polynomials was employed, $1, P^{(\ell)}_{\mathbf{k}1}, P^{(\ell)}_{\mathbf{k}2}, \ldots$. Since both bases are equivalent, the final result will be approximately the same in both methods, provided one includes a sufficiently large number of basis elements.

4.4 Summary

In this chapter we have presented a **relativistic generalization of the method of moments**. We constructed an orthonormal basis in terms of **irreducible tensors** $k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle}$ and **polynomials** in powers of $E_{\mathbf{k}}$, which allowed us to expand the deviation $\delta f_{\mathbf{k}}$ of the single-particle distribution function $f_{\mathbf{k}}$ from a reference state (which we took to be the distribution function $f_{0\mathbf{k}}$ in local thermodynamical equilibrium) in terms of the **irreducible moments** $\rho_n^{\mu_1\cdots\mu_\ell}$ of the deviations of the distribution function from equilibrium. We then proceeded to derive **exact equations of motion** for these moments. Finally, we also showed how to compute the moments $C_n^{\langle\mu_1\cdots\mu_\ell\rangle}$ of the **collision term** in the Boltzmann equation, which appear in the equations of motion for the irreducible moments. We restricted the discussion to the part of the collision integral which is **linear** in the irreducible moments. In Chapter 5 we will demonstrate how the fluid-dynamical equations can be derived from the equations of motion for the irreducible moments.

4.5 Appendix 1: Irreducible projection operators

In this appendix, we present the **irreducible projection operators** necessary to derive the irreducible moments of $\delta f_{\mathbf{k}}$. We start by recalling the definition of the irreducible projection operators [6, 20, 24],

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} = \sum_{q=0}^{[n/2]} C(n,q)\, \frac{1}{\mathcal{N}_{nq}} \sum_{\mathcal{P}^n_\mu \mathcal{P}^n_\nu} \Delta^{\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} \Delta_{\nu_1\nu_2} \cdots \Delta_{\nu_{2q-1}\nu_{2q}} \Delta^{\mu_{2q+1}}_{\nu_{2q+1}} \cdots \Delta^{\mu_n}_{\nu_n} \,. \tag{4.49}$$

Here, $[n/2]$ denotes the largest integer less than or equal to $n/2$, the coefficients $C(n,q)$ are defined as

$$C(n,q) = (-1)^q \frac{(n!)^2}{(2n)!} \frac{(2n-2q)!}{q!(n-q)!(n-2q)!} \,, \tag{4.50}$$

and the second sum in Eq. (4.49) runs over all **distinct** permutations $\mathcal{P}^n_\mu \mathcal{P}^n_\nu$ of μ- and ν-type indices. The coefficient in front of this sum is just the inverse of the total number of these distinct permutations,

$$\mathcal{N}_{nq} \equiv \frac{1}{(n-2q)!} \left(\frac{n!}{2^q q!}\right)^2 \,. \tag{4.51}$$

This number can be explained as follows: $(n!)^2$ is the number of **all** permutations of μ- and ν-type indices. In order to obtain the number of **distinct** permutations, one has to divide this by the number $(2^q)^2$ of permutations of μ- and ν-type indices on the same Δ projectors (where only projectors with only μ- and only ν-type indices are considered), and by the number $(q!)^2$ of trivial reorderings of the sequence of these projectors. Finally, one also has to divide by the number $(n-2q)!$ of trivial reorderings of the sequence of projectors with mixed indices.

The projectors (4.49) are symmetric under exchange of μ- and ν-type indices,

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} = \Delta^{(\mu_1\cdots\mu_n)}_{(\nu_1\cdots\nu_n)} \,, \tag{4.52}$$

and traceless with respect to contraction of either μ- or ν-type indices,

$$\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} g_{\mu_i\mu_j} = \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} g^{\nu_i\nu_j} = 0 \quad \text{for any } i,j \,. \tag{4.53}$$

Moreover, upon complete contraction,

$$\Delta^{\mu_1\cdots\mu_n}_{\mu_1\cdots\mu_n} \equiv \Delta^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n} g_{\mu_1\nu_1}\cdots g_{\mu_n\nu_n} = 2n+1 \,, \tag{4.54}$$

cf. Eq. (23) in Chapter VI.2 of Ref. [6]. Note that the relation (4.54) means that the projection of an arbitrary tensor of rank n with respect to Eq. (4.49), i.e., $A^{\nu_1\cdots\nu_n}\Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}$ has $2n+1$ independent tensor components.

In order to prove this, we note that an arbitrary tensor $A_d^{\mu_1\cdots\mu_n}$ of rank n in d-dimensional space-time has d^n independent components, because each of the n indices can assume d distinct values. Now consider a rank-n tensor which is completely symmetric with respect to the interchange of indices. This tensor can be constructed from the arbitrary tensor $A_d^{\mu_1\cdots\mu_n}$ via symmetrization,

$$A_d^{(\mu_1\cdots\mu_n)} = \frac{1}{n!}\sum_{\mathcal{P}_\mu} A_d^{\mu_1\cdots\mu_n} \,, \tag{4.55}$$

where the sum over $\mathcal{P}_\mu$ runs over all $n!$ permutations of the μ-type indices. The number of independent tensor components of such a symmetric tensor is given by the number of combinations with repetition to draw n elements from a set of d elements,

$$N_{dn}\left(A_d^{(\mu_1\cdots\mu_n)}\right) = \frac{(n+d-1)!}{n!\,(d-1)!} \,. \tag{4.56}$$

Let us now demand in addition that this tensor is traceless,

$$0 = A_d^{(\mu_1\cdots\mu_n)} g_{\mu_{n-1}\mu_n} \equiv A_d^{(\mu_1\cdots\mu_{n-2})} \,, \tag{4.57}$$

where the right-hand side defines a new symmetric tensor of rank $n-2$. According to Eq. (4.56), this tensor has

$$N_{dn}\left(A_d^{(\mu_1\cdots\mu_{n-2})}\right) = \frac{(n+d-3)!}{(n-2)!\,(d-1)!} \tag{4.58}$$

independent components. This is also the number of constraints by which the number of independent components of the original symmetric tensor $A_d^{(\mu_1\cdots\mu_n)}$ is reduced, if we demand that it is traceless in addition to being symmetric. Thus, the number of independent components of a symmetric traceless tensor is

$$\begin{aligned} N_{dn}\left(A_{d,\mathrm{tr-less}}^{(\mu_1\cdots\mu_n)}\right) &= N_{dn}\left(A_d^{(\mu_1\cdots\mu_n)}\right) - N_{dn}\left(A_d^{(\mu_1\cdots\mu_{n-2})}\right) \\ &= \frac{(n+d-1)!}{n!\,(d-1)!} - \frac{(n+d-3)!}{(n-2)!\,(d-1)!} = \frac{(n+d-3)!}{n!\,(d-2)!}(2n+d-2)\;. \end{aligned} \tag{4.59}$$

Let us now require in addition that such a symmetric traceless tensor is orthogonal to a given 4-vector u^μ,

$$0 = A_{d,\mathrm{tr-less}\perp}^{(\mu_1\cdots\mu_n)} u_{\mu_n} \equiv A_{d,\mathrm{tr-less}}^{(\mu_1\cdots\mu_{n-1})}\;. \tag{4.60}$$

The right-hand side defines a new symmetric traceless tensor of rank $n-1$ which, according to Eq. (4.59), has

$$N_{dn}\left(A_{d,\mathrm{tr-less}}^{(\mu_1\cdots\mu_{n-1})}\right) = \frac{(n+d-4)!}{(n-1)!\,(d-2)!}(2n+d-4) \tag{4.61}$$

independent components. This number reduces the number of independent components of the original symmetric traceless tensor, if we demand in addition that it is orthogonal to u^μ; thus the latter has

$$\begin{aligned} N_{dn}\left(A_{d,\mathrm{tr-less}\perp}^{(\mu_1\cdots\mu_n)}\right) &= N_{dn}\left(A_{d,\mathrm{tr-less}}^{(\mu_1\cdots\mu_n)}\right) - N_{dn}\left(A_{d,\mathrm{tr-less}}^{(\mu_1\cdots\mu_{n-1})}\right) \\ &= \frac{(n+d-3)!}{n!\,(d-2)!}(2n+d-2) - \frac{(n+d-4)!}{(n-1)!\,(d-2)!}(2n+d-4) \\ &= \frac{(n+d-4)!}{n!\,(d-3)!}(2n+d-3) \end{aligned} \tag{4.62}$$

independent components. Comparing this equation to Eq. (4.59) we realize that the orthogonality constraint (4.60) has effectively reduced the number of dimensions by one unit, $d \to d-1$.

Now taking $d=4$, Eq. (4.62) tells us that any symmetric traceless tensor of rank n, which is orthogonal to u^μ, has $N_{4n}(A_{4,\mathrm{tr-less}\perp}^{(\mu_1\cdots\mu_n)}) = 2n+1$ independent components. If this tensor is in addition orthogonal to another 4-vector l^μ, then Eq. (4.62) applies replacing $d=4$ by $d=3$, and we obtain $N_{3n}(A_{3,\mathrm{tr-less}\perp}^{(\mu_1\cdots\mu_n)}) = 2$ independent components. This result is independent of the tensor rank n.

4.6 Appendix 2: Thermodynamic integrals and properties

In this appendix, we compute the thermodynamic integrals $I_{i+n,q}$ in Eq. (4.31). They are obtained by suitable projections of the tensors

$$\mathcal{I}_i^{\mu_1\cdots\mu_n} = \left\langle E_{\mathbf{k}}^i\, k^{\mu_1}\cdots k^{\mu_n}\right\rangle_0\;, \tag{4.63}$$

where the angular brackets denote the average over momentum space defined in Eq. (3.24). The subscript i on this quantity reflects the power of $E_{\mathbf{k}}$ in the definition of the moment. Due to the fact that the equilibrium distribution function depends only on the quantities α_0, β_0, and the flow velocity u^μ, the equilibrium moments can be expanded in terms of u^μ and the projector $\Delta^{\mu\nu}$ as

$$\mathcal{I}_i^{\mu_1\cdots\mu_n} = \sum_{q=0}^{[n/2]} (-1)^q \, b_{nq} \, I_{i+n,q} \, \Delta^{(\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n)} \,, \tag{4.64}$$

where n, q are natural numbers while the sum runs over $0 \leq q \leq [n/2]$. Here, $[n/2]$ denotes the largest integer which is less than or equal to $n/2$.

The coefficient b_{nq} in Eq. (4.64) is defined as the number of distinct terms in the symmetrized tensor product

$$\Delta^{(\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n)} \equiv \frac{1}{b_{nq}} \sum_{\mathcal{P}_\mu^n} \Delta^{\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n} \,, \tag{4.65}$$

where the sum runs over all distinct permutations of the n indices $\mu_1, \ldots, \mu_n$. The total number of permutations of n indices is $n!$. There are q projection operators $\Delta^{\mu_i\mu_j}$ and $n-2q$ factors of u^{μ_k}. Permutations of the order of the $\Delta^{\mu_i\mu_j}$ and of the u^{μ_k} among themselves do not lead to distinct terms, so we need to divide the total number $n!$ by $q!(n-2q)!$. Finally, since $\Delta^{\mu_i\mu_j}$ is a symmetric projection operator, a permutation of its indices does not lead to a distinct term. Since there are q such projection operators, there are 2^q permutations that also do not lead to distinct terms. Hence, the total number of distinct terms in the symmetrized tensor product is

$$b_{nq} \equiv \frac{n!}{2^q q! \, (n-2q)!} = \frac{n! \, (2q-1)!!}{(2q)! \, (n-2q)!} \,, \tag{4.66}$$

which is identical to Eq. (A2) of Ref. [4].

In order to obtain the thermodynamic integrals $I_{i+n,q}$ by projection of the tensors $\mathcal{I}_i^{\mu_1\cdots\mu_n}$, it is advantageous to use the orthogonality relation [6]

$$\Delta^{(\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n)} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q'-1}\mu_{2q'}} u_{\mu_{2q'+1}} \cdots u_{\mu_n)} = \frac{(2q+1)!!}{b_{nq}} \delta_{qq'} \,. \tag{4.67}$$

Let us prove this relation. First, it is clear that if $q \neq q'$ there are terms where a u^{μ_i} gets contracted with a $\Delta_{\mu_i\mu_j}$, which gives zero. The existence of the Kronecker delta is thus easily explained and we only need to prove Eq. (4.67) for $q = q'$. Second, as the same set of indices is symmetrized on both tensor products, it actually suffices to keep the set of indices fixed on one tensor, say in the order $\mu_1, \ldots, \mu_{2q}, \mu_{2q+1}, \ldots, \mu_n$, and symmetrize only the one on the other,

$$\begin{aligned} &\Delta^{(\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n)} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q}} u_{\mu_{2q+1}} \cdots u_{\mu_n)} \\ &= \Delta^{\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n} \frac{1}{b_{nq}} \sum_{\mathcal{P}_\mu^n} \Delta_{\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q}} u_{\mu_{2q+1}} \cdots u_{\mu_n} \,, \end{aligned} \tag{4.68}$$

where we used Eq. (4.65). Among the terms in the sum over all distinct permutations, only those survive where the indices on the u's are $\mu_{2q+1}, \ldots, \mu_n$, just as in the term in front of the sum. (Otherwise, a u_{μ_i} will be contracted with a $\Delta^{\mu_i \mu_j}$, which gives zero.) Permutations among these indices do not lead to distinct terms. Using $u^\mu u_\mu = 1$, we thus obtain

$$\begin{aligned} &\Delta^{(\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n)} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q}} u_{\mu_{2q+1}} \cdots u_{\mu_n)} \\ &\quad = \frac{1}{b_{nq}} \Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q-1}\mu_{2q}} \sum_{\mathcal{P}_\mu^{2q}} \Delta_{\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q}} , \end{aligned} \tag{4.69}$$

where the sum now runs only over the distinct permutations of $2q$ indices $\mu_1, \ldots, \mu_{2q}$ on the Δ projectors. There are in total $(2q)!/(2^q q!) \equiv (2q-1)!!$ distinct terms, so that we obtain

$$\begin{aligned} &\Delta^{(\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n)} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q}} u_{\mu_{2q+1}} \cdots u_{\mu_n)} \\ &\quad = \frac{(2q-1)!!}{b_{nq}} \Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q-1}\mu_{2q}} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q})} . \end{aligned} \tag{4.70}$$

The proof of Eq. (4.67) is completed by proving that [6]

$$\Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q-1}\mu_{2q}} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q})} = 2q+1 . \tag{4.71}$$

This is done by complete induction. Since $\Delta^{\mu_1\mu_2}\Delta_{\mu_1\mu_2} = \Delta^{\mu_1}_{\mu_1} = 3$, Eq. (4.71) obviously holds for $q=1$. Now suppose it holds for q. Then we have to show that it also holds for $q+1$. In this case, using the definition of the symmetrized tensor,

$$\begin{aligned} &\Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q+1}\mu_{2q+2}} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q+1}\mu_{2q+2})} \\ &\quad = \frac{2^{q+1}(q+1)!}{(2q+2)!} \Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q+1}\mu_{2q+2}} \sum_{\mathcal{P}_\mu^{2q+2}} \Delta_{\mu_1\mu_2} \cdots \Delta_{\mu_{2q+1}\mu_{2q+2}} . \end{aligned} \tag{4.72}$$

Consider the contraction of $\Delta^{\mu_{2q+1}\mu_{2q+2}}$ with the sum over distinct permutations of $2q+2$ indices $\mu_1, \ldots, \mu_{2q+2}$. There is one term in the sum where both indices are on the same Δ projector. This term is $\sim \Delta^{\mu_{2q+1}\mu_{2q+2}}\Delta_{\mu_{2q+1}\mu_{2q+2}} \equiv 3$. Then, there are $2q$ terms where the indices μ_{2q+1} and μ_{2q+2} are on different projectors, say $\Delta_{\mu_{2q+1}\mu_j}\Delta_{\mu_i\mu_{2q+2}}$. Contracting with $\Delta^{\mu_{2q+1}\mu_{2q+2}}$ gives a term $\sim \Delta_{\mu_i\mu_j}$, where both indices are from the set $\mu_1, \ldots, \mu_{2q}$. Putting this together and using Eq. (4.71) gives

$$\begin{aligned} &\Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q+1}\mu_{2q+2}} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q+1}\mu_{2q+2})} \\ &\quad = \frac{2(q+1)}{(2q+2)(2q+1)} \frac{2^q q!}{(2q)!} \Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q-1}\mu_{2q}} (2q+3) \sum_{\mathcal{P}_\mu^{2q}} \Delta_{\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q}} \\ &\quad = \frac{2q+3}{2q+1} \Delta^{\mu_1\mu_2} \ldots \Delta^{\mu_{2q-1}\mu_{2q}} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q})} \equiv 2q+3 , \quad \text{q.e.d.} . \end{aligned} \tag{4.73}$$

With the orthogonality relation (4.67), we now easily find by projecting Eq. (4.64) that

$$\begin{aligned} I_{i+n,q} &\equiv \frac{(-1)^q}{(2q+1)!!} \mathcal{I}_i^{\mu_1 \cdots \mu_n} \Delta_{(\mu_1\mu_2} \cdots \Delta_{\mu_{2q-1}\mu_{2q}} u_{\mu_{2q+1}} \cdots u_{\mu_n)} \\ &= \frac{(-1)^q}{(2q+1)!!} \int dK \, E_{\mathbf{k}u}^{i+n-2q} \left(\Delta^{\mu\nu} k_\mu k_\nu\right)^q f_{0\mathbf{k}} , \end{aligned} \tag{4.74}$$

where we used the definition (4.63) of the tensor $\mathcal{I}_i^{\mu_1\cdots\mu_n}$. With the definition of the thermodynamic average $\langle\ldots\rangle_0$, the second line yields Eq. (4.31).

Other useful relations are obtained from contracting two indices of the tensors (4.63) with a Δ projector,

$$\mathcal{I}_i^{\mu_1\cdots\mu_n}\Delta_{\mu_{n-1}\mu_n} = m^2\,\mathcal{I}_i^{\mu_1\cdots\mu_{n-2}} - \mathcal{I}_{i+2}^{\mu_1\cdots\mu_{n-2}} \,. \tag{4.75}$$

Comparison of Eq. (4.63) for $n=0$ and Eq. (4.74) for $n=q=0$ yields the identity

$$I_{i,0} = \mathcal{I}_i \,, \tag{4.76}$$

and comparison of Eq. (4.63) for $n=2$ and Eq. (4.74) for $n=2, q=1$

$$I_{i+2,1} = -\frac{1}{3}\mathcal{I}_i^{\mu\nu}\Delta_{\mu\nu} = -\frac{1}{3}\left(m^2\mathcal{I}_i - \mathcal{I}_{i+2}\right) = -\frac{1}{3}\left(m^2 I_{i,0} - I_{i+2,0}\right) \,. \tag{4.77}$$

The thermodynamic integrals (4.31) obey useful recursion relations, which are given here. Replacing $\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^{q+1} = \left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^q\left(m^2 - E_{\mathbf{k}}^2\right)$ in Eq. (4.31) we obtain for $0 \leq q \leq n/2$,

$$I_{n+2,q} = m^2\,I_{nq} + (2q+3)\,I_{n+2,q+1} \,, \tag{4.78}$$

cf. Eq. (3.54). For $n=q=0$ this reads

$$I_{20} = m^2\,I_{00} + 3I_{21} \,, \tag{4.79}$$

which is consistent with Eq. (4.77) for $i=0$. In the massless limit this leads to the familiar relation $\varepsilon_0 = 3P_0$.

4.7 Appendix 3: Orthogonality of the irreducible tensors

In this appendix, we derive the **orthogonality condition** (4.7) for the irreducible tensors (4.4). The derivation utilizes the relation

$$k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} = \frac{\ell!}{(2\ell-1)!!}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^\ell \,. \tag{4.80}$$

This relation is proved as follows. We first note that

$$\begin{aligned} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} &= \Delta^{\mu_1\cdots\mu_\ell}_{\beta_1\cdots\beta_\ell}\Delta^{\alpha_1\cdots\alpha_\ell}_{\mu_1\cdots\mu_\ell}k_{\alpha_1}\cdots k_{\alpha_\ell}k^{\beta_1}\cdots k^{\beta_\ell} \\ &= \Delta^{\alpha_1\cdots\alpha_\ell}_{\beta_1\cdots\beta_\ell}k_{\alpha_1}\cdots k_{\alpha_\ell}k^{\beta_1}\cdots k^{\beta_\ell} \,. \end{aligned} \tag{4.81}$$

Now we insert the explicit form (4.49) of the projection operator and note that the contraction of all indices with the momenta reduces the second sum (including the prefactor $1/\mathcal{N}_{nq}$) to just a factor of $(\Delta^{\alpha\beta}k_\alpha k_\beta)^\ell$,

$$k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} = \sum_{q=0}^{[\ell/2]} C(\ell,q)\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^\ell \,. \tag{4.82}$$

The Legendre polynomials $P_\ell(z)$ have the representation [25]

$$P_\ell(z) = \frac{1}{2^\ell} \sum_{q=0}^{[\ell/2]} (-1)^q \frac{(2\ell-2q)!}{q!(\ell-q)!(\ell-2q)!} z^{\ell-2q} \equiv \frac{1}{2^\ell} \frac{(2\ell)!}{(\ell!)^2} \sum_{q=0}^{[\ell/2]} C(\ell,q)\, z^{\ell-2q} \, . \tag{4.83}$$

Since for all ℓ

$$1 \equiv P_\ell(1) = \frac{1}{2^\ell} \frac{(2\ell)!}{(\ell!)^2} \sum_{q=0}^{[\ell/2]} C(\ell,q) \, , \tag{4.84}$$

we derive the identity

$$\sum_{q=0}^{[\ell/2]} C(\ell,q) = \frac{2^\ell (\ell!)^2}{(2\ell)!} = \frac{\ell!\, 2^{\ell-1}(\ell-1)!}{(2\ell-1)!} \equiv \frac{\ell!}{(2\ell-1)!!} \, , \tag{4.85}$$

where we have used the definition of the double factorial for odd numbers. Inserting this into Eq. (4.82) proves Eq. (4.80).

