Relativistic hydrodynamics for heavy–ion collisions

– can a macroscopic approach be applied to a microscopic system?

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thanks to:

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Fundamentals of fluid dynamics (I)

Fluid dynamics is a theory that describes the motion of macroscopic fluids

Fluids are liquids (e.g. water $\Longrightarrow$ "hydrodynamics") or gases (e.g. air)

Equations of motion:

– non-relativistic, without dissipation:

Mass conservation:	$\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0\,,$	ρ : **mass density,** $\vec{v}$: **fluid velocity**
Euler's equation:	$\dfrac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\dfrac{1}{\rho} \nabla p\,,$	p : **pressure**

– relativistic, including dissipation:

Net-charge conservation:	$\partial_\mu N^\mu = 0\,,$	N^μ: **net-charge 4-current**
Energy-momentum conservation:	$\partial_\mu T^{\mu\nu} = 0\,,$	$T^{\mu\nu}$: **energy-momentum tensor**

Fundamentals of fluid dynamics (II)

Range of validity of fluid dynamics:

A fluid "element" contains typically $\sim 10^{23}$ particles per gram

$\Longrightarrow$ **interparticle distance $\lambda \ll L$,**

L: characteristic macroscopic length scale

(length scale of variation of fluid fields, $L^{-1} \sim |\nabla \vec{v}|/c\,,\ |\nabla p|/p$)

Mean-free path of particle interactions: $\ell \sim \lambda$

$\Longrightarrow$ **Knudsen number** $\boxed{\mathrm{Kn} \equiv \ell/L}$

$\Longrightarrow$ **Fluid dynamics is valid if $\mathrm{Kn} \ll 1$**

$\Longrightarrow$ **Fluid dynamics can be derived from an underlying microscopic theory as a power series in Kn (I)**

Where does microscopic information enter the fluid-dynamical eqs. of motion?

$\Longrightarrow$ **equation of state $p(\epsilon, n)$, transport coefficients $\zeta\,,\ \eta\,,\ \kappa\,, \ldots$**

$\Longrightarrow$ **"low-energy constants" (II)**

$\overset{\text{(I,II)}}{\Longrightarrow}$ **Fluid dynamics is long-distance, large-time effective theory of a given underlying microscopic theory**

Why heavy-ion collisions?

Fundamental theory of strong interactions:

$\Longrightarrow$ Quantum Chromodynamics (QCD)

$\Longrightarrow$ QCD phase diagram:

Hadronic phase:

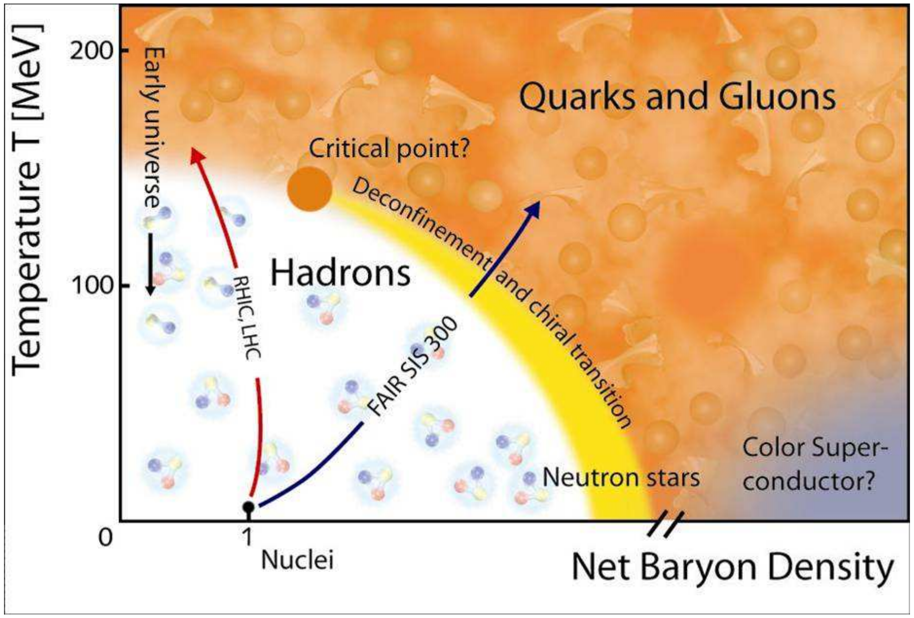
Confinement of quarks and gluons

(Chiral symmetry broken $\langle \bar{q} q \rangle \neq 0$)

Quark-Gluon Plasma (QGP):

Deconfined quarks and gluons

(Chiral symmetry restored $\langle \bar{q} q \rangle \simeq 0$)

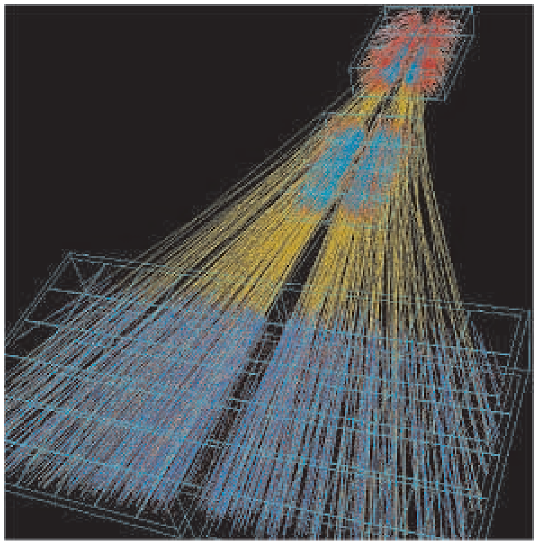


Heating and compressing QCD matter $\Longrightarrow$ heavy-ion collisions!