The orthogonality condition (4.7) is obtained from an integral of the type

$$M^{\langle\mu_1\cdots\mu_\ell\rangle}_{\langle\nu_1\cdots\nu_n\rangle} = \int dK\, \mathrm{F}(E_\mathbf{k})\; k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} k_{\langle\nu_1} \cdots k_{\nu_n\rangle} \, , \tag{4.86}$$

which is a tensor of rank $(\ell+n)$ that is (separately) symmetric under the permutation of μ-type and ν-type indices. In App. A of Ref. [20] it is proven that tensors of this type must obey the relation

$$M^{\langle\mu_1\cdots\mu_\ell\rangle}_{\langle\nu_1\cdots\nu_n\rangle} = \delta_{\ell n}\, \mathcal{M}\, \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_n} \, , \tag{4.87}$$

where $\mathcal{M}$ is an invariant scalar that can be computed by completely contracting the indices of $M^{\langle\mu_1\cdots\mu_\ell\rangle}_{\langle\nu_1\cdots\nu_n\rangle}$,

$$\begin{aligned} \mathcal{M} &\equiv \frac{1}{\Delta^{\mu_1\cdots\mu_\ell}_{\mu_1\cdots\mu_\ell}} \int dK\, \mathrm{F}(E_\mathbf{k})\, k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \\ &= \frac{\ell!}{(2\ell+1)!!} \int dK\; \mathrm{F}(E_\mathbf{k}) \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^\ell \, , \end{aligned} \tag{4.88}$$

where we have used Eqs. (4.54) and (4.80). This proves Eq. (4.7).

4.8 Appendix 4: Orthogonal polynomials

The **orthogonal polynomials** $P^{(\ell)}_{\mathbf{k}n}$ are defined by orthogonality condition (4.8) with the measure $\omega^{(\ell)}$ given by Eq. (4.9).

The polynomials $P^{(\ell)}_{\mathbf{k}m}$ are generated by the set 1, $E_\mathbf{k}$, $E^2_\mathbf{k}$, ... We follow Ref. [26] and construct this orthogonal set via the **Gram-Schmidt orthogonalization method** and using the measure (4.9). Without loss of generality, the first polynomial of each series is always set to one, i.e., $P^{(\ell)}_{\mathbf{k}0} = a^{(\ell)}_{00} = 1$. Then, the weights $W^{(\ell)}$ can be determined enforcing that the normalization condition (4.8) is satisfied for $m = n = 0$,

$$\int dK\, \omega^{(\ell)} = 1 \, . \tag{4.89}$$

This will lead to the following expressions for the weights,

$$W^{(\ell)} = \frac{(-1)^{\ell}}{\tilde{J}_{2\ell,\ell}} \,, \tag{4.90}$$

where it was convenient to define the thermodynamic integrals $\tilde{J}_{nq}$

$$\tilde{J}_{nq}\left(\alpha_0, \beta_0\right) = \int dK \; E_{\mathbf{k}}^{n-2q} \left(-\Delta_{\alpha\beta} k^{\alpha} k^{\beta}\right)^{q} f_{0\mathbf{k}} \mathcal{G}_{\mathbf{k}} \,. \tag{4.91}$$

With this knowledge, the remaining coefficients $a_{nm}^{(\ell)}$ can be extracted from Eq. (4.8). For a given value of n, the normalization/orthogonality conditions (4.8) lead to $n+1$ equations that can be used to calculate $a_{nm}^{(\ell)}$ for $m = 0, \ldots, n$. Inverting such coefficients from Eq. (4.8) is a cumbersome task, but can be achieved with the proper numerical resources. As an example, let us discuss the case of $n = 1$, for an arbitrary value of ℓ. In this case, Eqs. (4.5) and (4.8) lead to

$$a_{10}^{(\ell)} \int dK \, \omega^{(\ell)} + a_{11}^{(\ell)} \int dK \, \omega^{(\ell)} E_{\mathbf{k}} \;=\; 0 \,, \tag{4.92}$$

$$a_{10}^{(\ell)} \int dK \, \omega^{(\ell)} E_{\mathbf{k}} + a_{11}^{(\ell)} \int dK \, \omega^{(\ell)} E_{\mathbf{k}}^{2} \;=\; \frac{1}{a_{11}^{(\ell)}} \,. \tag{4.93}$$

The solution of Eqs. (4.92) and (4.93) is

$$\frac{a_{10}^{(0)}}{a_{11}^{(0)}} \;=\; -\frac{\tilde{J}_{10}}{\tilde{J}_{00}} \,, \tag{4.94}$$

$$\left[a_{11}^{(0)}\right]^{2} \;=\; \frac{\tilde{J}_{00}^{2}}{\tilde{J}_{20}\tilde{J}_{00} - \tilde{J}_{10}^{2}} \,, \tag{4.95}$$

As it turns out, it will always be sufficient to know, for a given n, the ratios, $a_{nm}^{(\ell)}/a_{nn}^{(\ell)}$, and the square of $a_{nn}^{(\ell)}$, $\left[a_{nn}^{(\ell)}\right]^{2}$, since only these quantities appear in the moment expansion of $\phi_{\mathbf{k}}$. Extracting the actual signs of the coefficients is irrelevant. General formulas for $a_{nm}^{(\ell)}/a_{nn}^{(\ell)}$ and $\left[a_{nn}^{(\ell)}\right]^{2}$ were derived in Ref. [20].

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5 Fluid Dynamics from the Method of Moments

05/20/2022

In this chapter, we present a **general derivation of relativistic fluid dynamics from the Boltzmann equation** [1] employing the **generalized method of moments presented** in Chapter 4. The main point of this approach, in contrast with the traditional 14-moment approximation, is that it does not close the fluid-dynamical equations of motion by an *ad hoc* truncation of the moment expansion. Instead, the reduction of the degrees of freedom is performed by **identifying the microscopic time scales** of the Boltzmann equation and considering only the **slowest ones**. In addition, the equations of motion for the dissipative quantities are truncated according to a systematic **power-counting scheme in Knudsen and inverse Reynolds number**. We conclude that the equations of motion can be closed in terms of only 14 dynamical variables, as long as we only keep terms of first order in either Knudsen or inverse Reynolds number, or the product of both. We show that, even though the equations of motion are closed in terms of these 14 fields, the transport coefficients carry information about **all** the moments of the distribution function.

This chapter is organized as follows. In Sec. 5.1 we introduce our **power-counting scheme in terms of Knudsen and inverse Reynolds numbers**. Then, by diagonalizing the linear part of the set of moment equations derived in Chapter 4, we demonstrate how to identify the **slowest microscopic time scale of the Boltzmann equation** for each dissipative current. We shall derive **dynamical equations for the slowest modes**, but **approximate faster modes by their asymptotic solution** for long times. This will then lead, in Sec. 5.3, to the **complete set of fluid-dynamical equations** which contains **all terms up to second order in Knudsen and inverse Reynolds numbers** R_i^{-1}, i.e., $\mathcal{O}(\mathrm{Kn}^2, \mathrm{R}_i^{-1}\mathrm{R}_j^{-1}, \mathrm{Kn}\,\mathrm{R}_i^{-1})$, where the indices $i, j = \Pi, n, \pi$ refer to the particular way of defining the inverse Reynolds numbers (see below). In Sec. 5.4 we first demonstrate the validity of our approach by restricting the calculation to the **14-moment approximation** and recovering the results derived in Chapter 3 (and of Ref. [2]) for the transport coefficients for the case of an ultrarelativistic, classical gas with constant cross section. We then show how to **successively improve** the expressions for the transport coefficients by **extending the number of moments** to $14 + 9 \times n$. We explicitly study the cases $n = 1, 2$, and 3. In Sec. 5.6 we derive hyperbolic equations of motion for transient fluid dynamics that are **valid up to second order in Knudsen number**. In Sec. 5.7 we compare the solutions of the derived fluid-dynamical equations to **numerical solutions of the Boltzmann equation** for certain shock-wave scenarios. We analyze the domain of validity of the derived equations, depending on the choice of cross section. We end this chapter with a brief discussion and summary of the results obtained.

5.1 Power counting

In Chapter 4 the single-particle distribution function $f_{\mathbf{k}}$ is expanded in terms of an orthogonal basis in 4-momentum space. The expansion basis contains two basic ingredients. The first are the **irreducible tensors**, 1, $k^{\langle\mu\rangle}$, $k^{\langle\mu_1}k^{\mu_2\rangle}, \ldots, k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}, \ldots$, which form a **complete and orthogonal** set, analogous to the spherical harmonics [1, 3]. We use the notation $A^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} A^{\nu_1\cdots\nu_\ell}$, with $\Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m}$ being symmetrized and, except for $m=1$, traceless projectors onto the subspaces orthogonal to u^μ, as defined in App. 4.5. In the local rest frame of the fluid, the irreducible tensors are formed from the spatial components of the 4-momentum k^μ. Note that the expansion of $f_{\mathbf{k}}$ in Israel-Stewart theory is **not** in terms of the irreducible tensors $k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}$, but in terms of the tensors $k^{\mu_1}\cdots k^{\mu_\ell}$, which are complete but neither irreducible nor orthogonal.

The second ingredient are **orthogonal polynomials in the energy** in the local rest frame, $E_{\mathbf{k}} = u^\mu k_\mu$, $P^{(\ell)}_{n\mathbf{k}} = \sum_{r=0}^n a^{(\ell)}_{nr} E^r_{\mathbf{k}}$, constructed in App. 4.8. Then, $f_{\mathbf{k}}$ is expanded as

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty}\sum_{n=0}^{N_\ell} \mathcal{H}^{(\ell)}_{n\mathbf{k}}\, \rho_n^{\mu_1\cdots\mu_\ell}\, k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} \,, \tag{5.1}$$

where $f_{0\mathbf{k}} = \left[\exp\left(\beta_0 E_{\mathbf{k}} - \alpha_0\right) + a\right]^{-1}$ is the local-equilibrium distribution function, with $\alpha_0 = \mu/T$ being the thermal potential and $\beta_0 = 1/T$ the inverse temperature, while $a = \pm 1$ for fermions/bosons and $a = 0$ for classical Boltzmann particles. We further introduced polynomials of order N_ℓ in energy, $\mathcal{H}^{(\ell)}_{n\mathbf{k}} \equiv \left(W^{(\ell)}/\ell!\right)\sum_{m=n}^{N_\ell} a^{(\ell)}_{mn} P^{(\ell)}_{m\mathbf{k}}$, with a normalization constant $W^{(\ell)}$, and the **irreducible moments** of $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}}$,

$$\rho_r^{\mu_1\cdots\mu_\ell} \equiv \int dK\, E^r_{\mathbf{k}}\, k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}} \,, \tag{5.2}$$

The values of α_0 and β_0 are defined by the matching conditions for particle number density and energy density, $n \equiv n_0 = \langle E_{\mathbf{k}}\rangle_0$, $\varepsilon \equiv \varepsilon_0 = \langle E^2_{\mathbf{k}}\rangle_0$, where $\langle\cdots\rangle_0 \equiv \int dK\,(\cdots)\, f_{0\mathbf{k}}$. The matching conditions and the definition of u^μ according to the Landau-frame choice imply that the following irreducible moments vanish: $\rho_1 = \rho_2 = \rho_1^\mu = 0$.

The equations of motion for ρ_r, ρ_r^μ, and $\rho_r^{\mu\nu}$ together with their respective transport coefficients were derived in Chapter 4, cf. Eqs. (4.21) – (4.23). For the sake of convenience, we again quote these equations of motion,

$$\begin{aligned}
\dot{\rho}_r - C_{r-1} = {}& \alpha_r^{(0)}\theta - \frac{G_{2r}}{D_{20}}\Pi\theta + \frac{G_{2r}}{D_{20}}\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{G_{3r}}{D_{20}}\partial_\mu n^\mu + (r-1)\,\rho^{\mu\nu}_{r-2}\sigma_{\mu\nu} \\
& + r\rho^\mu_{r-1}\dot{u}_\mu - \nabla_\mu\rho^\mu_{r-1} - \frac{1}{3}\left[(r+2)\,\rho_r - (r-1)\,m^2\rho_{r-2}\right]\theta \,,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\dot{\rho}_r^{\langle\mu\rangle} - C^{\langle\mu\rangle}_{r-1} = {}& \alpha_r^{(1)} I^\mu + \rho_r^\nu\omega^\mu_\nu + \frac{1}{3}\left[(r-1)\,m^2\rho^\mu_{r-2} - (r+3)\,\rho^\mu_r\right]\theta - \Delta^\mu_\lambda\nabla_\nu\rho^{\lambda\nu}_{r-1} + r\rho^{\mu\nu}_{r-1}\dot{u}_\nu \\
& + \frac{1}{5}\left[(2r-2)\,m^2\rho^\nu_{r-2} - (2r+3)\,\rho^\nu_r\right]\sigma^\mu_\nu + \frac{1}{3}\left[m^2 r\rho_{r-1} - (r+3)\,\rho_{r+1}\right]\dot{u}^\mu \\
& + \frac{\beta_0 J_{r+2,1}}{\varepsilon_0 + P_0}\left(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta^\mu_\nu\partial_\lambda\pi^{\lambda\nu}\right) - \frac{1}{3}\nabla^\mu\left(m^2\rho_{r-1} - \rho_{r+1}\right) \\
& + (r-1)\,\rho^{\mu\nu\lambda}_{r-2}\sigma_{\lambda\nu} \,,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} &= 2\alpha_r^{(2)}\sigma^{\mu\nu} - \frac{2}{7}\left[(2r+5)\,\rho_r^{\lambda\langle\mu} - 2m^2\,(r-1)\,\rho_{r-2}^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle} + 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}_{\lambda} \\ &+ \frac{2}{15}\left[(r+4)\,\rho_{r+2} - (2r+3)\,m^2\rho_r + (r-1)\,m^4\rho_{r-2}\right]\sigma^{\mu\nu} \\ &+ \frac{2}{5}\nabla^{\langle\mu}\left(\rho_{r+1}^{\nu\rangle} - m^2\rho_{r-1}^{\nu\rangle}\right) - \frac{2}{5}\left[(r+5)\,\rho_{r+1}^{\langle\mu} - m^2 r\rho_{r-1}^{\langle\mu}\right]\dot{u}^{\nu\rangle} \\ &- \frac{1}{3}\left[(r+4)\,\rho_r^{\mu\nu} - m^2\,(r-1)\,\rho_{r-2}^{\mu\nu}\right]\theta \\ &+ (r-1)\,\rho_{r-2}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda\,, \end{aligned} \tag{5.5}$$

where $C_r^{\langle\mu_1\cdots\mu_\ell\rangle}$ are the irreducible moments of the collision term

$$C_r^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \int dK\, E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} C\left[f\right]\ . \tag{5.6}$$

We also employ the usual notation, where $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}$ is the shear tensor, $\theta \equiv \nabla_\mu u^\mu$ the expansion scalar, and $\omega^{\mu\nu} \equiv \left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right)/2$ the vorticity tensor. The thermodynamic functions $\alpha_r^{(0)}$, $\alpha_r^{(1)}$, $\alpha_r^{(2)}$, G_{nq}, and D_{nq} were introduced and defined in Eqs. (4.28), (4.29), (4.30), and (4.32). Finally, in deriving these equations, Landau matching conditions [4] were employed.

In Chapter 4 we further expressed the irreducible moments of the collision term, $C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle}$, in terms of the irreducible moments of $\delta f_{\mathbf{k}}$,

$$C_{r-1}^{\langle\mu_1\ldots\mu_\ell\rangle} = -\sum_{n=0}^{N_\ell}\mathcal{A}_{rn}^{(\ell)}\,\rho_n^{\mu_1\cdots\mu_\ell} + (\text{terms nonlinear in }\ \delta f)\ , \tag{5.7}$$

where the scalars $\mathcal{A}_{rn}^{(\ell)}$ emerged from the linear part of the collision term, cf. Eq. (4.48),

$$\begin{aligned}\mathcal{A}_{rn}^{(\ell)} &= \frac{1}{\nu\,(2\ell+1)}\int dKdK'dPdP' W_{\mathbf{kk'}\to\mathbf{pp'}} f_{0\mathbf{k}} f_{0\mathbf{k'}}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p'}} E_{\mathbf{k}}^{r-1} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} \\ &\times\left(\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle\nu_1}\cdots k_{\nu_\ell\rangle} + \mathcal{H}_{\mathbf{k'}n}^{(\ell)} k'_{\langle\nu_1}\cdots k'_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle\nu_1}\cdots p_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p'}n}^{(\ell)} p'_{\langle\nu_1}\cdots p'_{\nu_\ell\rangle}\right)\ ,\end{aligned} \tag{5.8}$$

while (terms nonlinear in δf) correspond to terms that are nonlinear in the irreducible moments of $\delta f_{\mathbf{k}}$.

It was found that there is an **infinite number of coupled equations** (labeled by the index r) for these moments, and the equations for the moments up to rank two contain moments of rank **higher than two**. In general, one would have to solve this infinite set of coupled equations in order to determine the time evolution of the system. However, in the fluid-dynamical limit, it is expected that the macroscopic dynamics of a given system simplifies, and therefore it can be described by the conserved currents N^μ and $T^{\mu\nu}$ alone. From the kinetic point of view, it is usually believed that the **validity of the fluid-dynamical limit** can be quantified by the **Knudsen number**,

$$\mathrm{Kn} = \frac{\lambda}{L}\ , \tag{5.9}$$

where λ and L are typical microscopic and macroscopic length or time scales of the system, respectively, as already defined in Chapter 1. The relevant **macroscopic scales** are usually estimated from the **gradients of fluid-dynamical quantities**, while the **microscopic scales** are of the order of the **mean free path or time** between collisions. It is generally assumed that when there is a **clear separation** of the microscopic and macroscopic scales, i.e., when $\mathrm{Kn} \ll 1$, the **microscopic details can be safely integrated out** and the dynamics of the system can be described using **only a few macroscopic fields**.

Furthermore, we also expect fluid dynamics to be **valid near local thermodynamical equilibrium**, i.e., when $\delta f_{\mathbf{k}} \ll f_{0\mathbf{k}}$. We can quantify the **deviation from equilibrium** in terms of the macroscopic variables by defining a set of **ratios of dissipative quantities to the equilibrium pressure or density**. These can be understood as generalizations of the **inverse Reynolds number** and will be denoted as

$$\mathrm{R}_{\Pi}^{-1} \equiv \frac{|\Pi|}{P_0} , \quad \mathrm{R}_{n}^{-1} \equiv \frac{|n^{\mu}|}{n_0} , \quad \mathrm{R}_{\pi}^{-1} \equiv \frac{|\pi^{\mu\nu}|}{P_0} . \tag{5.10}$$

Since the non-equilibrium moments are integrals of $\delta f_{\mathbf{k}}$, while the equilibrium pressure and particle number density are integrals over the equilibrium distribution function $f_{0\mathbf{k}}$, these ratios may be used to quantify **how significantly the system deviates from equilibrium**.

With this in mind, it is clear that these two measures, the Knudsen number and the inverse Reynolds number, should be used to quantify the proximity of the system to the fluid-dynamical limit. In general, these two measures are **independent** of each other, e.g., a system can be initialized in such way that the Knudsen number is large, but the inverse Reynolds number is small or vice versa. When deriving transient fluid dynamics, one should not *a priori* assume that $\mathrm{Kn} \sim \mathrm{R}_i^{-1}$: while the inverse Reynolds and Knudsen numbers are certainly related, their relation is in principle dynamical and is precisely what we aim to find. Only for **asymptotically long times**, when the solutions of the dynamical equations **approach their Navier-Stokes values**, one typically has $\mathrm{Kn} \sim \mathrm{R}_i^{-1}$, as will be discussed in more detail below.

In the traditional 14-moment approximation introduced by Israel and Stewart [5], the fluid-dynamical limit is implemented by a truncation of the expansion of the distribution function, which corresponds **neither** to a truncation in Knudsen **nor** in inverse Reynolds number. In this sense, the domain of validity of the equations of motion obtained via the traditional 14-moment approximation is not clear, because it is not possible to determine the order of the terms that were neglected. In order to obtain a closed set of macroscopic equations with a **clear domain of validity in both Kn and $\mathbf{R}_i^{-1}$**, another truncation procedure is necessary. Its derivation is the main purpose of this chapter.

5.2 Resummed transient relativistic fluid dynamics

The exact equations of motion (5.3) – (5.5) contain **infinitely many degrees of freedom**, given by the irreducible moments of the distribution function, and also **infinitely many microscopic time scales**, related to the coefficients $\mathcal{A}_{rn}^{(\ell)}$. As was argued in Ref. [6], the **slowest** microscopic time scale should dominate the dynamics at **long times**, i.e., in the fluid-dynamical limit. In order to extract the relevant relaxation time scales, we have to determine the **normal modes** of Eqs. (5.3) – (5.5), i.e., we **diagonalize** the part which is **linear** in the irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$. These are the linear terms on the left-hand sides arising from Eq. (5.7) and the first terms on the right-hand sides. The nonlinear terms from Eq. (5.7) as well as the remaining terms on the right-hand sides of Eqs. (5.3) – (5.5), which are products of the moments and gradients of the primary fluid-dynamical variables α_0, β_0, and u^μ, or which are gradients of the moments, are not considered in the diagonalization procedure. Identifying and separating the microscopic time scales of the Boltzmann equation is also the basic step for obtaining general relations between the irreducible moments and the dissipative currents and, as we shall see, closing the equations of motion in terms of N^μ and $T^{\mu\nu}$.

To this end, we shall introduce the matrix $\Omega^{(\ell)}$, which **diagonalizes** $\mathcal{A}^{(\ell)}$,

$$\left(\Omega^{-1}\right)^{(\ell)} \mathcal{A}^{(\ell)}\Omega^{(\ell)} = \mathrm{diag}\left(\chi_0^{(\ell)}, \ldots, \chi_j^{(\ell)}, \ldots\right) , \tag{5.11}$$

where $\chi_j^{(\ell)}$ are the **eigenvalues** of $\mathcal{A}^{(\ell)}$. We further define the tensors $X_i^{\mu_1\cdots\mu_\ell}$ as

$$X_i^{\mu_1\cdots\mu_\ell} \equiv \sum_{j=0}^{N_\ell} \left(\Omega^{-1}\right)_{ij}^{(\ell)} \rho_j^{\mu_1\cdots\mu_\ell} . \tag{5.12}$$

These are the **eigenmodes** of the linearized Boltzmann equation. Multiplying Eq. (5.7) with $\left(\Omega^{-1}\right)^{(\ell)}$ from the left and using Eqs. (5.11) and (5.12) we obtain

$$\sum_{j=0}^{N_\ell} \left(\Omega^{-1}\right)_{ij}^{(\ell)} C_{j-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\chi_i^{(\ell)} X_i^{\mu_1\cdots\mu_\ell} + (\text{terms nonlinear in } \delta f) , \tag{5.13}$$

where we do not sum over the index i on the right-hand side of the equation. Then we multiply Eqs. (5.3) – (5.5) with $\left(\Omega^{-1}\right)_{ir}^{(\ell)}$ and sum over r. Using Eq. (5.13), we obtain the equations of motion for the new variables $X_i^{\mu_1\cdots\mu_\ell}$,

$$\begin{aligned} \dot{X}_i + \chi_i^{(0)} X_i &= \beta_i^{(0)}\theta \quad + \ (\text{higher-order terms}) , \\ \dot{X}_i^{\langle\mu\rangle} + \chi_i^{(1)} X_i^\mu &= \beta_i^{(1)} I^\mu \quad + \ (\text{higher-order terms}) , \\ \dot{X}_i^{\langle\mu\nu\rangle} + \chi_i^{(2)} X_i^{\mu\nu} &= \beta_i^{(2)}\sigma^{\mu\nu} \ + \ (\text{higher-order terms}) , \end{aligned} \tag{5.14}$$

where we introduced the transport coefficients

$$\beta_i^{(0)} = \sum_{j=0,\neq 1,2}^{N_0} \left(\Omega^{-1}\right)_{ij}^{(0)} \alpha_j^{(0)} , \quad \beta_i^{(1)} = \sum_{j=0,\neq 1}^{N_1} \left(\Omega^{-1}\right)_{ij}^{(1)} \alpha_j^{(1)} , \quad \beta_i^{(2)} = 2\sum_{j=0}^{N_2} \left(\Omega^{-1}\right)_{ij}^{(2)} \alpha_j^{(2)} . \tag{5.15}$$

With “higher-order terms” in Eqs. (5.14) we refer to the terms nonlinear in δf from Eq. (5.13) as well as to the higher-order terms on the right-hand sides of Eqs. (5.3) – (5.5). As expected, the equations of motion for the tensors $X_i^{\mu_1\cdots\mu_\ell}$ **decouple** in the linear regime, displaying a **relaxation-type behavior**. Without loss of generality, we order the tensors $X_r^{\mu_1\cdots\mu_\ell}$ according to increasing $\chi_r^{(\ell)}$, i.e., $\chi_r^{(\ell)} < \chi_{r+1}^{(\ell)}\ \forall\ \ell$.

By diagonalizing Eqs. (5.3) – (5.5) we were able to identify the **microscopic time scales** of the Boltzmann equation given by the **inverse of the coefficients** $\chi_r^{(\ell)}$. It is clear that, if the nonlinear terms in Eqs. (5.14) are sufficiently small, each tensor $X_r^{\mu_1\cdots\mu_\ell}$ **relaxes exponentially** and **independently** to its respective **asymptotic value**, given by the first term on the right-hand sides of Eqs. (5.14) (divided by the corresponding $\chi_r^{(\ell)}$), on a time scale $\sim 1/\chi_r^{(\ell)}$. We will refer to these asymptotic solutions as **Navier-Stokes values**. By neglecting all relaxation scales, i.e., taking the limit $\chi_r^{(\ell)} \to \infty$ with the ratio $\beta_r^{(\ell)}/\chi_r^{(\ell)}$ held fixed, all irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$ become proportional to gradients of α_0, β_0, and u^μ, and we obtain a Chapman-Enskog-type solution, which at first order in the Knudsen number results in the relativistic Navier-Stokes equations of fluid dynamics. As already discussed, these equations provide solutions which are unstable and acausal, hence they cannot provide a proper description of relativistic fluids.

The solution for this problem was already mentioned in Chapter 1. To obtain **causal and stable** equations one must take into account the **characteristic time scales** over which the bulk-viscous pressure, the particle-diffusion current, and the shear-stress tensor relax towards their asymptotic Navier-Stokes values. As shown in Ref. [6], in the fluid-dynamical limit these are given by the **slowest microscopic time scales of the underlying microscopic theory**.

In practice, this is implemented by assuming that only the **slowest modes** with rank 2 and smaller, X_0, X_0^μ, and $X_0^{\mu\nu}$, **remain in the transient regime** and satisfy the partial differential equations (5.14),

$$\begin{aligned}
\dot{X}_0 + \chi_0^{(0)} X_0 &= \beta_0^{(0)}\theta \quad + \ (\text{higher-order terms}) \ , \\
\dot{X}_0^{\langle\mu\rangle} + \chi_0^{(1)} X_0^\mu &= \beta_0^{(1)} I^\mu \quad + \ (\text{higher-order terms}) \ , \\
\dot{X}_0^{\langle\mu\nu\rangle} + \chi_0^{(2)} X_0^{\mu\nu} &= \beta_0^{(2)}\sigma^{\mu\nu} \ + \ (\text{higher-order terms}) \ ,
\end{aligned} \tag{5.16}$$

while the modes described by **faster relaxation scales**, i.e., X_r, X_r^μ, and $X_r^{\mu\nu}$, for any $r > 0$, will be **approximated by their asymptotic solutions**,

$$\begin{aligned}
X_r &\simeq \frac{\beta_r^{(0)}}{\chi_r^{(0)}}\theta \quad + \ (\text{higher-order terms}) \ , \\
X_r^\mu &\simeq \frac{\beta_r^{(1)}}{\chi_r^{(1)}} I^\mu \quad + \ (\text{higher-order terms}) \ , \\
X_r^{\mu\nu} &\simeq \frac{\beta_r^{(2)}}{\chi_r^{(2)}}\sigma^{\mu\nu} \ + \ (\text{higher-order terms}) \ .
\end{aligned} \tag{5.17}$$

While this approximation is similar to the Chapman-Enskog expansion, Eqs. (5.16) go beyond the Chapman-Enskog expansion since they include the **transient dynamics of the slowest eigenmodes**.

Note that, for $r \geq 1$, X_r, X_r^μ, and $X_r^{\mu\nu}$ are of **first order in Knudsen number**, $\mathcal{O}(\mathrm{Kn})$. The reason is that the gradient terms θ, I^μ, and $\sigma^{\mu\nu}$ are proportional to L^{-1}, while $1/\chi_r^{(\ell)}$ is proportional to λ. The other coefficients $\beta_r^{(\ell)}$ are simply functions of the thermodynamic variables α_0, β_0.

Furthermore, in order to obtain the traditional equations of fluid dynamics given in terms of the conserved currents, it was already mentioned that there should not appear any tensor $X_r^{\mu\nu\lambda\cdots}$ of rank higher than 2. Neglecting such tensors is justified because they have asymptotic solutions which are at least $\mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1})$, i.e., beyond the order we consider here. This can be seen by noting that it is **impossible to construct irreducible tensors of rank larger than 2** in terms of **single powers of space-like gradients** of temperature, chemical potential or velocity, i.e., it is impossible to construct irreducible tensors of rank larger than 2 that are only of first order in Knudsen number.

Equations (5.17) enable us to approximate, up to a given order in Knudsen number, the irreducible moments that do not appear in the conserved currents in terms of those that do occur, namely the particle-diffusion current, the bulk-viscous pressure, and the shear-stress tensor. Using relations (5.17) it is possible to prove that, for all $r, n \geq 1$,

$$X_n = \frac{\chi_r^{(0)}}{\chi_n^{(0)}} \frac{\beta_n^{(0)}}{\beta_r^{(0)}} X_r + \text{(higher-order terms)} , \tag{5.18}$$

$$X_n^\mu = \frac{\chi_r^{(1)}}{\chi_n^{(1)}} \frac{\beta_n^{(1)}}{\beta_r^{(1)}} X_r^\mu + \text{(higher-order terms)} , \tag{5.19}$$

$$X_n^{\mu\nu} = \frac{\chi_r^{(2)}}{\chi_n^{(2)}} \frac{\beta_n^{(2)}}{\beta_r^{(2)}} X_r^{\mu\nu} + \text{(higher-order terms)} . \tag{5.20}$$

Then, choosing, e.g., $r = 3$ for the scalar modes, $r = 2$ for the vector modes, and $r = 1$ for the tensor modes in the above relations, we can write,

$$X_n = \frac{\chi_3^{(0)}}{\chi_n^{(0)}} \frac{\beta_n^{(0)}}{\beta_3^{(0)}} X_3 + \text{(higher-order terms)} , \tag{5.21}$$

$$X_n^\mu = \frac{\chi_2^{(1)}}{\chi_n^{(1)}} \frac{\beta_n^{(1)}}{\beta_2^{(1)}} X_2^\mu + \text{(higher-order terms)} , \tag{5.22}$$

$$X_n^{\mu\nu} = \frac{\chi_1^{(2)}}{\chi_n^{(2)}} \frac{\beta_n^{(2)}}{\beta_1^{(2)}} X_1^{\mu\nu} + \text{(higher-order terms)} . \tag{5.23}$$

Next, we invert Eq. (5.12),

$$\rho_r^{\mu_1\cdots\mu_\ell} = \sum_{n=0}^{N_\ell} \Omega_{rn}^{(\ell)} X_n^{\mu_1\cdots\mu_\ell} , \tag{5.24}$$

and, using Eqs. (5.17), obtain

$$\rho_r \simeq \Omega_{r0}^{(0)} X_0 + \left(\frac{\chi_3^{(0)}}{\beta_3^{(0)}} \sum_{n=3}^{N_0} \Omega_{rn}^{(0)} \frac{\beta_n^{(0)}}{\chi_n^{(0)}} \right) X_3 = \Omega_{r0}^{(0)} X_0 + \mathcal{O}(\mathrm{Kn}) ,$$

$$\rho_r^\mu \simeq \Omega_{r0}^{(1)} X_0^\mu + \left(\frac{\chi_2^{(1)}}{\beta_2^{(1)}} \sum_{n=2}^{N_1} \Omega_{rn}^{(1)} \frac{\beta_n^{(1)}}{\chi_n^{(1)}} \right) X_2^\mu = \Omega_{r0}^{(1)} X_0^\mu + \mathcal{O}(\mathrm{Kn}) ,$$

$$\rho_r^{\mu\nu} \simeq \Omega_{r0}^{(2)} X_0^{\mu\nu} + \left(\frac{\chi_1^{(2)}}{\beta_1^{(2)}} \sum_{n=1}^{N_2} \Omega_{rn}^{(2)} \frac{\beta_n^{(2)}}{\chi_n^{(2)}} \right) X_1^{\mu\nu} = \Omega_{r0}^{(2)} X_0^{\mu\nu} + \mathcal{O}(\mathrm{Kn}) . \tag{5.25}$$

Here, we indicated that the contributions from the modes X_3, X_2^μ, and $X_1^{\mu\nu}$ are of order $\mathcal{O}(\mathrm{Kn})$, cf. Eq. (5.17).

The dissipative quantities appearing in the conservation laws can be exactly identified with the irreducible moments ρ_0, ρ_0^μ, and $\rho_0^{\mu\nu}$ in the following way,

$$\rho_0 = -\frac{3}{m^2}\,\Pi\;, \qquad \rho_0^\mu = n^\mu\;, \qquad \rho_0^{\mu\nu} = \pi^{\mu\nu}\;. \tag{5.26}$$

Thus, taking $r = 0$ in Eqs. (5.25) and, without loss of generality, setting $\Omega_{00}^{(\ell)} = 1$ we obtain,

$$\begin{aligned}
-\frac{3}{m^2}\Pi &\simeq X_0 + \left(\frac{\chi_3^{(0)}}{\beta_3^{(0)}}\sum_{n=3}^{N_0}\Omega_{0n}^{(0)}\frac{\beta_n^{(0)}}{\chi_n^{(0)}}\right) X_3 = X_0 + \mathcal{O}(\mathrm{Kn})\;,\\
n^\mu &\simeq X_0^\mu + \left(\frac{\chi_2^{(1)}}{\beta_2^{(1)}}\sum_{n=2}^{N_1}\Omega_{0n}^{(1)}\frac{\beta_n^{(1)}}{\chi_n^{(1)}}\right) X_2^\mu = X_0^\mu + \mathcal{O}(\mathrm{Kn})\;,\\
\pi^{\mu\nu} &\simeq X_0^{\mu\nu} + \left(\frac{\chi_1^{(2)}}{\beta_1^{(2)}}\sum_{n=1}^{N_2}\Omega_{0n}^{(2)}\frac{\beta_n^{(2)}}{\chi_n^{(2)}}\right) X_1^{\mu\nu} = X_0^{\mu\nu} + \mathcal{O}(\mathrm{Kn})\;.
\end{aligned} \tag{5.27}$$

Substituting Eqs. (5.27) into Eqs. (5.25) and using Eq. (5.15), we obtain

$$\begin{aligned}
\frac{m^2}{3}\rho_r &\simeq -\Omega_{r0}^{(0)}\Pi + \frac{\chi_3^{(0)}}{\beta_3^{(0)}}\left(\zeta_r - \Omega_{r0}^{(0)}\zeta_0\right) X_3 = -\Omega_{r0}^{(0)}\Pi + \mathcal{O}(\mathrm{Kn})\;,\\
\rho_r^\mu &\simeq \Omega_{r0}^{(1)} n^\mu + \frac{\chi_2^{(1)}}{\beta_2^{(1)}}\left(\varkappa_r - \Omega_{r0}^{(1)}\varkappa_0\right) X_2^\mu = \Omega_{r0}^{(1)} n^\mu + \mathcal{O}(\mathrm{Kn})\;,\\
\rho_r^{\mu\nu} &\simeq \Omega_{r0}^{(2)}\pi^{\mu\nu} + 2\frac{\chi_1^{(2)}}{\beta_1^{(2)}}\left(\eta_r - \Omega_{r0}^{(2)}\eta_0\right) X_1^{\mu\nu} = \Omega_{r0}^{(2)}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn})\;,\\
\rho_r^{\mu\nu\lambda\cdots} &\simeq \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1})\;,
\end{aligned} \tag{5.28}$$

where we defined the transport coefficients

$$\zeta_r \equiv \frac{m^2}{3}\sum_{n=0,\neq 1,2}^{N_0}\tau_{rn}^{(0)}\alpha_n^{(0)}\;, \qquad \varkappa_r \equiv \sum_{n=0,\neq 1}^{N_1}\tau_{rn}^{(1)}\alpha_n^{(1)}\;, \qquad \eta_r \equiv \sum_{n=0}^{N_2}\tau_{rn}^{(2)}\alpha_n^{(2)}\;, \tag{5.29}$$

with

$$\tau_{rn}^{(\ell)} \equiv \sum_{m=0}^{N_\ell}\Omega_{rm}^{(\ell)}\frac{1}{\chi_m^{(\ell)}}\left(\Omega^{-1}\right)_{mn}^{(\ell)}\;. \tag{5.30}$$

These coefficients define the inverse of $\mathcal{A}^{(\ell)}$, $\tau^{(\ell)} \equiv (\mathcal{A}^{-1})^{(\ell)}$, cf. Eq. (5.11). Note that, for $\ell = 0$, one excludes the $m = 1, 2$ terms in the sum, and for $\ell = 1$ the $m = 1$ term. In order to obtain the fourth equation (5.28), we further used that $X_r^{\mu_1\cdots\mu_\ell} \sim \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1})$ for $\ell \geq 3$. In the next subsection, we shall identify the coefficients ζ_0, $\varkappa_0$, and η_0 as the bulk-viscosity, particle-diffusion, and shear-viscosity coefficient, respectively.

So far we have proved that, by taking into account only the **slowest relaxation time scales**, **all irreducible moments** of the deviation of the single-particle distribution function from the equilibrium one can be related, **up to first order in Knudsen number**, $\mathcal{O}(\mathrm{Kn})$, to the **dissipative currents**, Π, n^μ, and $\pi^{\mu\nu}$. This demonstrates that in this limit, it is possible to **reduce the number of dynamical variables** in Eqs. (5.3) – (5.5) to quantities **appearing in the conserved currents**. This will be explicitly shown in the next section. We note that so far we also explicitly kept the terms $\sim X_3$, X_2^μ, $X_1^{\mu\nu}$, for the purpose of extending traditional fluid dynamics, which only considers the 14 quantities appearing in the conservation equations as dynamical variables, to a fluid-dynamical theory with a larger number of dynamical variables, see Sec. 5.6.

We remark that similar relations between the irreducible moments and the dissipative currents can also be obtained with the 14-moment approximation, but with a different set of proportionality coefficients [7]. However, in the traditional 14-moment approximation such relations are obtained by **explicitly truncating** the moment expansion and, as a result, they are **not** of a definite order in powers of Knudsen number. This is the reason why the 14-moment approximation does not give rise to equations of motion with a definite domain of validity in Knudsen and inverse Reynolds numbers.

Note, however, that the relations (5.28) are only valid for the moments $\rho_r^{\mu_1\cdots\mu_\ell}$ with **positive** r. This is not a problem since similar relations can also be obtained for the irreducible moments with **negative** r. The moment expansion developed in Sec. 4.1 was constructed in terms of a **complete basis** and, therefore, any moment that does not appear in this expansion must be linearly related to those that do appear. This means that, using this moment expansion, it is possible to express the moments with negative r in terms of the ones with positive r. This was already done in Sec. 4.1, cf. Eq. (4.17). Here, we re-express this formula using a different notation,

$$\rho_{-r}^{\nu_1\cdots\nu_\ell} = \sum_{n=0}^{N_\ell} \mathcal{F}_{rn}^{(\ell)} \rho_n^{\nu_1\cdots\nu_\ell} \, , \tag{5.31}$$

where we defined the following thermodynamic integral

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!} \int dK \, f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-r} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^\ell \, . \tag{5.32}$$

Therefore, Eq. (5.28) reads for negative values of r

$$\begin{aligned}
\frac{m^2}{3}\rho_{-r} &= -\gamma_r^{(0)}\Pi + \left[\frac{\chi_3^{(0)}}{\beta_3^{(0)}} \sum_{n=0,\neq 1,2}^{N_0} \mathcal{F}_{rn}^{(0)} \left(\zeta_n - \Omega_{n0}^{(0)}\zeta_0\right)\right] X_3 = -\gamma_r^{(0)}\Pi + \mathcal{O}(\mathrm{Kn}) \, , \\
\rho_{-r}^\mu &= \gamma_r^{(1)} n^\mu + \left[\frac{\chi_2^{(1)}}{\beta_2^{(1)}} \sum_{n=0,\neq 1}^{N_1} \mathcal{F}_{rn}^{(1)} \left(\varkappa_n - \Omega_{n0}^{(1)}\varkappa_0\right)\right] X_2^\mu = \gamma_r^{(1)} n^\mu + \mathcal{O}(\mathrm{Kn}) \, , \\
\rho_{-r}^{\mu\nu} &= \gamma_r^{(2)}\pi^{\mu\nu} + 2\left[\frac{\chi_1^{(2)}}{\beta_1^{(2)}} \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)} \left(\eta_n - \Omega_{n0}^{(2)}\eta_0\right)\right] X_1^{\mu\nu} = \gamma_r^{(2)}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}) \, , \\
\rho_{-r}^{\mu\nu\cdots} &= \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1}) \, ,
\end{aligned} \tag{5.33}$$

where we introduced the coefficients

$$\gamma_r^{(0)} = \sum_{n=0,\neq 1,2}^{N_0} \mathcal{F}_{rn}^{(0)} \Omega_{n0}^{(0)} \; , \quad \gamma_r^{(1)} = \sum_{n=0,\neq 1}^{N_1} \mathcal{F}_{rn}^{(1)} \Omega_{n0}^{(1)} \; , \quad \gamma_r^{(2)} = \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)} \Omega_{n0}^{(2)} \; . \tag{5.34}$$

5.3 Resummed transient relativistic fluid dynamics: 14 dynamical variables

Now we are ready to close Eqs. (5.3) – (5.5) in terms of the 14 quantities appearing in N^μ and $T^{\mu\nu}$, i.e., the five primary fluid-dynamical quantities α_0, β_0, and u^μ, as well as the nine dissipative currents Π, n^μ, and $\pi^{\mu\nu}$, and to derive the fluid-dynamical equations of motion for these 14 dynamical variables.

For this purpose, it is convenient to use the inverse of $\mathcal{A}^{(\ell)}$, $\tau^{(\ell)} = (\mathcal{A}^{-1})^{(\ell)}$, cf. Eq. (5.30), which by definition satisfies $\tau^{(\ell)}\mathcal{A}^{(\ell)} = \mathbf{1}$. Hence, it is straightforward to rewrite Eq. (5.7) as

$$\sum_{j=0}^{N_\ell} \tau_{ij}^{(\ell)} C_{j-1}^{\langle \mu_1 \cdots \mu_\ell \rangle} = -\rho_i^{\mu_1 \cdots \mu_\ell} + (\text{terms nonlinear in } \delta f) \; . \tag{5.35}$$

Then we multiply Eqs. (5.3) – (5.5) by $\tau_{nr}^{(\ell)}$, sum over r, and substitute Eq. (5.35). Next, we use Eqs. (5.28) and (5.33) to replace all irreducible moments $\rho_i^{\mu_1 \cdots \mu_\ell}$ appearing in the equations by the fluid-dynamical variables. Additionally, all covariant time derivatives of α_0, β_0, and u^μ are replaced by spatial gradients of fluid-dynamical variables using the conservation laws in the form shown in Eqs. (4.25), (4.26), and (4.27). The resulting equations of motion can be written in the following form,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R} \; , \tag{5.36}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \varkappa I^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu \; , \tag{5.37}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu} \; . \tag{5.38}$$

We remark that in order to derive these equations of motion, we substituted X_3, X_2^μ, and $X_1^{\mu\nu}$ in Eqs. (5.28) and (5.33) by their Navier-Stokes values, i.e.,

$$X_3 = \frac{\beta_3^{(0)}}{\chi_3^{(0)}}\theta \; , \quad X_2^\mu = \frac{\beta_2^{(1)}}{\chi_2^{(1)}} I^\mu \; , \quad X_1^{\mu\nu} = \frac{\beta_1^{(2)}}{\chi_1^{(2)}} \sigma^{\mu\nu} \; , \tag{5.39}$$

and used Eq. (5.11) in the following form,

$$\sum_{j=0}^{N_\ell} \tau_{ij}^{(\ell)} \Omega_{jm}^{(\ell)} = \Omega_{im}^{(\ell)} \frac{1}{\chi_m^{(\ell)}} \; . \tag{5.40}$$

We further made use of the following relations,

$$\begin{aligned}\dot{\theta} &= -\sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - \frac{1}{3}\theta^2 + \frac{1}{\varepsilon_0 + P_0}\nabla_\mu F^\mu - \frac{2\left(\varepsilon_0 + P_0\right) + \beta_0 J_{30}}{\left(\varepsilon_0 + P_0\right)^3} F_\mu F^\mu \\ &\quad - \frac{\left(\varepsilon_0 + P_0\right) J_{20} - n_0 J_{30}}{\left(\varepsilon_0 + P_0\right)^3} F_\mu I^\mu + \mathcal{O}\left(\mathrm{Kn}^2 \mathrm{R}_i^{-1}\right) , \qquad (5.41)\\ \dot{I}^{\langle\mu\rangle} &= \frac{\left(\varepsilon_0 + P_0\right) J_{20} - n_0 J_{30}}{D_{20}} \nabla^\mu \theta - \left\{ \frac{\partial}{\partial \beta_0}\left[\beta_0 \frac{\left(\varepsilon_0 + P_0\right) J_{20} - n_0 J_{30}}{D_{20}}\right]\right\} \frac{1}{\varepsilon_0 + P_0}\theta F^\mu \\ &\quad + \left\{\left[\left(\frac{\partial}{\partial \alpha_0} + h_0^{-1}\frac{\partial}{\partial \beta_0}\right) \frac{\left(\varepsilon_0 + P_0\right) J_{20} - n_0 J_{30}}{D_{20}}\right] - \frac{1}{3}\right\}\theta I^\mu \\ &\quad - \sigma^{\mu\alpha} I_\alpha - \omega^{\mu\alpha} I_\alpha - \frac{1}{3}\theta I^\mu + \mathcal{O}\left(\mathrm{Kn}^2 \mathrm{R}_i^{-1}\right) , \qquad (5.42)\\ \dot{\sigma}^{\langle\mu\nu\rangle} &= -\sigma_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + 2\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \frac{2}{3}\sigma^{\mu\nu}\theta + \frac{1}{\varepsilon_0 + P_0}\nabla^{\langle\mu}F^{\nu\rangle} \\ &\quad - \frac{2(\varepsilon_0 + P_0) + \beta_0 J_{30}}{\left(\varepsilon_0 + P_0\right)^3} F^{\langle\mu}F^{\nu\rangle} - \frac{(\varepsilon_0 + P_0) J_{20} - n_0 J_{30}}{\left(\varepsilon_0 + P_0\right)^3} F^{\langle\mu}I^{\nu\rangle} + \mathcal{O}\left(\mathrm{Kn}^2 \mathrm{R}_i^{-1}\right) , \\ &\qquad (5.43)\end{aligned}$$

where we defined $F^\mu \equiv \nabla^\mu P_0$.

In the above equations of motion all nonlinear terms and couplings to other currents were collected in the tensors $\mathcal{J}$, $\mathcal{K}$, $\mathcal{R}$, $\mathcal{J}^\mu$, $\mathcal{K}^\mu$, $\mathcal{R}^\mu$, $\mathcal{J}^{\mu\nu}$, $\mathcal{K}^{\mu\nu}$, and $\mathcal{R}^{\mu\nu}$. The tensors $\mathcal{J}$, $\mathcal{J}^\mu$, and $\mathcal{J}^{\mu\nu}$ contain all terms of **first order in the product of Knudsen and inverse Reynolds numbers**,

$$\begin{aligned}\mathcal{J} &= -\ell_{\Pi n}\nabla_\mu n^\mu - \tau_{\Pi n} n^\mu F_\mu - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n} n^\mu I_\mu + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} , \\ \mathcal{J}^\mu &= -\tau_n n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta - \ell_{n\Pi}\nabla^\mu \Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_\lambda \pi_\nu^\lambda + \tau_{n\Pi}\Pi F^\mu - \tau_{n\pi}\pi^{\mu\nu}F_\nu \\ &\quad - \lambda_{nn} n_\nu \sigma^{\mu\nu} + \lambda_{n\Pi}\Pi I^\mu - \lambda_{n\pi}\pi^{\mu\nu} I_\nu , \\ \mathcal{J}^{\mu\nu} &= 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu}F^{\nu\rangle} \\ &\quad + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}I^{\nu\rangle} . \qquad (5.44)\end{aligned}$$

The tensors $\mathcal{K}$, $\mathcal{K}^\mu$, and $\mathcal{K}^{\mu\nu}$ contain all terms of **second order in Knudsen number**,

$$\begin{aligned}\mathcal{K} &= \tilde{\zeta}_1 \omega_{\mu\nu}\omega^{\mu\nu} + \tilde{\zeta}_2 \sigma_{\mu\nu}\sigma^{\mu\nu} + \tilde{\zeta}_3 \theta^2 + \tilde{\zeta}_4 I_\mu I^\mu + \tilde{\zeta}_5 F_\mu F^\mu + \tilde{\zeta}_6 I_\mu F^\mu + \tilde{\zeta}_7 \nabla_\mu I^\mu + \tilde{\zeta}_8 \nabla_\mu F^\mu , \\ \mathcal{K}^\mu &= \tilde{\varkappa}_1 \sigma^{\mu\nu} I_\nu + \tilde{\varkappa}_2 \sigma^{\mu\nu} F_\nu + \tilde{\varkappa}_3 I^\mu \theta + \tilde{\varkappa}_4 F^\mu \theta + \tilde{\varkappa}_5 \omega^{\mu\nu} I_\nu + \tilde{\varkappa}_6 \Delta_\lambda^\mu \partial_\nu \sigma^{\lambda\nu} + \tilde{\varkappa}_7 \nabla^\mu \theta , \\ \mathcal{K}^{\mu\nu} &= \tilde{\eta}_1 \omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \tilde{\eta}_2 \theta\sigma^{\mu\nu} + \tilde{\eta}_3 \sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \tilde{\eta}_4 \sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} \\ &\quad + \tilde{\eta}_5 I^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_6 F^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_7 I^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_8 \nabla^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_9 \nabla^{\langle\mu}F^{\nu\rangle} . \qquad (5.45)\end{aligned}$$

The tensors $\mathcal{R}$, $\mathcal{R}^\mu$, and $\mathcal{R}^{\mu\nu}$ contain all terms of **second order in inverse Reynolds number**,

$$\begin{aligned}\mathcal{R} &= \varphi_1 \Pi^2 + \varphi_2 n_\mu n^\mu + \varphi_3 \pi_{\mu\nu}\pi^{\mu\nu} , \\ \mathcal{R}^\mu &= \varphi_4 n_\nu \pi^{\mu\nu} + \varphi_5 \Pi n^\mu , \\ \mathcal{R}^{\mu\nu} &= \varphi_6 \Pi\pi^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} + \varphi_8 n^{\langle\mu}n^{\nu\rangle} . \qquad (5.46)\end{aligned}$$

In Eqs. (5.36) – (5.38), terms of order $\mathcal{O}(\mathrm{Kn}^3)$, $\mathcal{O}(\mathrm{R}_i^{-1}\mathrm{R}_j^{-1}\mathrm{R}_k^{-1})$, $\mathcal{O}(\mathrm{Kn}^2\mathrm{R}_i^{-1})$ and $\mathcal{O}(\mathrm{Kn}\,\mathrm{R}_i^{-1}\mathrm{R}_j^{-1})$ were omitted.

Note that the equations of motion are closed in terms of 14 dynamical variables, even without making use of the 14-moment approximation. This means that the reduction of degrees of freedom was **not obtained by a direct truncation** of the moment expansion, but by a **separation of the microscopic time scales and the power-counting scheme** itself. The information about all other moments are actually **included** in the transport coefficients, as will be shown later. If we neglect the terms of second order in Knudsen and inverse Reynolds number, Eqs. (5.45) and (5.46), respectively, we recover the equations of motion that are of the same form as those derived via the 14-moment approximation [2]. However, even in this case, the coefficients in Eq. (5.44) and the relaxation times in Eqs. (5.36) – (5.38) are not the same as those calculated from the 14-moment approximation of Israel and Stewart.

The resulting equations of motion (5.36) – (5.38) contain **57 transport coefficients**. In particular, the **viscosity coefficients and relaxation times** of the dissipative currents are found to be

$$\zeta \equiv \zeta_0 = \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_{0r}^{(0)} \alpha_r^{(0)} , \quad \varkappa \equiv \varkappa_0 = \sum_{r=0,\neq 1}^{N_1} \tau_{0r}^{(1)} \alpha_r^{(1)} , \quad \eta \equiv \eta_0 = \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \alpha_r^{(2)} ,$$
$$\tau_\Pi \equiv \frac{1}{\chi_0^{(0)}} , \quad \tau_n \equiv \frac{1}{\chi_0^{(1)}} , \quad \tau_\pi \equiv \frac{1}{\chi_0^{(2)}} , \tag{5.47}$$

cf. Eqs. (5.16) and (5.29). Note that in general these transport coefficients depend not only on one moment of the distribution function but on **all** moments of corresponding rank ℓ, i.e., the contribution of higher moments of the distribution function are **resummed** as contributions to the microscopic formulas for the transport coefficients. For this reason, we shall refer to this formalism as **Resummed Transient Relativistic Fluid Dynamics (RTRFD)**.

As in Chapman-Enskog theory, the viscosity coefficients can only be obtained by inverting $\mathcal{A}^{(\ell)}$. As a matter of fact, the microscopic formulas for ζ, $\varkappa$, and η obtained above are equivalent to those obtained from the Chapman-Enskog expansion, see Sec. 3.2. However, to obtain the transient dynamics of the fluid, characterized by the relaxation times, it is also necessary to find the **eigenvalues and eigenvectors** of $\mathcal{A}^{(\ell)}$.

In practice, the moment expansion of the single-particle distribution function is always **truncated** and the matrices $\mathcal{A}^{(\ell)}$, $\Omega^{(\ell)}$, and $\tau^{(\ell)}$ will actually be of **finite dimension**. The truncation of this expansion was already introduced as an upper limit, N_ℓ, in the corresponding summations. In principle, one should only truncate the moment expansion when the values of all relevant transport coefficients have converged. Note that different transport coefficients may require a different number of moments to converge.

5.4 Transport coefficients

In this section, we compute the transport coefficients for several cases. First, we consider the **lowest possible truncation scheme** for the moment expansion, with $N_0 = 2$,

$N_1 = 1$, and $N_2 = 0$. In this case, the distribution function is expanded in terms of **14 moments** and is actually **equivalent** to the one obtained via Israel-Stewart's 14-moment Ansatz. Second, we consider the next simplest case and take $N_0 = 3$, $N_1 = 2$, and $N_2 = 1$. Then, the distribution function is characterized by **23 moments**, and consequently we shall refer to this case as **23-moment approximation**. Finally, we include 32 and 41 moments and verify the **convergence of the transport coefficients**.

We also compute the transport coefficients of the terms appearing in $\mathcal{J}$, $\mathcal{J}^\mu$, and $\mathcal{J}^{\mu\nu}$, the explicit expressions of which are given in App. 5.9. Note, however, that we are using a **linear approximation to the collision term**. Nonlinear contributions could in principle also enter the transport coefficients in the equations of motion (5.36) – (5.38), but will not be calculated here. For this reason we also do not compute any coefficient of the terms of order $\mathcal{O}(\mathrm{R}_i^{-1}\mathrm{R}_j^{-1})$, i.e., those entering $\mathcal{R}$, $\mathcal{R}^\mu$, and $\mathcal{R}^{\mu\nu}$, cf. Eq. (5.46), since all of them originate **exclusively from nonlinear contributions to the collision term**. An investigation of these terms can be found in Ref. [8]. App. I of that reference also contains explicit expressions for the transport coefficients contained in $\mathcal{K}$, $\mathcal{K}^\mu$, and $\mathcal{K}^{\mu\nu}$, cf. Eq. (5.45). One should note that they vanish in the 14-moment approximation $(N_0 = 2, N_1 = 1, N_2 = 0)$.

5.4.1 14-moment approximation

The 14-moment approximation is recovered by truncating all summations at $N_0 = 2$, $N_1 = 1$, and $N_2 = 0$. For this specific truncation $\mathcal{A}^{(\ell)}$ is just a number (because for $\mathcal{A}^{(0)}$ we have to exclude the second and third rows and columns and for $\mathcal{A}^{(1)}$ the second row and column), and thus

$$\tau^{(\ell)} = \frac{1}{\mathcal{A}^{(\ell)}} , \quad \Omega^{(\ell)} = 1 , \quad \chi^{(\ell)} = \mathcal{A}^{(\ell)} .$$

Then, the equations of motion and transport coefficients reduce to those derived in Refs. [2, 9] and reproduced in Sec. 3.3.

For a **classical gas of hard spheres** with **constant** total cross section σ, in the massless limit the integrals $\mathcal{A}^{(1)} = \mathcal{A}^{(1)}_{00}$ and $\mathcal{A}^{(2)} = \mathcal{A}^{(2)}_{00}$ can be computed and have the following simple form

$$\mathcal{A}^{(1)} = \frac{4}{9\lambda} , \tag{5.48}$$

$$\mathcal{A}^{(2)} = \frac{3}{5\lambda} , \tag{5.49}$$

where we defined the mean free path $\lambda \equiv 1/(n_0\sigma)$. The details of this calculation are shown in App. 5.10. The coefficients in the ultrarelativistic limit, $m\beta_0 \to 0$, can then be calculated analytically. The coefficients of order $\mathcal{O}(\mathrm{Kn}\,\mathrm{R}_i^{-1})$ are collected for the shear viscosity and particle diffusion in Tables 5.1 and 5.2, respectively. Note that, in this limit, the bulk-viscous pressure vanishes, and thus we do not need to compute $\mathcal{A}^{(0)}$ or any coefficient related to bulk-viscous pressure.

$\varkappa$	$\tau_n[\lambda]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
$3/(16\sigma)$	$9/4$	1	$3/5$	$\beta_0/20$	$\beta_0/20$	$\beta_0/(80P_0)$

Table 5.1: The transport coefficients for the equation of motion of the particle-diffusion current, calculated for a classical gas with constant cross section in the ultra-relativistic limit and in the 14-moment approximation.

η	$\tau_\pi[\lambda]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
$4/(3\sigma\beta_0)$	$5/3$	$10/7$	0	$4/3$	0	0

Table 5.2: The transport coefficients for the equation of motion of the shear-stress tensor, calculated for a classical gas with constant cross section in the ultrarelativistic limit and in the 14-moment approximation.

5.4.2 23-moment approximation and beyond

In order to better understand our result (5.47), we compute the first correction to the expressions in Tables 5.1 and 5.2. For this purpose, we consider $N_0 = 3$, $N_1 = 2$, and $N_2 = 1$. Then, $\mathcal{A}^{(\ell)}$, $\Omega^{(\ell)}$, and $\tau^{(\ell)}$ are 2×2 matrices that can be computed from the collision integral (4.48). We obtain the elements of $\mathcal{A}^{(1,2)}$, its inverse $\tau^{(1,2)}$, and $\Omega^{(1,2)}$ as

$$\mathcal{A}^{(1)} = \frac{1}{3\lambda}\begin{pmatrix} 2 & \beta_0^2/30 \\ -4/\beta_0^2 & 1 \end{pmatrix}, \quad \mathcal{A}^{(2)} = \frac{1}{\lambda}\begin{pmatrix} 9/10 & -\beta_0/20 \\ 4/(3\beta_0) & 1/3 \end{pmatrix}, \tag{5.50}$$

$$\tau^{(1)} = \frac{3}{8}\lambda\begin{pmatrix} 15/4 & -\beta_0^2/8 \\ 15/\beta_0^2 & 15/2 \end{pmatrix}, \quad \tau^{(2)} = \frac{1}{11}\lambda\begin{pmatrix} 10 & 3\beta_0/2 \\ -40/\beta_0 & 27 \end{pmatrix}, \tag{5.51}$$

$$\Omega^{(1)} = \begin{pmatrix} 1 & 1 \\ -\left(15+\sqrt{105}\right)/\beta_0^2 & \left(-15+\sqrt{105}\right)/\beta_0^2 \end{pmatrix}, \quad \Omega^{(2)} = \begin{pmatrix} 1 & 1 \\ 8/\beta_0 & 10/(3\beta_0) \end{pmatrix}, \tag{5.52}$$

see App. 5.10 for details. For all matrices with $\ell = 1$, the second row and column have been removed. The eigenvalues of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are

$$\chi_0^{(1)} = \frac{1}{2\lambda}\left(1 - \sqrt{\frac{7}{135}}\right), \quad \chi_2^{(1)} = \frac{1}{2\lambda}\left(1 + \sqrt{\frac{7}{135}}\right), \tag{5.53}$$

$$\chi_0^{(2)} = \frac{1}{2\lambda}, \quad \chi_1^{(2)} = \frac{11}{15\lambda}. \tag{5.54}$$

Note that the next largest eigenvalue following $\chi_0^{(1)}$ is $\chi_2^{(1)}$, not $\chi_1^{(1)}$ (following our convention to erase all rows and columns with index 1 in the matrices for $\ell = 1$).

Using Eq. (5.47), we calculate the corrected values for the particle-number diffusion coefficient and diffusion relaxation time and for the shear viscosity and shear relaxation time,

$$\varkappa = \frac{21}{128} n_0 \lambda \simeq 0.164\, n_0 \lambda \;, \tag{5.55}$$

$$\tau_n = \frac{90}{45 - \sqrt{105}} \lambda \simeq 2.5897\, \lambda \;, \tag{5.56}$$

$$\eta = \frac{14}{11} P_0 \lambda \simeq 1.2727\, P_0 \lambda \;, \tag{5.57}$$

$$\tau_\pi = 2\lambda \;, \tag{5.58}$$

where we used that, in the massless and classical limit,

$$\alpha_0^{(1)} = \frac{1}{12} n_0 \;, \quad \alpha_2^{(1)} = -\frac{1}{\beta_0} P_0 \;, \tag{5.59}$$

$$\alpha_0^{(2)} = \frac{4}{5} P_0 \;, \quad \alpha_1^{(2)} = \frac{4}{\beta_0} P_0 \;. \tag{5.60}$$

As before, the coefficients in the ultrarelativistic limit, $m\beta_0 \to 0$, can then be calculated analytically. The coefficients of order $\mathcal{O}(\mathrm{Kn}\,\mathrm{R}_i^{-1})$ are collected for the equations of motion of particle-diffusion current and shear-stress tensor in Tables 5.3 and 5.4, respectively.

$\varkappa$	$\tau_n[\lambda]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
$21/(128\sigma)$	2.59	1.00	0.96	$0.054\beta_0$	$0.118\beta_0$	$0.0295\beta_0/P_0$

Table 5.3: The transport coefficients for the equation of motion of the particle-diffusion current, calculated for a classical gas with constant cross section in the ultra-relativistic limit and in the 23-moment approximation.

η	$\tau_\pi[\lambda]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
$14/(11\sigma\beta_0)$	2	134/77	$0.344/\beta_0$	4/3	$-0.689/\beta_0$	$-0.689/n_0$

Table 5.4: The transport coefficients for the equation of motion of the shear-stress tensor, calculated for a classical gas with constant cross section in the ultrarelativistic limit and in the 23-moment approximation.

In order to obtain these expressions we used the results from App. 5.11 and that, in the massless and classical limit, $D_{20} = 3P_0^2$. Note that most of the transport coefficients are corrected by the inclusion of more moments in the computation. The coefficients related to the shear-stress tensor are less affected by the additional moments, when compared to the particle-diffusion coefficients. This might explain the poor agreement between Israel-Stewart theory and numerical solutions of the Boltzmann equation in Ref. [10] regarding heat flow and fugacity.

We further checked the convergence of this approach by taking 32 and 41 moments. In this case, the matrices $\mathcal{A}^{(1,2)}$, $\tau^{(1,2)}$, and $\Omega^{(1,2)}$ were computed numerically. There is a clear **convergence** as we increase the number of moments. For the particular case of classical particles with constant cross section, 32 moments seem sufficient, see Tables 5.5 and 5.6 for the results.

number of moments	$\varkappa$	$\tau_n[\lambda]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
14	$3/(16\sigma)$	$9/4$	1	$3/5$	$\beta_0/20$	$\beta_0/20$	$\beta_0/(80P_0)$
23	$21/(128\sigma)$	2.59	1.0	0.96	$0.054\beta_0$	$0.118\beta_0$	$0.0295\beta_0/P_0$
32	$0.1605/\sigma$	2.57	1.0	0.93	$0.052\beta_0$	$0.119\beta_0$	$0.0297\beta_0/P_0$
41	$0.1596/\sigma$	2.57	1.0	0.92	$0.052\beta_0$	$0.119\beta_0$	$0.0297\beta_0/P_0$

Table 5.5: The transport coefficients for the equation of motion of the particle-diffusion current, calculated for a classical gas with constant cross section in the ultra-relativistic limit, in the 14, 23, 32, and 41-moment approximations.

number of moments	η	$\tau_\pi[\lambda]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
14	$4/(3\sigma\beta_0)$	$5/3$	$10/7$	0	$4/3$	0	0
23	$14/(11\sigma\beta_0)$	2	$134/77$	$0.344/\beta_0$	$4/3$	$-0.689/\beta_0$	$-0.689/n_0$
32	$1.268/(\sigma\beta_0)$	2	1.69	$0.254/\beta_0$	$4/3$	$-0.687/\beta_0$	$-0.687/n_0$
41	$1.267/(\sigma\beta_0)$	2	1.69	$0.244/\beta_0$	$4/3$	$-0.685/\beta_0$	$-0.685/n_0$

Table 5.6: The transport coefficients for the equation of motion of the shear-stress tensor, calculated for a classical gas with constant cross section in the ultrarelativistic limit, in the 14, 23, 32, and 41-moment approximations.

5.5 Discussion: Navier-Stokes limit and causality

Note that one of the main features of transient theories of fluid dynamics is the relaxation of the dissipative currents towards their Navier-Stokes values, with time scales given by the transport coefficients τ_Π, τ_n, and τ_π. From the Boltzmann equation, Navier-Stokes theory is obtained by means of the Chapman-Enskog expansion, cf. Chapter 3, which describes an asymptotic solution of the single-particle distribution function. It is already clear from the previous section that the equations of motion derived in this chapter approach Navier-Stokes-type solutions at asymptotically long times, in which the dissipative currents are solely expressed in terms of gradients of fluid-dynamical variables.

It is interesting to investigate, however, if our equations approach the correct Navier-Stokes theory, i.e., if the viscosity coefficients obtained via our method are equivalent to the ones obtained via Chapman-Enskog theory. It should be noted that this is neither the case for Grad's theory nor for Israel's and Stewart's theory [2, 3, 5, 11, 12]. The transport coefficients computed within these theories only coincide, if we use the **minimal truncation scheme** in Chapman-Enskog theory, as explained in Chapter 3. We remark that, after taking into account further corrections to the shear-viscosity coefficient, see Eq.

(5.57) and Table 5.6, our results approach the solution obtained using Chapman-Enskog theory, $\eta_{NS} = 1.2654/(\beta_0\sigma)$ [3]. In principle there is no reason for the method of moments to attain a different Navier-Stokes limit than Chapman-Enskog theory. As a matter of fact, if the same basis of irreducible tensors $k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}$ and polynomials $P^{(\ell)}_{\mathbf{k}n}$ is used in both calculations, they both yield the same result, even order by order.

It is important to mention that the terms $\mathcal{K}$, $\mathcal{K}^\mu$, and $\mathcal{K}^{\mu\nu}$, which are of second order in Knudsen number, lead to several problems. The terms which contain second-order spatial derivatives of u^μ, α_0, and P_0, e.g., $\nabla_\mu I^\mu$, $\nabla_\mu F^\mu$, $\nabla^{\langle\mu}I^{\nu\rangle}$, $\nabla^{\langle\mu}F^{\nu\rangle}$, $\Delta^\mu_\alpha\partial_\nu\sigma^{\alpha\nu}$, and $\nabla^\mu\theta$, are especially problematic since they change the boundary conditions of the equations. In relativistic systems these derivatives, even though they are squares of space-like vectors, also contain time derivatives and thus require initial values. This means that, by including them, one would have to specify not only the initial spatial distribution of the fluid-dynamical variables but also the spatial distribution of their time derivatives. In practice, this implies that we would be increasing the number of fluid-dynamical degrees of freedom.

There is an even more serious problem. By including terms of order higher than one in Knudsen number, the transport equations become **parabolic**. In a relativistic theory, this comes with disastrous consequences, since the solutions are acausal and consequently unstable [13, 14, 15]. For this reason we do not compute the transport coefficients for these higher-order terms in this chapter.

If one wants to include terms of higher order in Knudsen number, it is mandatory to include also second-order co-moving time derivatives of the dissipative quantities. Or, equivalently, one could promote the moments ρ_3, $\rho_2^\mu, \rho_1^{\mu\nu}$, or further ones, to dynamical variables. We will show how to do this in the next section.

5.6 Resummed transient relativistic fluid dynamics: 23 dynamical variables

As already discussed, the terms of higher order in Knudsen number render the equations of motion parabolic, **despite** the existence of a relaxation time. Therefore, describing the fluid up to a higher order in Knudsen number using Eqs. (5.36) – (5.38) is problematic since, in order to do so, one would have to solve parabolic equations in a covariant setup. In this section, we show how to solve this problem and derive transient fluid-dynamical equations of motion that are **hyperbolic even up to second order in the Knudsen number**.

The parabolic and, thus, acausal nature of the equations of motion (5.36) – (5.38) can be understood as follows. The main assumption of RTRFD is to approximate the quickly varying eigenmodes of the Boltzmann equation by their asymptotic (i.e., Navier-Stokes) values. This approximation happened in Eq. (5.17), while the substitution of the eigenmodes X_3, X_2^μ, and $X_1^{\mu\nu}$ by their Navier-Stokes values occurred in Eq. (5.39). It was this last step that rendered Eqs. (5.36) – (5.38) parabolic since in this substitution it is implicitly assumed that these eigenmodes relax **instantaneously** to their corresponding Navier-Stokes values, consequently leading to acausal behavior.

In order to obtain hyperbolic equations of motion which do not simply neglect terms of order $\mathcal{O}(\mathrm{Kn}^2)$, it is necessary to refrain from the substitution (5.39). This can be simply done by keeping X_3, X_2^μ, and $X_1^{\mu\nu}$ in Eqs. (5.28) as **independent dynamical variables** instead of replacing them by their Navier-Stokes values.

In this case, it is convenient to replace the modes X_3, X_2^μ, and $X_1^{\mu\nu}$ in Eqs. (5.28) and (5.33) by irreducible moments of the distribution function. In principle, any of the irreducible moments ρ_r, ρ_r^μ, and $\rho_r^{\mu\nu}$ can be used to replace these modes as independent dynamical variables. Without loss of generality, we choose ρ_3, ρ_2^μ, and $\rho_1^{\mu\nu}$. After Π ($\ell = 0$), n^μ ($\ell = 1$), and $\pi^{\mu\nu}$ ($\ell = 2$), these are the irreducible moments with the lowest power of $E_{\mathbf{k}}$ under the integral (5.2) which appear in the expansion of the single-particle distribution function (5.1). We can relate X_3, X_2^μ, and $X_1^{\mu\nu}$ to Π, n^μ, $\pi^{\mu\nu}$, ρ_3, ρ_2^μ, and $\rho_1^{\mu\nu}$, by taking in Eq. (5.28) $r = 3$ for the scalar irreducible moment, $r = 2$ for the first-rank irreducible moment, and $r = 1$ for the second-rank irreducible moment, obtaining

$$\begin{aligned}
\frac{\chi_3^{(0)}}{\beta_3^{(0)}}\left(\zeta_3 - \Omega_{30}^{(0)}\zeta_0\right) X_3 &= \frac{m^2}{3}\rho_3 + \Omega_{30}^{(0)}\Pi \,, \\
\frac{\chi_2^{(1)}}{\beta_2^{(1)}}\left(\varkappa_2 - \Omega_{20}^{(1)}\varkappa_0\right) X_2^\mu &= \rho_2^\mu - \Omega_{20}^{(1)} n^\mu \,, \\
2\frac{\chi_1^{(2)}}{\beta_1^{(2)}}\left(\eta_1 - \Omega_{10}^{(2)}\eta_0\right) X_1^{\mu\nu} &= \rho_1^{\mu\nu} - \Omega_{10}^{(2)}\pi^{\mu\nu} \,. \qquad (5.61)
\end{aligned}$$

Then, we substitute the relations (5.61) into Eqs. (5.28) and (5.33), effectively removing the dependence of the irreducible moments on the eigenmodes X_3, X_2^μ, and $X_1^{\mu\nu}$. We obtain the following new relations for the irreducible moments with positive r,

$$\begin{aligned}
\frac{m^2}{3}\rho_r &\simeq \lambda_{r0}^{(0)}\Pi + \lambda_{r3}^{(0)}\rho_3 + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1}, \mathrm{R}_i^{-2}) \,, \\
\rho_r^\mu &= \lambda_{r0}^{(1)} n^\mu + \lambda_{r2}^{(1)}\rho_2^\mu + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1}, \mathrm{R}_i^{-2}) \,, \\
\rho_r^{\mu\nu} &= \lambda_{r0}^{(2)}\pi^{\mu\nu} + \lambda_{r1}^{(2)}\rho_1^{\mu\nu} + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1}, \mathrm{R}_i^{-2}) \,, \qquad (5.62)
\end{aligned}$$

and for the irreducible moments with negative r,

$$\begin{aligned}
\frac{m^2}{3}\rho_{-r} &= \left(\sum_{n=0,\neq 1,2}^{N_0}\mathcal{F}_{rn}^{(0)}\lambda_{n0}^{(0)}\right)\Pi + \left(\sum_{n=0,\neq 1,2}^{N_0}\mathcal{F}_{rn}^{(0)}\lambda_{n3}^{(0)}\right)\rho_3 + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1}, \mathrm{R}_i^{-2}) \,, \\
\rho_{-r}^\mu &= \left(\sum_{n=0,\neq 1}^{N_1}\mathcal{F}_{rn}^{(1)}\lambda_{n0}^{(1)}\right) n^\mu + \left(\sum_{n=0,\neq 1}^{N_1}\mathcal{F}_{rn}^{(1)}\lambda_{n2}^{(1)}\right)\rho_2^\mu + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1}, \mathrm{R}_i^{-2}) \,, \\
\rho_{-r}^{\mu\nu} &= \left(\sum_{n=0}^{N_2}\mathcal{F}_{rn}^{(2)}\lambda_{n0}^{(2)}\right)\pi^{\mu\nu} + \left(\sum_{n=0}^{N_2}\mathcal{F}_{rn}^{(2)}\lambda_{n1}^{(2)}\right)\rho_1^{\mu\nu} + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{R}_i^{-1}, \mathrm{R}_i^{-2}) \,. \qquad (5.63)
\end{aligned}$$

The thermodynamic functions $\mathcal{F}_{rn}^{(\ell)}$ were defined in Eq. (5.32) and we introduced the auxiliary functions

$$\lambda_{r0}^{(0)} = \frac{\Omega_{30}^{(0)}\zeta_r - \Omega_{r0}^{(0)}\zeta_3}{\zeta_3 - \Omega_{30}^{(0)}\zeta_0} , \quad \lambda_{r3}^{(0)} = \frac{m^2}{3}\frac{\zeta_r - \Omega_{r0}^{(0)}\zeta_0}{\zeta_3 - \Omega_{30}^{(0)}\zeta_0} , \tag{5.64}$$

$$\lambda_{r0}^{(1)} = \frac{\Omega_{20}^{(1)}\varkappa_r - \Omega_{r0}^{(1)}\varkappa_2}{\Omega_{20}^{(1)}\varkappa_0 - \varkappa_2} , \quad \lambda_{r2}^{(1)} = \frac{\varkappa_0\Omega_{r0}^{(1)} - \varkappa_r}{\varkappa_0\Omega_{20}^{(1)} - \varkappa_2} , \tag{5.65}$$

$$\lambda_{r0}^{(2)} = \frac{\Omega_{10}^{(2)}\eta_r - \Omega_{r0}^{(2)}\eta_1}{\Omega_{10}^{(2)}\eta_0 - \eta_1} , \quad \lambda_{r1}^{(2)} = \frac{\Omega_{r0}^{(2)}\eta_0 - \eta_r}{\Omega_{10}^{(2)}\eta_0 - \eta_1} . \tag{5.66}$$

Using the relations (5.62), (5.63), it is possible to close Eqs. (5.3) – (5.5) in terms of Π, n^μ, $\pi^{\mu\nu}$, ρ_3, ρ_2^μ, and $\rho_1^{\mu\nu}$. The resulting equations of motion will be hyperbolic up to a higher order in Knudsen number when compared with Eqs. (5.36) – (5.38).

The equations of motion are obtained as explained in Sec. 5.3. We multiply Eqs. (5.3) – (5.5) by $\tau_{nr}^{(\ell)}$, substitute Eq. (5.35), and sum over r. The only difference is that now we use Eqs. (5.62) and (5.63) to replace all irreducible moments $\rho_i^{\mu_1\cdots\mu_\ell}$ appearing in the equations by Π, n^μ, $\pi^{\mu\nu}$, ρ_3, ρ_2^μ, and $\rho_1^{\mu\nu}$. The resulting equations of motion can be written as

$$\begin{aligned}
\hat{\tau}_\Pi\dot{\vec{\Pi}} + \vec{\Pi} &= -\vec{\zeta}\theta - \hat{\delta}_{\Pi\Pi}\vec{\Pi}\theta - \hat{\ell}_{\Pi n}\nabla_\mu\vec{n}^\mu - \hat{\tau}_{\Pi n}\vec{n}_\mu F^\mu - \hat{\lambda}_{\Pi n}\vec{n}_\mu I^\mu + \hat{\lambda}_{\Pi\pi}\vec{\pi}^{\mu\nu}\sigma_{\mu\nu} , \\
\hat{\tau}_n\dot{\vec{n}}^{\langle\mu\rangle} + \vec{n}^\mu &= \vec{\varkappa}I^\mu - \hat{\tau}_n\vec{n}_\nu\omega^{\nu\mu} - \hat{\delta}_{nn}\vec{n}^\mu\theta + \hat{\ell}_{n\pi}\Delta^{\mu\nu}\partial_\lambda\vec{\pi}_\nu^\lambda - \hat{\ell}_{n\Pi}\nabla^\mu\vec{\Pi} - \hat{\tau}_{n\pi}\vec{\pi}^{\mu\nu}F_\nu \\
&\quad + \hat{\tau}_{n\Pi}\vec{\Pi}F^\mu - \hat{\lambda}_{nn}\vec{n}_\nu\sigma^{\mu\nu} - \hat{\lambda}_{n\pi}\vec{\pi}^{\mu\nu}I_\nu + \hat{\lambda}_{n\Pi}\vec{\Pi}I^\mu , \\
\hat{\tau}_\pi\dot{\vec{\pi}}^{\langle\mu\nu\rangle} + \vec{\pi}^{\mu\nu} &= 2\vec{\eta}\sigma^{\mu\nu} + 2\hat{\tau}_\pi\vec{\pi}_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \hat{\delta}_{\pi\pi}\vec{\pi}^{\mu\nu}\theta - \hat{\tau}_{\pi\pi}\vec{\pi}^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} - \hat{\tau}_{\pi n}\vec{n}^{\langle\mu}F^{\nu\rangle} \\
&\quad + \hat{\ell}_{\pi n}\nabla^{\langle\mu}\vec{n}^{\nu\rangle} + \hat{\lambda}_{\pi n}\vec{n}^{\langle\mu}I^{\nu\rangle} + \hat{\lambda}_{\pi\Pi}\vec{\Pi}\sigma^{\mu\nu} ,
\end{aligned} \tag{5.67}$$

where we defined the vectors

$$\vec{\Pi} \equiv \begin{pmatrix} \Pi \\ \rho_3 \end{pmatrix} , \quad \vec{n}^\mu \equiv \begin{pmatrix} n^\mu \\ \rho_2^\mu \end{pmatrix} , \quad \vec{\pi}^{\mu\nu} \equiv \begin{pmatrix} \pi^{\mu\nu} \\ \rho_1^{\mu\nu} \end{pmatrix} . \tag{5.68}$$

In order to obtain the above equations, all covariant time derivatives of α_0, β_0, and u^μ were replaced by spatial gradients of fluid-dynamical variables using the conservation laws in the form shown in Eqs. (4.25), (4.26), and (4.27).

In this approximation, RTRFD becomes a theory with **23 dynamical variables** while, in the previous approximation, i.e., Eqs. (5.36) – (5.38), there were only 14 dynamical variables. These equations of motion are hyperbolic and neglect terms of order $\mathcal{O}(\mathrm{R}_i^{-1}\mathrm{Kn}^2,\mathrm{Kn}^3,\mathrm{R}_i^{-2})$, in contrast to Eqs. (5.36) – (5.38) that become hyperbolic by neglecting terms of $\mathcal{O}(\mathrm{Kn}^2)$. Above, $\hat{\tau}_\Pi$, $\hat{\tau}_n$, $\hat{\tau}_\pi$, $\hat{\ell}_{\Pi n}$, $\hat{\ell}_{n\Pi}$, $\hat{\ell}_{n\pi}$, $\hat{\ell}_{\pi n}$, $\hat{\delta}_{\Pi\Pi}$, $\hat{\delta}_{nn}$, $\hat{\delta}_{\pi\pi}$, $\hat{\tau}_{\Pi n}$, $\hat{\tau}_{n\pi}$, $\hat{\tau}_{n\Pi}$, $\hat{\tau}_{\pi\pi}$, $\hat{\tau}_{\pi n}$, $\hat{\lambda}_{\Pi n}$, $\hat{\lambda}_{\Pi\pi}$, $\hat{\lambda}_{nn}$, $\hat{\lambda}_{n\Pi}$, $\hat{\lambda}_{n\pi}$, $\hat{\lambda}_{\pi n}$, and $\hat{\lambda}_{\pi\Pi}$ are 2×2 matrices, while $\vec{\zeta}$, $\vec{\varkappa}$, and $\vec{\eta}$ are two-component vectors.

As happened before, even though closed in terms of 23 moments, the transport coefficients will depend on all the moments of the distribution function. The microscopic formulas for these transport coefficients were computed for a gas of massless particles and are shown in App. 5.12. For a gas of classical particles with a constant cross section σ, the values for the diffusion and viscosity coefficients, $\vec{\varkappa}$ and $\vec{\eta}$, and for the relaxation-time matrices, $\hat{\tau}_n$ and $\hat{\tau}_\pi$, are,

$$\frac{\vec{\varkappa}}{\lambda n_0} = \begin{pmatrix} 0.1596 \\ -2.3616/\beta_0^2 \end{pmatrix}, \quad \frac{\vec{\eta}}{\lambda P_0} = \begin{pmatrix} 1.268 \\ 6.929/\beta_0 \end{pmatrix},$$
$$\frac{\hat{\tau}_n}{\lambda} = \begin{pmatrix} 1.295 & -0.053\beta_0^2 \\ 5.18/\beta_0^2 & 2.787 \end{pmatrix}, \quad \frac{\hat{\tau}_\pi}{\lambda} = \begin{pmatrix} 0.912 & 0.136\beta_0 \\ -3.647/\beta_0 & 2.456 \end{pmatrix}. \tag{5.69}$$

The transport coefficients of the nonlinear terms in the equation of motion for $\vec{n}^\mu$ are

$$\frac{\hat{\delta}_{nn}}{\lambda} = \begin{pmatrix} 1.295 & -0.0883\beta_0^2 \\ 5.18/\beta_0^2 & 4.645 \end{pmatrix}, \quad \frac{\hat{\lambda}_{nn}}{\lambda} = \begin{pmatrix} 0.524 & -0.0341\beta_0^2 \\ 2.096/\beta_0^2 & 2.863 \end{pmatrix},$$
$$\frac{\hat{\lambda}_{n\pi}}{\lambda} = \begin{pmatrix} 0.1677\beta_0 & -0.0288\beta_0^2 \\ 0.6708/\beta_0 & -0.1147 \end{pmatrix}, \quad \frac{\hat{\tau}_{n\pi}}{\lambda} = \frac{1}{4P_0}\begin{pmatrix} 0 & 0.0973\beta_0^2 \\ 0 & -2.6106 \end{pmatrix},$$
$$\frac{\hat{\ell}_{n\pi}}{\lambda} = \begin{pmatrix} -0.4723\beta_0 & 0.0973\beta_0^2 \\ 13.111/\beta_0 & -2.611 \end{pmatrix}, \tag{5.70}$$

while those in the equation of motion for $\vec{\pi}^{\mu\nu}$ are

$$\frac{\hat{\delta}_{\pi\pi}}{\lambda} = -\frac{4}{3}\begin{pmatrix} 0.912 & 0.17\beta_0 \\ -3.647/\beta_0 & 3.0698 \end{pmatrix}, \quad \frac{\hat{\tau}_{\pi\pi}}{\lambda} = \begin{pmatrix} 1.5688 & 0.2261\beta_0 \\ -6.2751/\beta_0 & 5.0956 \end{pmatrix}, \tag{5.71}$$
$$\frac{\hat{\tau}_{\pi n}}{\lambda} = \frac{1}{P_0}\begin{pmatrix} 0.2228/\beta_0 & 0.0714\beta_0 \\ 0.8913/\beta_0^2 & 1.5144 \end{pmatrix}, \quad \frac{\hat{\ell}_{\pi n}}{\lambda} = \begin{pmatrix} 0.2228/\beta_0 & 0.0476\beta_0 \\ 0.8913/\beta_0^2 & 1.0096 \end{pmatrix},$$
$$\frac{\hat{\lambda}_{\pi n}}{\lambda} = \begin{pmatrix} 0.1186/\beta_0 & 0.0084\beta_0 \\ -0.4744/\beta_0^2 & -0.0338 \end{pmatrix}. \tag{5.72}$$

In the massless limit, the scalar moments become less important (Π is exactly zero and ρ_3 is small) and, for this reason, we did not compute the microscopic formulas nor the transport coefficients associated with these moments. These transport coefficients were computed including a total of 41 moments, as was done for the transport coefficients of the terms $\mathcal{J}^\mu$ and $\mathcal{J}^{\mu\nu}$ in the last section.

As already mentioned, this approach increases the domain of validity of the equations of motion without making them parabolic. However, this does not solve the intrinsic problem of the coarse-graining procedure: if one attempts to go to an even higher order in Kn, the equations become once more parabolic. This can be solved in a similar fashion, by again increasing the number of dynamical variables describing the system. However, a complete and causal description of the system can only be obtained by the microscopic theory itself, i.e., by solving the Boltzmann equation or, equivalently, considering an infinite number of moments.

5.7 Comparisons with microscopic theory

In the last sections, a systematic derivation of transient relativistic fluid dynamics from the Boltzmann equation was introduced. The main difference between Israel-Stewart theory and the theory derived in the last chapter is that the latter does not truncate the moment expansion of the single-particle distribution function. Instead, dynamical equations for all its moments are considered and solved by separating the slowest microscopic time scale from the faster ones. Then, the resulting fluid-dynamical equations are truncated according to a systematic power-counting scheme using the inverse Reynolds number $\mathrm{R}^{-1} \sim |n^\mu|/n \sim |\pi^{\mu\nu}|/P_0$ and the Knudsen number $\mathrm{Kn} = \lambda/L$, with λ being the mean free path and L a characteristic macroscopic distance scale, e.g., $L^{-1} \sim \partial_\mu u^\mu$. The values of the transport coefficients of fluid dynamics are obtained by resumming the contributions from all moments of the single-particle distribution function, similar to what happens in the Chapman-Enskog expansion [16]. This theory was referred to as resummed transient relativistic fluid dynamics (RTRFD).

In this section, we reproduce the findings of Ref. [17] and we test the validity of RTRFD by comparing it to solutions of the relativistic Boltzmann equation, as was also previously done in Refs. [10, 18, 19, 20]. We then demonstrate that this method is able to handle problems with strong initial gradients in pressure or particle-number density. This resolves the differences between the solution of Israel-Stewart theory and of the Boltzmann equation observed in Ref. [20]. We conclude that these differences were caused by the uncontrolled truncation procedure of the expansion of the single-particle distribution function in terms of Lorentz tensors in 4-momentum as employed in Israel-Stewart theory.

5.7.1 Stationary shock solutions

We consider a (3+1)-dimensional gas of classical massless particles with a constant cross section σ. For the sake of simplicity, the distribution of particles is assumed to be homogeneous in the (y, z)-plane, being allowed to vary only in the longitudinal x-direction. This effectively leads to a (1+1)-dimensional problem. We consider two different types of initial conditions:

In **case I**, the system is initialized with a homogeneous fugacity distribution, $\lambda \equiv e^{\alpha_0} \equiv 1$, but with an inhomogeneous pressure profile in the longitudinal direction, i.e., the system is in chemical, but not mechanical, equilibrium. The pressure profile is constructed by smoothly connecting two temperature states, $T(-\infty) = 0.4\,\mathrm{GeV}$ (the temperature at $x \to -\infty$) and $T(+\infty) = 0.25\,\mathrm{GeV}$ (the temperature at $x \to \infty$) using a Woods-Saxon parametrization with a thickness parameter $D = 0.3$ fm. In this scheme, by taking the limit $D \to 0$, we obtain the pressure profile of the Riemann problem.

In **case II**, the pressure is assumed to be homogeneous, i.e., the system is initially in mechanical equilibrium, with a value of $P_0 = d_{\mathrm{dof}} T^4(-\infty)/\pi^2$ (the degeneracy factor is taken to be $d_{\mathrm{dof}} = 16$, as appropriate for a gas of gluons, the gauge bosons of the theory of the strong interaction, quantum chromodynamics, and $T(-\infty) = 0.4\,\mathrm{GeV}$). On the other hand, the system is not in chemical equilibrium and the fugacity distribution is obtained by smoothly connecting two fugacity states, $\lambda(-\infty) = 1$ (the fugacity at $x \to -\infty$) and $\lambda(+\infty) = 0.2$ (the fugacity at $x \to \infty$) using a Woods-Saxon parametrization with a thickness parameter $D = 0.3$ fm.

In both cases, matter is initialized in **local** thermodynamic equilibrium, i.e., with all dissipative currents and eigenmodes of the Boltzmann equation set to zero, and at rest, i.e., with a vanishing collective velocity $u^\mu = 0$. These initial conditions are shown in Fig.

5.1.

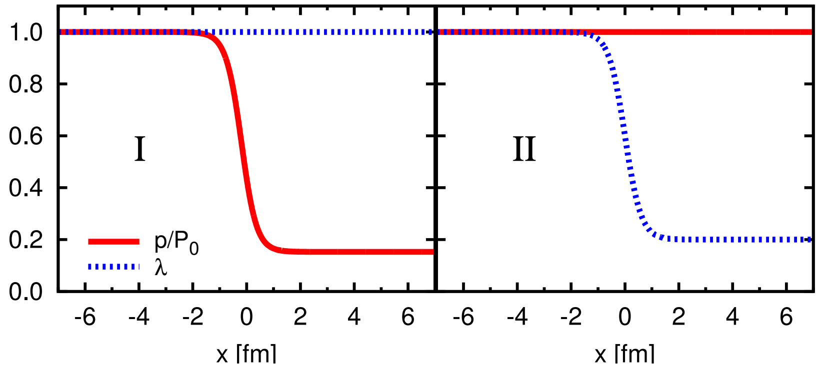

Figure 5.1: Initial conditions for cases I and II. Figure taken from Ref. [17].

In both cases, we consider two exemplary values for the cross section, $\sigma = 2$ and 8 mb, and consider the solutions after the system has evolved for 6 fm in time. We compare the solution of the Boltzmann equation with that of traditional Israel-Stewart theory (including terms omitted in the original work [5, 11] but quoted in Ref. [21]), as well as with RTRFD at various levels of approximation. Equations (5.36) – (5.38) contain 13 moments as independent dynamical variables (14, if we include the bulk-viscous pressure). The calculation of the transport coefficients in these equations can be done with increasing accuracy, as more irreducible moments are considered in the moment expansion. The lowest possible accuracy is reached if no more than the original 13 (14, in the case of non-vanishing bulk-viscous pressure) irreducible moments are considered for the calculation of the transport coefficients. At the next level, we include one more set of irreducible moments of tensor-rank one and two (and one more scalar moment in the case of non-vanishing bulk viscous pressure), which leads to a total of 21 (23, in the case of non-vanishing bulk-viscous pressure) irreducible moments. In this way, the number of irreducible moments entering the transport coefficients increases by 8 (9) at each successive level of approximation. For the purpose of this comparison, we found that going to the third level of iteration, i.e., considering $13 + 8 \times 3 = 37$ moments ($14 + 9 \times 3 = 41$ in the case of non-vanishing bulk-viscous pressure) is sufficient to reach the desired accuracy in the values of the transport coefficients. In the following, we shall compare RTRFD with 13 dynamical degrees of freedom and with the transport coefficients computed with 13 and with 37 moments. We shall term these variants of RTRFD "13/13" and "13/37", respectively. In addition, we also solve Eqs. (5.67). These contain 21 dynamical degrees of freedom. We compute the corresponding transport coefficients using 37 moments. We shall refer to this variant of RTRFD as "21/37". In the following figures, the numerical solutions of the Boltzmann equation is always displayed by open dots, the results of Israel-Stewart theory by black dash-dotted lines, the solution of RTRFD "13/13" by green dashed lines, that of RTRFD "13/37" by blue dotted lines, and that of RTRFD "21/37" by solid red curves.

In Fig. 5.2 we show the fugacity (top) and thermodynamic pressure (bottom) and in Fig. 5.3 the heat flow $q^\mu \equiv -(\varepsilon + P_0)n^\mu/n$ (top) and shear-stress tensor (bottom) for case I. The Boltzmann equation and the fluid-dynamical theories were solved for $\sigma = 2$ mb (shown in the left panels of each figure) and for $\sigma = 8$ mb (shown in the right panels). For $\sigma = 8$ mb, the thermodynamic pressure and shear-stress tensor computed in all fluid-dynamical theories are in good agreement with the numerical solutions of the Boltzmann equation. As we decrease the cross section we expect the agreement between macroscopic and microscopic theory to become worse. This explains why, for $\sigma = 2$ mb, the pressure and shear-stress tensor computed within fluid-dynamical theories deviate more strongly from those computed via the microscopic theory. Nevertheless, compared to the fugacity and heat-flow profiles, the agreement is not too bad, even for the smaller value of the cross section.

The initial pressure gradient in case I drives, via conservation of momentum, the creation of large velocity gradients. On the other hand, the gradient of fugacity is initially zero and turns out to remain small throughout the evolution. In this situation, higher-order terms involving gradients of velocity and of the shear-stress tensor, e.g. $\varkappa_6 \Delta^\mu_\lambda \partial_\nu \sigma^{\lambda\nu} \subset \mathcal{K}^\mu$ and $\ell_{n\pi} \Delta^{\mu\nu} \nabla_\lambda \pi^\lambda_\nu \subset \mathcal{J}^\mu$ in the particle-diffusion equation (5.37), become of the same order as the respective (first-order) Navier-Stokes term $\varkappa I^\mu$. Therefore, if terms of this type are not properly taken into account, we expect large deviations from the solution of the Boltzmann equation. This can be seen in Figs. 5.2 and 5.3 when comparing Israel-Stewart theory, RTRFD "13/13", as well as RTRFD "13/37" with the Boltzmann result. In all of these variants, the parabolic term $\sim \varkappa_6$ is either absent (Israel-Stewart theory and RTRFD "13/13") or has to be dismissed (RTRFD "13/37") for reasons of causality. In addition, Israel-Stewart theory and RTRFD "13/13" do not have the correct value for $\ell_{n\pi}$, because we did not include a sufficiently large number of irreducible moments in its computation. Although RTRFD "13/37" features (within the desired accuracy) the correct value for this transport coefficient (as well as for $\varkappa_6$), it does even worse in describing the fugacity and heat-flow profiles than the previous two theories. This is because the term $\sim \varkappa_6$ could not be taken into account for reasons of causality, although it is of the same order of magnitude as the term $\sim \ell_{n\pi}$. These problems of fluid-dynamical theories with only 13 dynamical variables are resolved by RTRFD "21/37" which is the only fluid-dynamical theory considered here that contains **all** contributions of second-order in the Knudsen number in a hyperbolic fashion.

In Fig. 5.4 we show the fugacity (top) and thermodynamic pressure (bottom) and in Fig. 5.5 the heat-flow (top) and shear-stress tensor profiles (bottom) for case II. As before, the Boltzmann equation and the fluid-dynamical theories considered were solved for $\sigma = 2$ mb (shown in the left panels) and for $\sigma = 8$ mb (shown in the right panels). Again, we expect, and see, better agreement between fluid dynamics and the Boltzmann equation for the larger value of the cross section. While the fugacity profiles are in good agreement with the solution of the Boltzmann equation for all fluid-dynamical theories and both values of the cross section, the heat flow is not well described in Israel-Stewart theory and in RTRFD "13/13": Israel-Stewart theory predicts values for the heat flow which are smaller in magnitude than the Boltzmann equation, while RTRFD "13/13" predicts larger values, even for $\sigma = 8$ mb. On the other hand, both RTRFD "13/37" and RTRFD "21/37" describe the heat flow very well or even perfectly, respectively, for both values of the cross section. The reason is that the diffusion coefficient $\varkappa$ has the correct value in these theories (while it deviates by $\sim 30\%$ in both Israel-Stewart theory and RTRFD "13/13").

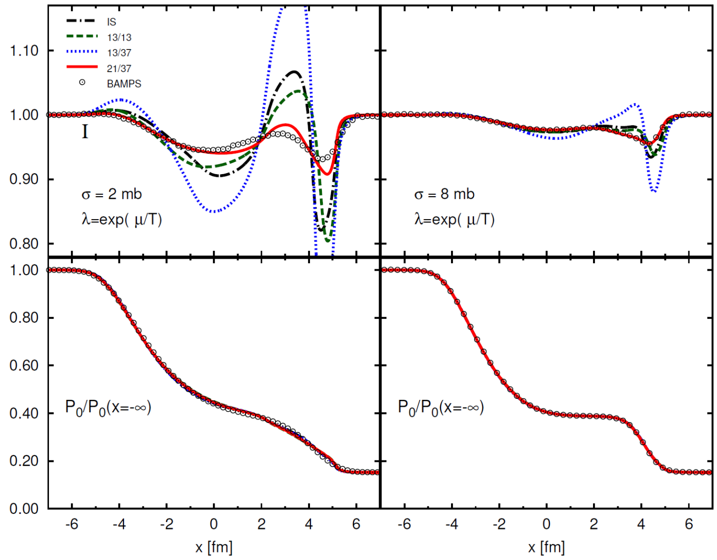

Figure 5.2: Fugacity and pressure profiles at $t = 6$ fm for case I, for $\sigma = 2$ mb (left panels) and $\sigma = 8$ mb (right panels). Figure taken from Ref. [17].

Since in case II the initial pressure gradient is zero and turns out to remain small throughout the evolution, the velocity gradients remain small as well. In this situation, it is important to include higher-order terms that couple the shear-stress tensor to heat flow. This is the reason why the solutions of Israel-Stewart theory and RTRFD "13/13" (where these higher-order terms vanish in the massless limit) are not in good agreement with that of the Boltzmann equation for the thermodynamic pressure and the shear-stress tensor, for both values of the cross section. On the other hand, RTRFD "13/37" does a better job in matching the Boltzmann equation. It is not perfect, because the higher-order terms $\sim \eta_5$ and $\sim \eta_8$ were dropped. The best agreement is, again, found within RTRFD "21/37" where all second-order terms in the Knudsen number are taken into account.

5.8 Summary

In this chapter we have presented a **general and consistent derivation of relativistic fluid dynamics** from the Boltzmann equation using the method of moments. The main difference of our approach, termed **Resummed Transient Relativistic Fluid Dynamics (RTRFD)**, to Israel-Stewart theory is that we did not close the fluid-dynamical equations of motion by truncating the expansion of the distribution function. Instead, we kept **all** terms in the moment expansion and truncated the exact equations of motion according to a **power-counting scheme in Knudsen and inverse Reynolds numbers**. Contrary to many calculations, we did not assume that the inverse Reynolds and Knudsen numbers are of the same order. As a matter of fact, in order to obtain relaxation-type equations, we had to explicitly include the **slowest microscopic time scales**, which are shown to be the characteristic times over which dissipative currents relax towards their

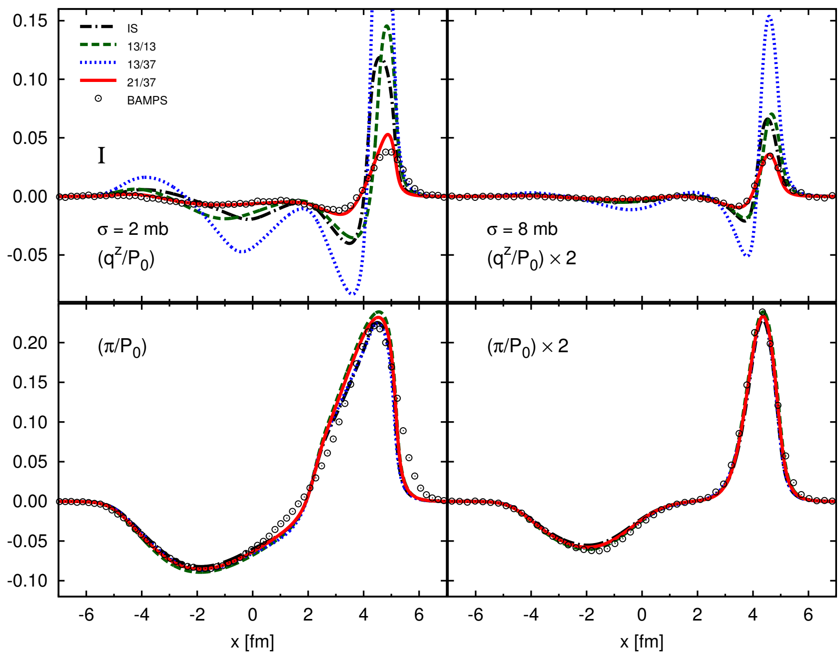

Figure 5.3: Shear-stress tensor and heat-flow profiles at $t = 6$ fm for case I, for $\sigma = 2$ mb (left panels) and $\sigma = 8$ mb (right panels). Figure taken from Ref. [17].

asymptotic Navier-Stokes solutions. Thus, Navier-Stokes theory, or the Chapman-Enskog expansion, is already included in our formulation as an asymptotic limit of the dynamical equations.

We concluded that the equations of motion can be closed in terms of only 14 dynamical variables, as long as we only keep terms of second order in Knudsen and/or inverse Reynolds number. Even though the equations of motion are closed in terms of these 14 fields, the transport coefficients carry information about **all** moments of the distribution function (all different relaxation scales of the irreducible moments). The bulk-viscosity, particle-diffusion, and shear-viscosity coefficients agree with the values obtained via Chapman-Enskog theory. We then showed how to use this formalism to derive equations of motion that are hyperbolic and, at the same time, include terms up to second order in the Knudsen number.

Finally, we compared the derived equations of motion of RTRFD at various levels of approximation with numerical solutions of the Boltzmann equation for two different types of shock solution (labeled case I and II). The initial conditions, cases I and II, were chosen in such a way that considerably different spatial profiles are generated throughout the fluid-dynamical evolution. In case I, the pressure gradient is initially large, which gives rise to large velocity gradients in the later stages of the evolution. This means that, in case I, the shear-stress tensor is mainly generated by its corresponding Navier-Stokes term, i.e., by gradients of velocity. On the other hand, the fugacity gradient is initially zero in case I, and remains relatively small throughout the evolution of the fluid. Therefore, in case I the heat flow is not mainly created by its Navier-Stokes term, i.e., by the gradient of fugacity, but by the coupling term to the shear tensor and shear-stress tensor, i.e., the terms $\Delta^{\mu\nu}\nabla_\lambda \pi_\nu^\lambda$ and $\Delta^{\mu\nu}\nabla_\lambda \sigma_\nu^\lambda$ in Eq. (5.37). Therefore, in this case the higher-order

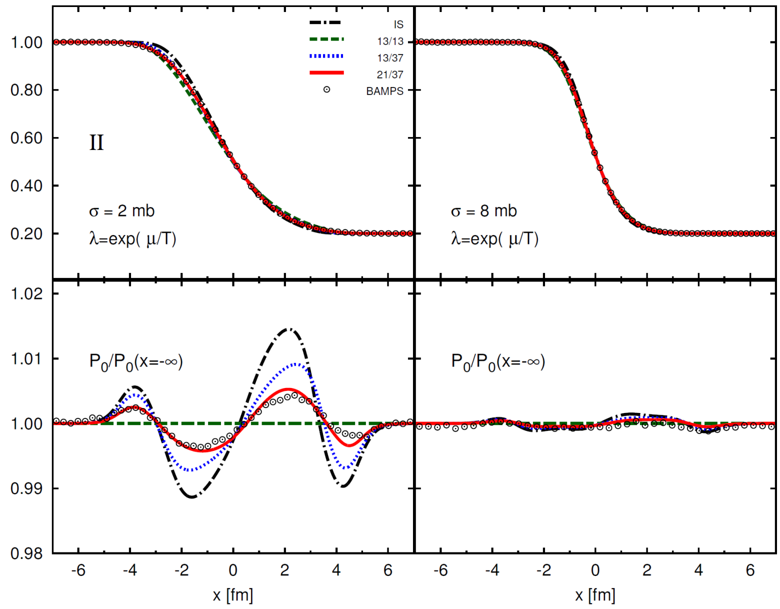

Figure 5.4: Fugacity and thermodynamic pressure profiles at $t = 6$ fm for case II, for $\sigma = 2$ mb (left panels) and $\sigma = 8$ mb (right panels). Figure taken from Ref. [17].

terms in Knudsen number must be included and one really needs to solve the hyperbolic equations derived in this section to obtain a good agreement. The fact that Israel-Stewart theory is always deviating from the microscopic theory when it concerns heat flow means that it does not predict correctly the terms of order one and two in Knudsen number. On the other hand, in RTRFD "13/37" and RTRFD "21/37" all transport coefficients are computed with a sufficiently large number of irreducible moments. This guarantees that all terms of the desired order are included and is the reason for the better agreement of these fluid-dynamical theories with the microscopic theory. The reason why RTRFD "13/37" fails in certain situations is that important terms have to be neglected in order to preserve hyperbolicity and causality.

In case II, the fugacity gradient is initially large while the pressure gradient is zero. This means that the heat flow originates mainly from its Navier-Stokes term, while the shear-tress tensor originates mainly from its coupling to heat flow, i.e., the terms $\nabla^{<\mu} n^{\nu>}$, $\nabla^{<\mu} I^{\nu>}$, $I^{<\mu} I^{\nu>}$, and $n^{<\mu} I^{\nu>}$ in Eq. (5.38). The fact that the heat flow calculated from Israel-Stewart theory deviates from the solution given by the microscopic theory even in this case is evidence that the Navier-Stokes term of this theory does not contain the correct transport coefficient. The coupling of the shear-stress tensor with the heat flow in Israel-Stewart theory is also not correctly taken into account.

In conclusion, the **resummation of irreducible moments** for the computation of the transport coefficients is essential to obtain a good agreement with the microscopic theory. It provides not only the correct values for the shear-viscosity and heat-conduction coefficients, but also for the transport coefficients that couple the respective dissipative currents. Moreover, in situations where higher-order terms are important, one has to

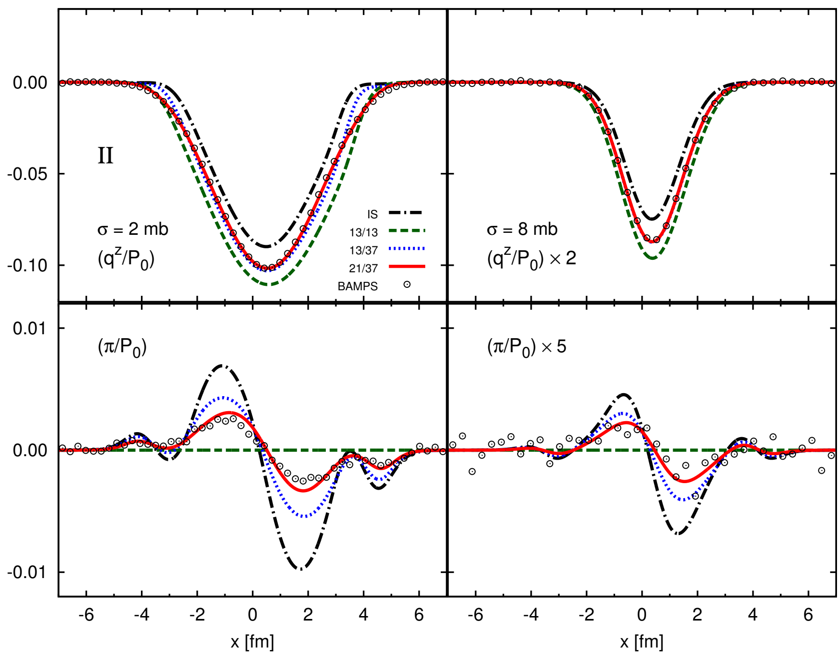

Figure 5.5: Shear-stress tensor and heat-flow profiles at $t = 6$ fm for case II, for $\sigma = 2$ mb (left panels) and $\sigma = 8$ mb (right panels). Figure taken from Ref. [17].

make sure to include them in a hyperbolic way, and not simply drop relevant contributions because they are parabolic. These two factors resolve differences between the solution of Israel-Stewart theory and of the Boltzmann equation observed in Ref. [20].

As expected, and explicitly demonstrated in this chapter, the agreement between solutions of RTRFD and the Boltzmann equation depends on the value of the cross section σ. For the cases considered here, we obtain a good agreement for $\sigma = 8$ mb, while for $\sigma = 2$ mb we start to notice small deviations. In order to improve the agreement for smaller values of the cross section, we would have to include more moments of the Boltzmann equation to describe the state of the system, i.e., such moments would have to contribute not only to the values of the transport coefficients but also as independent dynamical variables.

5.9 Appendix 1: Transport coefficients in Eq. (5.44)

In this appendix we list all transport coefficients appearing in Eq. (5.44). The transport coefficients in the equation for the bulk-viscous pressure are

$$\ell_{\Pi n} = -\frac{m^2}{3}\left(\gamma_1^{(1)}\tau_{00}^{(0)} - \sum_{r=0,\neq 1,2}^{N_0} \tau_{0r}^{(0)}\frac{G_{3r}}{D_{20}} + \sum_{r=0}^{N_0-3} \tau_{0,r+3}^{(0)}\Omega_{r+2,0}^{(1)}\right) , \tag{5.73}$$

$$\tau_{\Pi n} = \frac{m^2}{3\left(\varepsilon_0 + P_0\right)}\left\{\tau_{00}^{(0)}\frac{\partial\gamma_1^{(1)}}{\partial\ln\beta_0} - \sum_{r=0,\neq 1,2}^{N_0} \tau_{0r}^{(0)}\frac{G_{3r}}{D_{20}} + \sum_{r=0}^{N_0-3} \tau_{0,r+3}^{(0)}\left[\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\ln\beta_0} + (r+3)\,\Omega_{r+2,0}^{(1)}\right]\right\} , \tag{5.74}$$

$$\begin{aligned}\delta_{\Pi\Pi} = {} & \frac{1}{3}\left(2 + m^2\gamma_2^{(0)}\right)\tau_{00}^{(0)} - \frac{m^2}{3}\sum_{r=0,\neq 1,2}^{N_0} \tau_{0r}^{(0)}\frac{G_{2r}}{D_{20}} + \frac{1}{3}\sum_{r=0}^{N_0-3}(r+5)\,\tau_{0,r+3}^{(0)}\Omega_{r+3,0}^{(0)} \\ & + \sum_{r=3}^{N_0}\tau_{0r}^{(0)}\left[\frac{\left(\varepsilon_0 + P_0\right)J_{20} - n_0 J_{30}}{D_{20}}\frac{\partial\Omega_{r0}^{(0)}}{\partial\alpha_0} + \frac{\left(\varepsilon_0 + P_0\right)J_{10} - n_0 J_{20}}{D_{20}}\frac{\partial\Omega_{r0}^{(0)}}{\partial\beta_0}\right] \\ & - \frac{m^2}{3}\sum_{r=0}^{N_0-5}(r+4)\,\tau_{0,r+5}^{(0)}\Omega_{r+3,0}^{(0)} ,\end{aligned} \tag{5.75}$$

$$\lambda_{\Pi n} = -\frac{m^2}{3}\left[\left(\frac{\partial\gamma_1^{(1)}}{\partial\alpha_0} + \frac{1}{h_0}\frac{\partial\gamma_1^{(1)}}{\partial\beta_0}\right)\tau_{00}^{(0)} + \sum_{r=0}^{N_0-3}\tau_{0,r+3}^{(0)}\left(\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\alpha_0} + \frac{1}{h_0}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\beta_0}\right)\right] , \tag{5.76}$$

$$\lambda_{\Pi\pi} = \frac{m^2}{3}\left[\gamma_2^{(2)}\tau_{00}^{(0)} - \sum_{r=0,\neq 1,2}^{N_0}\tau_{0r}^{(0)}\frac{G_{2r}}{D_{20}} - \sum_{r=0}^{N_0-3}(r+2)\,\tau_{0,r+3}^{(0)}\Omega_{r+1,0}^{(2)}\right] , \tag{5.77}$$

where $h_0 = (\varepsilon_0 + P_0)/n_0$ is the enthalpy per particle.

The transport coefficients in the equation for the particle-diffusion current are

$$\delta_{nn} = \left(1 + \frac{1}{3}m^2\gamma_2^{(1)}\right)\tau_{00}^{(1)} - \frac{1}{3}m^2\sum_{r=0}^{N_1-2}(r+1)\,\tau_{0,r+2}^{(1)}\Omega_{r0}^{(1)} + \frac{1}{3}\sum_{r=2}^{N_1}(r+3)\,\tau_{0r}^{(1)}\Omega_{r0}^{(1)}$$
$$+ \sum_{r=2}^{N_1}\tau_{0r}^{(1)}\left[\frac{(\varepsilon_0+P_0)\,J_{20} - n_0J_{30}}{D_{20}}\frac{\partial\Omega_{r0}^{(1)}}{\partial\alpha_0} + \frac{(\varepsilon_0+P_0)\,J_{10} - n_0J_{20}}{D_{20}}\frac{\partial\Omega_{r0}^{(1)}}{\partial\beta_0}\right], \tag{5.78}$$

$$\ell_{n\Pi} = \left(\frac{1}{h_0} - \gamma_1^{(0)}\right)\tau_{00}^{(1)} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\frac{\beta_0 J_{r+4,1}}{\varepsilon_0+P_0} + \frac{1}{m^2}\sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\Omega_{r+3,0}^{(0)}$$
$$- \sum_{r=0}^{N_1-4}\tau_{0,r+4}^{(1)}\Omega_{r+3,0}^{(0)}, \tag{5.79}$$

$$\tau_{n\Pi} = \frac{1}{\varepsilon_0+P_0}\left(\left(\frac{1}{h_0} - \frac{\partial\gamma_1^{(0)}}{\partial\ln\beta_0}\right)\tau_{00}^{(1)} - \sum_{r=0}^{N_1-4}\tau_{0,r+4}^{(1)}\left[(r+4)\,\Omega_{r+3,0}^{(0)} + \frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\ln\beta_0}\right]\right.$$
$$\left.+ \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\left\{\frac{\beta_0J_{r+4,1}}{\varepsilon_0+P_0} + \frac{1}{m^2}\left[(r+5)\,\Omega_{r+3,0}^{(0)} + \frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\ln\beta_0}\right]\right\}\right), \tag{5.80}$$

$$\ell_{n\pi} = \left(\frac{1}{h_0} - \gamma_1^{(2)}\right)\tau_{00}^{(1)} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\left(\frac{\beta_0J_{r+4,1}}{\varepsilon_0+P_0} - \Omega_{r+1,0}^{(2)}\right), \tag{5.81}$$

$$\tau_{n\pi} = \frac{1}{\varepsilon_0+P_0}\left\{\left(\frac{1}{h_0} - \frac{\partial\gamma_1^{(2)}}{\partial\ln\beta_0}\right)\tau_{00}^{(1)}\right.$$
$$\left.+ \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\left[\frac{\beta_0J_{r+4,1}}{\varepsilon_0+P_0} - \frac{\partial\Omega_{r+1,0}^{(2)}}{\partial\ln\beta_0} - (r+2)\Omega_{r+1,0}^{(2)}\right]\right\}, \tag{5.82}$$

$$\lambda_{nn} = \frac{1}{5}\left(3 + 2m^2\gamma_2^{(1)}\right)\tau_{00}^{(1)} - \frac{2}{5}m^2\sum_{r=0,r\neq1}^{N_1-2}(r+1)\,\tau_{0,r+2}^{(1)}\Omega_{r0}^{(1)} + \frac{1}{5}\sum_{r=2}^{N_1}(2r+3)\,\tau_{0r}^{(1)}\Omega_{r0}^{(1)}, \tag{5.83}$$

$$\lambda_{n\Pi} = \tau_{00}^{(1)}\left(\frac{1}{h_0}\frac{\partial\gamma_1^{(0)}}{\partial\beta_0} + \frac{\partial\gamma_1^{(0)}}{\partial\alpha_0}\right) - \frac{1}{m^2}\sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\left(\frac{1}{h_0}\frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\beta_0} + \frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\alpha_0}\right)$$
$$+ \sum_{r=0}^{N_1-4}\tau_{0,r+4}^{(1)}\left(\frac{1}{h_0}\frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\beta_0} + \frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\alpha_0}\right), \tag{5.84}$$

$$\lambda_{n\pi} = \left(\frac{1}{h_0}\frac{\partial\gamma_1^{(2)}}{\partial\beta_0} + \frac{\partial\gamma_1^{(2)}}{\partial\alpha_0}\right)\tau_{00}^{(1)} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\left(\frac{1}{h_0}\frac{\partial\Omega_{r+1,0}^{(2)}}{\partial\beta_0} + \frac{\partial\Omega_{r+1,0}^{(2)}}{\partial\alpha_0}\right). \tag{5.85}$$

The transport coefficients in the equation for the shear-stress tensor are

$$\delta_{\pi\pi} = \frac{1}{3}m^2\gamma_2^{(2)}\tau_{00}^{(2)} + \frac{1}{3}\sum_{r=0}^{N_2}(r+4)\,\tau_{0r}^{(2)}\Omega_{r0}^{(2)} - \frac{1}{3}m^2\sum_{r=0}^{N_2-2}(r+1)\,\tau_{0,r+2}^{(2)}\Omega_{r0}^{(2)}$$
$$+\sum_{r=0}^{N_2}\tau_{0r}^{(2)}\left[\frac{(\varepsilon_0+P_0)\,J_{10}-n_0J_{20}}{D_{20}}\frac{\partial\Omega_{r0}^{(2)}}{\partial\beta_0} + \frac{(\varepsilon_0+P_0)\,J_{20}-n_0J_{30}}{D_{20}}\frac{\partial\Omega_{r0}^{(2)}}{\partial\alpha_0}\right], \tag{5.86}$$

$$\tau_{\pi\pi} = \frac{2}{7}\sum_{r=0}^{N_2}(2r+5)\,\tau_{0r}^{(2)}\Omega_{r0}^{(2)} + \frac{4}{7}m^2\gamma_2^{(2)}\tau_{00}^{(2)} - \frac{4}{7}m^2\sum_{r=0}^{N_2-2}(r+1)\,\tau_{0,r+2}^{(2)}\Omega_{r0}^{(2)}, \tag{5.87}$$

$$\lambda_{\pi\Pi} = \frac{6}{5}\tau_{00}^{(2)} + \frac{2}{5}m^2\gamma_2^{(0)}\tau_{00}^{(2)} - \frac{2}{5m^2}\sum_{r=0}^{N_2-1}(r+5)\,\tau_{0,r+1}^{(2)}\Omega_{r+3,0}^{(0)}$$
$$+\frac{2}{5}\sum_{r=3}^{N_2}(2r+3)\,\tau_{0r}^{(2)}\Omega_{r0}^{(0)} - \frac{2}{5}m^2\sum_{r=0,\neq 1,2}^{N_2-2}(r+1)\,\tau_{0,r+2}^{(2)}\Omega_{r0}^{(0)}, \tag{5.88}$$

$$\tau_{\pi n} = \frac{2}{5(\varepsilon_0+P_0)}\left\{-m^2\tau_{00}^{(2)}\frac{\partial\gamma_1^{(1)}}{\partial\ln\beta_0} - m^2\sum_{r=0}^{N_2-3}\tau_{0,r+3}^{(2)}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\ln\beta_0}\right.$$
$$\left.-m^2\sum_{r=0,\neq 1}^{N_2-1}(r+1)\,\tau_{0,r+1}^{(2)}\Omega_{r0}^{(1)} + \sum_{r=0}^{N_2-1}\tau_{0,r+1}^{(2)}\left[(r+6)\,\Omega_{r+2,0}^{(1)} + \frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\ln\beta_0}\right]\right\}, \tag{5.89}$$

$$\ell_{\pi n} = -\frac{2}{5}m^2\gamma_1^{(1)}\tau_{00}^{(2)} + \frac{2}{5}\sum_{r=0}^{N_2-1}\tau_{0,r+1}^{(2)}\Omega_{r+2,0}^{(1)} - \frac{2}{5}m^2\sum_{r=0,\neq 1}^{N_2-1}\tau_{0,r+1}^{(2)}\Omega_{r0}^{(1)}, \tag{5.90}$$

$$\lambda_{\pi n} = -\frac{2}{5}m^2\tau_{00}^{(2)}\left(\frac{1}{h_0}\frac{\partial\gamma_1^{(1)}}{\partial\beta_0} + \frac{\partial\gamma_1^{(1)}}{\partial\alpha_0}\right) + \frac{2}{5}\sum_{r=0}^{N_2-1}\tau_{0,r+1}^{(2)}\left(\frac{1}{h_0}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\beta_0} + \frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\alpha_0}\right)$$
$$-\frac{2}{5}m^2\sum_{r=0}^{N_2-3}\tau_{0,r+3}^{(2)}\left(\frac{1}{h_0}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\beta_0} + \frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\alpha_0}\right). \tag{5.91}$$

5.10 Appendix 2: Calculation of the collision integrals

In this appendix, we calculate the collision integrals (4.48) for a classical gas, i.e., $\tilde{f}_{0\mathbf{k}} = 1$, of hard spheres in the ultrarelativistic limit, $m\beta_0 \ll 1$. Then, Eq. (4.48) becomes

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu\,(2\ell+1)}\int dKdK'dPdP'W_{\mathbf{kk'}\to\mathbf{pp'}}f_{0\mathbf{k}}f_{0\mathbf{k'}}E_{\mathbf{k}}^{r-1}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}$$
$$\times\left(\mathcal{H}_{\mathbf{k}n}^{(\ell)}k_{\langle\nu_1}\cdots k_{\nu_\ell\rangle} + \mathcal{H}_{\mathbf{k'}n}^{(\ell)}k'_{\langle\nu_1}\cdots k'_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)}p_{\langle\nu_1}\cdots p_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p'}n}^{(\ell)}p'_{\langle\nu_1}\cdots p'_{\nu_\ell\rangle}\right). \tag{5.92}$$

The functions $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ were defined in Eq. (4.14). The transition rate $W_{\mathbf{kk'}\to\mathbf{pp'}}$ is written in terms of the differential cross section $\sigma(s,\Theta)$ as

$$W_{\mathbf{kk'}\to\mathbf{pp'}} = s\sigma(s,\Theta_s)\,(2\pi)^6\,\delta^{(4)}\left(k^\mu + k'^\mu - p^\mu - p'^\mu\right). \tag{5.93}$$

The variable s and Θ_s are defined as

$$s = (k + k')^2 \ , \quad \cos\Theta_s = \frac{(k - k')_\mu \, (p - p')^\mu}{(k - k')^2} \ . \tag{5.94}$$

We further define the total cross section as the integral

$$\sigma(s) = \frac{2\pi}{\nu} \int d\Theta_s \, \sin\Theta_s \, \sigma(s, \Theta_s) \ . \tag{5.95}$$

In order to calculate $\mathcal{A}_{rn}^{(\ell)}$ it is convenient to first define the tensors

$$X^n_{\mu\nu\gamma_1\cdots\gamma_m} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} k_\mu k_\nu \times \left(k_{\gamma_1} \cdots k_{\gamma_m} + k'_{\gamma_1} \cdots k'_{\gamma_m} - p_{\gamma_1} \cdots p_{\gamma_m} - p'_{\gamma_1} \cdots p'_{\gamma_m} \right) \ . \tag{5.96}$$

The collision integrals $\mathcal{A}_{rn}^{(\ell)}$ can always be expressed as linear combinations of contractions/projections of $X^n_{\mu\nu\gamma_1\cdots\gamma_m}$. For the purpose of this appendix, we only need $X^n_{\mu\nu\gamma_1\cdots\gamma_m}$ for $m = 2$ and 3. For now we concentrate on calculating these integrals. We separate $X^n_{\mu\nu\gamma_1\cdots\gamma_m}$ as

$$X^n_{\mu\nu\gamma_1\cdots\gamma_m} = A^n_{\mu\nu\gamma_1\cdots\gamma_m} + B^n_{\mu\nu\gamma_1\cdots\gamma_m} \ , \tag{5.97}$$

with

$$\begin{aligned} A^n_{\mu\nu\gamma_1\cdots\gamma_m} &= \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} \, k_\mu k_\nu \left(k_{\gamma_1} \cdots k_{\gamma_m} + k'_{\gamma_1} \cdots k'_{\gamma_m} \right) \ , \\ B^n_{\mu\nu\gamma_1\cdots\gamma_m} &= -\frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} \, k_\mu k_\nu \left(p_{\gamma_1} \cdots p_{\gamma_m} + p'_{\gamma_1} \cdots p'_{\gamma_m} \right) \ . \end{aligned} \tag{5.98}$$

The $dPdP'$ integration in the first tensor, $A^n_{\mu\nu\gamma_1\cdots\gamma_m}$, can be immediately performed and written in terms of the total cross section, $\sigma(s)$, as

$$A^n_{\mu\nu\gamma_1\cdots\gamma_m} = \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} k_\mu k_\nu \left(k_{\gamma_1} \cdots k_{\gamma_m} + k'_{\gamma_1} \cdots k'_{\gamma_m} \right) \frac{s}{2} \sigma(s) \ . \tag{5.99}$$

The calculation of the second tensor, $B^n_{\mu\nu\gamma_1\ldots\gamma_m}$, is cumbersome. First, we write it in the general form

$$B^n_{\mu\nu\gamma_1\cdots\gamma_m} = -\int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} k_\mu k_\nu \Theta_{\gamma_1\cdots\gamma_m} \ , \tag{5.100}$$

where we introduced the tensor

$$\Theta^{\gamma_1\cdots\gamma_m} = \frac{2}{\nu} \int dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} p^{\gamma_1} \cdots p^{\gamma_m} \ . \tag{5.101}$$

The integral $\Theta^{\gamma_1\cdots\gamma_m}$ is an m-th rank tensor. For isotropic cross sections, this tensor can only depend on the normalized total momentum of the collision $\tilde{P}^\mu_T \equiv s^{-1/2}(k^\mu + k'^\mu) \equiv$

$s^{-1/2}P_T^\mu$. Thus, the tensor structure of $\Theta^{\gamma_1\cdots\gamma_m}$ must be constructed from combinations of $\tilde{P}_T^\mu$ and the projection operator orthogonal to $\tilde{P}_T^\mu$, $\Delta_P^{\mu\nu} = g^{\mu\nu} - \tilde{P}_T^\mu \tilde{P}_T^\nu$. In general,

$$\Theta^{\gamma_1\cdots\gamma_m} = \sum_{q=0}^{[m/2]} (-1)^q \, a_{mq} \mathcal{C}_{mq} C_q^{\gamma_1\cdots\gamma_m} \; , \tag{5.102}$$

where we defined

$$\begin{aligned} a_{mq} &= \frac{m!}{(m-2q)!2q!} (2q-1)!! \; , \\ C_q^{\gamma_1\cdots\gamma_m} &= \Delta_P^{(\gamma_1\gamma_2} \cdots \Delta_P^{\gamma_{2q-1}\gamma_{2q}} \tilde{P}_T^{\gamma_{2q+1}} \cdots \tilde{P}_T^{\gamma_m)} \; , \\ \mathcal{C}_{mq} &= \frac{2}{\nu\,(2q+1)!!} \int dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} \left(\tilde{P}_T^\mu p_\mu\right)^{m-2q} \left(-\Delta_P^{\alpha\beta} p_\alpha p_\beta\right)^q \; . \end{aligned} \tag{5.103}$$

As usual, the parentheses () around the indices denote the symmetrization of the tensor. For example,

$$\begin{aligned} \Theta_{\mu\nu} &= \mathcal{C}_{20} \tilde{P}_{T\mu} \tilde{P}_{T\nu} - \mathcal{C}_{21} \Delta_{P\mu\nu} \; , \\ \Theta_{\mu\nu\lambda} &= \mathcal{C}_{30} \tilde{P}_{T\mu} \tilde{P}_{T\nu} \tilde{P}_{T\lambda} - \mathcal{C}_{31} \left(\Delta_{P\mu\nu} \tilde{P}_{T\lambda} + \Delta_{P\mu\lambda} \tilde{P}_{T\nu} + \Delta_{P\nu\lambda} \tilde{P}_{T\mu}\right) \; . \end{aligned} \tag{5.104}$$

The integrals $\mathcal{C}_{nq}$ are scalars and can be computed in any frame. It is most convenient to calculate them in the center-of-momentum frame, where, $\tilde{P}_T^\mu = (1,0,0,0)$ and $\Delta_P^{\mu\nu} = \mathrm{diag}\,(0,-1,-1,-1)$. Then, it is straightforward to prove that

$$\mathcal{C}_{nq} = \frac{\sigma(s)}{2^n\,(2q+1)!!} s^{(n-2q+1)/2} \left(s - 4m^2\right)^{(2q+1)/2} \underset{m\to 0}{\longrightarrow} \frac{\sigma(s)}{2^n\,(2q+1)!!} s^{(n+2)/2} \; . \tag{5.105}$$

In the massless limit, the tensors $X^n_{\mu\nu\alpha\beta}$ and $X^n_{\mu\nu\alpha\beta\gamma}$ become

$$\begin{aligned} X^n_{\mu\nu\alpha\beta} &= \frac{1}{2} \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} k_\mu k_\nu \, s\sigma(s) \left(k_\alpha k_\beta + k'_\alpha k'_\beta - \frac{2}{3} P_{T\alpha} P_{T\beta} + \frac{s}{6} g_{\alpha\beta}\right) \; , \\ X^n_{\mu\nu\alpha\beta\gamma} &= \frac{1}{2} \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} k_\mu k_\nu \, s\sigma(s) \left[k_\alpha k_\beta k_\gamma + k'_\alpha k'_\beta k'_\gamma \right. \\ &\quad \left. - \frac{1}{2} P_{T\alpha} P_{T\beta} P_{T\gamma} + \frac{s}{12} \left(g_{\alpha\beta} P_{T\gamma} + g_{\alpha\gamma} P_{T\beta} + g_{\beta\gamma} P_{T\alpha}\right)\right] \; , \end{aligned} \tag{5.106}$$

where we used that, in the massless limit, $s = 2k^\lambda k'_\lambda$.

5.10.1 Particle-diffusion current

For the collision integrals related to the particle-number diffusion current, we need the following two contractions

$$\begin{aligned} \Delta^{\mu\alpha} u^\nu u^\beta X^n_{\mu\nu\alpha\beta} &= -\sigma \left(I_{10} I_{n+5,1} - 4 I_{21} I_{n+4,1} - I_{31} I_{n+3,1}\right) \; , \\ \Delta^{\mu\alpha} u^\nu u^\beta u^\gamma X^n_{\mu\nu\alpha\beta\gamma} &= -\frac{\sigma}{2} \left(3 I_{10} I_{n+6,1} - 11 I_{21} I_{n+5,1} - 5 I_{31} I_{n+4,1} - 3 I_{41} I_{n+3,1}\right) \; . \end{aligned} \tag{5.107}$$

To obtain the above relations, we used the definitions (4.31) and Eq. (4.46). In the massless and classical limits the integrals $I_{nq} = J_{nq}$ can be calculated analytically

$$I_{nq} = d_{\text{dof}} \frac{e^{\alpha_0}}{(2q+1)!!} \frac{1}{2\pi^2} \frac{(n+1)!}{\beta_0^{n+2}} = \frac{(n+1)!}{(2q+1)!!} \frac{P_0}{2\beta_0^{n-2}} \ . \tag{5.108}$$

Then,

$$\begin{aligned}
\Delta^{\mu\alpha} u^\nu u^\beta X^{-2}_{\mu\nu\alpha\beta} &= \frac{4}{3} n_0 \sigma \frac{P_0}{\beta_0} \ , \\
\Delta^{\mu\alpha} u^\nu u^\beta X^{0}_{\mu\nu\alpha\beta} &= -24 n_0 \sigma \frac{P_0}{\beta_0^3} \ , \\
\Delta^{\mu\alpha} u^\nu u^\beta u^\gamma X^{-2}_{\mu\nu\alpha\beta\gamma} &= 12 n_0 \sigma \frac{P_0}{\beta_0^2} \ , \\
\Delta^{\mu\alpha} u^\nu u^\beta u^\gamma X^{0}_{\mu\nu\alpha\beta\gamma} &= -280 n_0 \sigma \frac{P_0}{\beta_0^4} \ .
\end{aligned} \tag{5.109}$$

As a consistency check, we confirm that $\Delta^{\mu\alpha} u^\nu u^\beta X^{-1}_{\mu\nu\alpha\beta} = \Delta^{\mu\alpha} u^\nu u^\beta u^\gamma X^{-1}_{\mu\nu\alpha\beta\gamma} = 0$.

The components of $\mathcal{A}^{(1)}$ change according to the number of moments included. In the 14-moment approximation, using Eqs. (4.14) and (4.5), we obtain

$$\mathcal{A}^{(1)}_{00} = \frac{W^{(1)}}{3} a^{(1)}_{10} a^{(1)}_{11} \Delta^{\mu\alpha} u^\nu u^\beta X^{-2}_{\mu\nu\alpha\beta} = \frac{4}{9} n_0 \sigma \ . \tag{5.110}$$

In the 23-moment approximation, e.g., considering three polynomials in the expansion (5.1) for $\ell = 1$,

$$\begin{aligned}
\mathcal{A}^{(1)}_{r0} &= \frac{W^{(1)}}{3} \left[\left(a^{(1)}_{10} a^{(1)}_{11} + a^{(1)}_{20} a^{(1)}_{21} \right) \Delta^{\mu\alpha} u^\nu u^\beta X^{r-2}_{\mu\nu\alpha\beta} + a^{(1)}_{20} a^{(1)}_{22} \Delta^{\mu\alpha} u^\nu u^\beta u^\gamma X^{r-2}_{\mu\nu\alpha\beta\gamma} \right] \ , \\
\mathcal{A}^{(1)}_{r2} &= \frac{W^{(1)}}{3} \left(a^{(1)}_{22} a^{(1)}_{21} \Delta^{\mu\alpha} u^\nu u^\beta X^{r-2}_{\mu\nu\alpha\beta} + a^{(1)}_{22} a^{(1)}_{22} \Delta^{\mu\alpha} u^\nu u^\beta u^\gamma X^{r-2}_{\mu\nu\alpha\beta\gamma} \right) \ .
\end{aligned} \tag{5.111}$$

Then, using the results from App. 4.8 for the coefficients $a^{(\ell)}_{nq}$ together with Eqs. (5.108) and (5.109), we obtain

$$\begin{aligned}
\mathcal{A}^{(1)}_{00} &= \frac{2}{3} n_0 \sigma \ , \quad \mathcal{A}^{(1)}_{02} = \frac{\beta_0^2}{90} n_0 \sigma \ , \\
\mathcal{A}^{(1)}_{20} &= -\frac{4}{3\beta_0^2} n_0 \sigma \ , \quad \mathcal{A}^{(1)}_{22} = \frac{1}{3} n_0 \sigma \ .
\end{aligned} \tag{5.112}$$

5.10.2 Shear-stress tensor

For the collision integrals related to the shear-stress tensor, we need the following two contractions

$$\begin{aligned}
\Delta^{\mu\nu\alpha\beta} X^{n}_{\mu\nu\alpha\beta} &= \frac{10}{3} \sigma \left(I_{10} I_{n+5,2} + 4 I_{21} I_{n+4,2} \right) \ , \\
\Delta^{\mu\nu\alpha\beta} u^\gamma X^{n}_{\mu\nu\alpha\beta\gamma} &= 5\sigma \left(I_{10} I_{n+6,2} - I_{21} I_{n+5,2} + 2 I_{31} I_{n+4,2} \right) \ .
\end{aligned} \tag{5.113}$$

In order to obtain the above relations, we used the definitions (4.31) and Eq. (4.46). Using Eq. (5.108),

$$\begin{aligned}
\Delta^{\mu\nu\alpha\beta} X^{-1}_{\mu\nu\alpha\beta} &= 24\sigma \frac{P_0^2}{\beta_0} \,, \\
\Delta^{\mu\nu\alpha\beta} X^{0}_{\mu\nu\alpha\beta} &= \frac{400}{3}\sigma \frac{P_0^2}{\beta_0^2} \,, \\
\Delta^{\mu\nu\alpha\beta} u^{\gamma_1} X^{-1}_{\mu\nu\alpha\beta\gamma} &= 132\sigma \frac{P_0^2}{\beta_0^2} \,, \\
\Delta^{\mu\nu\alpha\beta} u^{\gamma} X^{0}_{\mu\nu\alpha\beta\gamma} &= 880\sigma \frac{P_0^2}{\beta_0^3} \,.
\end{aligned} \tag{5.114}$$

The components of $\mathcal{A}^{(2)}$ change according to the number of moments included. In the 14-moment approximation, using Eqs. (4.14) and (4.5), we obtain

$$\mathcal{A}^{(2)}_{00} = \frac{W^{(2)}}{10} \Delta^{\mu\nu\alpha\beta} X^{-1}_{\mu\nu\alpha\beta} = \frac{3}{5} n_0 \sigma \,, \tag{5.115}$$

where we used Eqs. (5.108) and (5.114), together with the results from App. 4.8.

In the 23-moment approximation, e.g., considering two polynomials in the expansion (5.1), for $\ell = 2$,

$$\begin{aligned}
\mathcal{A}^{(2)}_{r0} &= \frac{W^{(2)}}{10} \left(1 + a^{(2)}_{10} a^{(2)}_{10}\right) \Delta^{\mu\nu\alpha\beta} X^{r-1}_{\mu\nu\alpha\beta} + \frac{W^{(2)}}{10} a^{(2)}_{10} a^{(2)}_{11} \Delta^{\mu\nu\alpha\beta} u^{\gamma} X^{r-1}_{\mu\nu\alpha\beta\gamma} \,, \\
\mathcal{A}^{(2)}_{r1} &= \frac{W^{(2)}}{10} a^{(2)}_{11} a^{(2)}_{10} \Delta^{\mu\nu\alpha\beta} X^{r-1}_{\mu\nu\alpha\beta} + \frac{W^{(2)}}{10} a^{(2)}_{11} a^{(2)}_{11} \Delta^{\mu\nu\alpha\beta} u^{\gamma} X^{r-1}_{\mu\nu\alpha\beta\gamma} \,.
\end{aligned} \tag{5.116}$$

Then, using once more the results from App. 4.8 and Eqs. (5.108) and (5.114), we obtain

$$\begin{aligned}
\mathcal{A}^{(2)}_{00} &= \frac{9}{10} n_0 \sigma \,, \quad \mathcal{A}^{(2)}_{01} = -\frac{1}{20} \beta_0 n_0 \sigma \,, \\
\mathcal{A}^{(2)}_{10} &= \frac{4}{3\beta_0} n_0 \sigma \,, \quad \mathcal{A}^{(2)}_{11} = \frac{1}{3} n_0 \sigma \,.
\end{aligned} \tag{5.117}$$

We did not calculate the coefficients related to the bulk-viscous pressure, since this quantity vanishes in the massless limit. Also, if the mass was taken to be finite, some of the steps taken in this appendix would not be possible.

5.11 Appendix 3: Calculation of $\gamma_1^{(2)}$

In this appendix we compute the quantity $\gamma_1^{(2)}$ in the 14-moment approximation and the 23-moment approximation. Among all the $\gamma_i^{(\ell)}$ appearing in the transport coefficients listed in App. 5.9 this is the only one that survives in the ultrarelativistic limit, all the others are accompanied by factors of m^2 or couple to Π, which vanishes in this limit. The variable $\gamma_1^{(2)}$ was defined in the main text,

$$\gamma_1^{(2)} = \sum_{n=0}^{N_2} \mathcal{F}^{(2)}_{rn} \Omega^{(2)}_{n0} \,. \tag{5.118}$$

The first step is to compute the thermodynamic integral

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!} \int dK\, f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-r} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^\ell \ . \tag{5.119}$$

5.11.1 14-moment approximation

In this case, $N_1 = 1$ and $N_2 = 0$, and

$$\gamma_1^{(2)} = \mathcal{F}_{10}^{(2)} \ . \tag{5.120}$$

Also, in the 14-moment approximation,

$$\mathcal{H}_{\mathbf{k}0}^{(2)} \equiv \frac{W^{(2)}}{2!} a_{00}^{(2)} P_{\mathbf{k}0}^{(2)} = \frac{W^{(2)}}{2!} \ . \tag{5.121}$$

In the massless and classical limit,

$$\mathcal{H}_{\mathbf{k}0}^{(2)} = \frac{\beta_0^2}{8P_0} \ , \tag{5.122}$$

and finally

$$\gamma_1^{(2)} = \frac{\beta_0^2}{4P_0} \frac{1}{5!!} \int dK\, f_{0\mathbf{k}} E_{\mathbf{k}}^{-1} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^2 = \frac{\beta_0}{5} \ . \tag{5.123}$$

5.11.2 23-moment approximation

In this case, $N_1 = 2$ and $N_2 = 1$, and

$$\gamma_1^{(2)} = \mathcal{F}_{10}^{(2)} + \Omega_{10}^{(2)} \mathcal{F}_{11}^{(2)} \ . \tag{5.124}$$

Also, in the 23-moment approximation,

$$\begin{aligned} \mathcal{H}_{\mathbf{k}0}^{(2)} &= \frac{W^{(2)}}{2!} \left(1 + a_{10}^{(2)} P_{\mathbf{k}1}^{(2)}\right) = \frac{W^{(2)}}{2!} \left[1 + \left(a_{10}^{(2)}\right)^2 + a_{10}^{(2)} a_{11}^{(2)} E_{\mathbf{k}}\right] \ , \\ \mathcal{H}_{\mathbf{k}1}^{(2)} &= \frac{W^{(2)}}{2!} a_{11}^{(2)} P_{\mathbf{k}1}^{(2)} = \frac{W_{(2)}}{2!} \left[a_{10}^{(2)} a_{11}^{(2)} + \left(a_{11}^{(2)}\right)^2 E_{\mathbf{k}}\right] \ . \end{aligned} \tag{5.125}$$

We know that

$$W^{(2)} = \frac{\beta_0^2}{4P_0} \ , \quad \left(a_{11}^{(2)}\right)^2 = \frac{\beta_0^2}{6} \ , \quad \frac{a_{10}^{(2)}}{a_{11}^{(2)}} = -\frac{6}{\beta_0} \ . \tag{5.126}$$

Thus,

$$\begin{aligned} \mathcal{H}_{\mathbf{k}0}^{(2)} &= \frac{\beta_0^2}{8P_0} \left(7 - \beta_0 E_{\mathbf{k}}\right) \ , \\ \mathcal{H}_{\mathbf{k}1}^{(2)} &= \frac{\beta_0^3}{8P_0} \left(-1 + \frac{1}{6} \beta_0 E_{\mathbf{k}}\right) \ , \end{aligned} \tag{5.127}$$

and

$$\mathcal{F}_{10}^{(2)} = \frac{\beta_0^2}{4P_0}\frac{1}{5!!}\int dK\, f_{0\mathbf{k}} E_{\mathbf{k}}^{-1} (7-\beta_0 E_{\mathbf{k}}) \left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^2 = \frac{2}{5}\beta_0 \,,$$
$$\mathcal{F}_{11}^{(2)} = \frac{\beta_0^3}{4P_0}\frac{1}{5!!}\int dK\, f_{0\mathbf{k}} E_{\mathbf{k}}^{-1} \left(-1+\frac{1}{6}\beta_0 E_{\mathbf{k}}\right) \left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^2 = -\frac{\beta_0^2}{30} \,. \tag{5.128}$$

Substituting $\Omega^{(2)}$ from Eq. (5.50) we obtain

$$\gamma_1^{(2)} = \frac{2}{15}\beta_0 = 0.133\beta_0 \,. \tag{5.129}$$

5.12 Appendix 4: Transport coefficients in Sec. 5.6

In this appendix we list all transport coefficients appearing in the extension of fluid dynamics discussed in Sec. 5.6. The microscopic formulas for the diffusion and viscosity coefficients, $\vec{\varkappa}$ and $\vec{\eta}$, and for the relaxation-time matrices, $\hat{\tau}_n$ and $\hat{\tau}_\pi$, are

$$\vec{\varkappa} = \sum_{k=0,\neq 1}^{N_1} \alpha_k^{(1)} \begin{pmatrix} \tau_{0k}^{(1)} \\ \tau_{2k}^{(1)} \end{pmatrix} \,, \quad \vec{\eta} = \sum_{k=0}^{N_2} \alpha_k^{(2)} \begin{pmatrix} \tau_{0k}^{(2)} \\ \tau_{1k}^{(2)} \end{pmatrix} \,, \tag{5.130}$$

$$\hat{\tau}_n = \sum_{r=0,\neq 1}^{N_1} \begin{pmatrix} \tau_{0r}^{(1)}\lambda_{r0}^{(1)} & \tau_{0r}^{(1)}\lambda_{r2}^{(1)} \\ \tau_{2r}^{(1)}\lambda_{r0}^{(1)} & \tau_{2r}^{(1)}\lambda_{r2}^{(1)} \end{pmatrix} \,, \quad \hat{\tau}_\pi = \sum_{r=0}^{N_2} \begin{pmatrix} \tau_{0r}^{(2)}\lambda_{r0}^{(2)} & \tau_{0r}^{(2)}\lambda_{r1}^{(2)} \\ \tau_{1r}^{(2)}\lambda_{r0}^{(2)} & \tau_{1r}^{(2)}\lambda_{r1}^{(2)} \end{pmatrix} \,. \tag{5.131}$$

The transport coefficients of the nonlinear terms in the equation of motion for $\vec{n}^\mu$ are

$$\hat{\delta}_{nn} = \frac{1}{3}\sum_{r=0,\neq 1}^{N_1} \begin{pmatrix} 3\tau_{0r}^{(1)}\lambda_{r0}^{(1)} & 5\tau_{0r}^{(1)}\lambda_{r2}^{(1)} \\ 3\tau_{2r}^{(1)}\lambda_{r0}^{(1)} & 5\tau_{2r}^{(1)}\lambda_{r2}^{(1)} \end{pmatrix} \,, \tag{5.132}$$

$$\hat{\lambda}_{nn} = \frac{1}{5}\sum_{r=0,\neq 1}^{N_1} (2r+3) \begin{pmatrix} \tau_{0r}^{(1)}\lambda_{r0}^{(1)} & \tau_{0r}^{(1)}\lambda_{r2}^{(1)} \\ \tau_{2r}^{(1)}\lambda_{r0}^{(1)} & \tau_{2r}^{(1)}\lambda_{r2}^{(1)} \end{pmatrix} \,, \tag{5.133}$$

$$\hat{\lambda}_{n\pi} = \frac{1}{4}\left[\sum_{r=0}^{N_2} \begin{pmatrix} \tau_{00}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r0}^{(2)} & 2\tau_{00}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r1}^{(2)} \\ \tau_{20}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r0}^{(2)} & 2\tau_{20}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r1}^{(2)} \end{pmatrix} + \sum_{r=2}^{N_1} \begin{pmatrix} (1-r)\,\tau_{0r}^{(1)}\lambda_{r-1,0}^{(2)} & (2-r)\,\tau_{0r}^{(1)}\lambda_{r-1,1}^{(2)} \\ (1-r)\,\tau_{2r}^{(1)}\lambda_{r-1,0}^{(2)} & (2-r)\,\tau_{2r}^{(1)}\lambda_{r-1,1}^{(2)} \end{pmatrix}\right] \,, \tag{5.134}$$

$$\hat{\tau}_{n\pi} = -4P_0\left[\sum_{r=2}^{N_1} \begin{pmatrix} 0 & \tau_{0r}^{(1)}\lambda_{r-1,1}^{(2)} \\ 0 & \tau_{2r}^{(1)}\lambda_{r-1,1}^{(2)} \end{pmatrix} + \sum_{r=0}^{N_2} \begin{pmatrix} 0 & \tau_{00}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r1}^{(2)} \\ 0 & \tau_{20}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r1}^{(2)} \end{pmatrix}\right] \,, \tag{5.135}$$

$$\hat{\ell}_{n\pi} = -\sum_{r=0}^{N_2} \begin{pmatrix} \tau_{00}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r0}^{(2)} & \tau_{00}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r1}^{(2)} \\ \tau_{20}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r0}^{(2)} & \tau_{20}^{(1)}\mathcal{F}_{1r}^{(2)}\lambda_{r1}^{(2)} \end{pmatrix} + \frac{\beta_0}{4P_0}\sum_{r=0,\neq 1}^{N_1} \begin{pmatrix} \tau_{0r}^{(1)} I_{r+2,1} & 0 \\ \tau_{2r}^{(1)} I_{r+2,1} & 0 \end{pmatrix} - \sum_{r=2}^{N_1} \begin{pmatrix} \tau_{0r}^{(1)}\lambda_{r-1,0}^{(2)} & \tau_{0r}^{(1)}\lambda_{r-1,1}^{(2)} \\ \tau_{2r}^{(1)}\lambda_{r-1,0}^{(2)} & \tau_{2r}^{(1)}\lambda_{r-1,1}^{(2)} \end{pmatrix} \,, \tag{5.136}$$

while those in the equation of motion for $\vec{\pi}^{\mu\nu}$ are

$$\hat{\delta}_{\pi\pi} = \frac{1}{3}\sum_{r=0}^{N_2} \begin{pmatrix} 4\tau_{0r}^{(2)}\lambda_{r0}^{(2)} & 5\tau_{0r}^{(2)}\lambda_{r1}^{(2)} \\ 4\tau_{1r}^{(2)}\lambda_{r0}^{(2)} & 5\tau_{1r}^{(2)}\lambda_{r1}^{(2)} \end{pmatrix} , \tag{5.137}$$

$$\hat{\tau}_{\pi\pi} = \frac{2}{7}\sum_{r=0}^{N_2} (2r+5) \begin{pmatrix} \tau_{0r}^{(2)}\lambda_{r0}^{(2)} & \tau_{0r}^{(2)}\lambda_{r1}^{(2)} \\ \tau_{1r}^{(2)}\lambda_{r0}^{(2)} & \tau_{1r}^{(2)}\lambda_{r1}^{(2)} \end{pmatrix} , \tag{5.138}$$

$$\hat{\tau}_{\pi n} = \frac{1}{5P_0}\sum_{r=1}^{N_2} \begin{pmatrix} 2\tau_{0r}^{(2)}\lambda_{r+1,0}^{(1)} & 3\tau_{0r}^{(2)}\lambda_{r+1,2}^{(1)} \\ 2\tau_{1r}^{(2)}\lambda_{r+1,0}^{(1)} & 3\tau_{1r}^{(2)}\lambda_{r+1,2}^{(1)} \end{pmatrix} , \tag{5.139}$$

$$\hat{\ell}_{\pi n} = \frac{2}{5}\sum_{r=1}^{N_2} \begin{pmatrix} \tau_{0r}^{(2)}\lambda_{r+1,0}^{(1)} & \tau_{0r}^{(2)}\lambda_{r+1,2}^{(1)} \\ \tau_{1r}^{(2)}\lambda_{r+1,0}^{(1)} & \tau_{1r}^{(2)}\lambda_{r+1,2}^{(1)} \end{pmatrix} , \tag{5.140}$$

$$\hat{\lambda}_{\pi n} = -\frac{1}{10}\sum_{r=2}^{N_2} \begin{pmatrix} (1+r)\,\tau_{0r}^{(2)}\lambda_{r+1,0}^{(1)} & \tau_{0r}^{(2)}\,(r-1)\,\lambda_{r+1,2}^{(1)} \\ (1+r)\,\tau_{1r}^{(2)}\lambda_{r+1,0}^{(1)} & \tau_{1r}^{(2)}\,(r-1)\,\lambda_{r+1,2}^{(1)} \end{pmatrix} . \tag{5.141}$$

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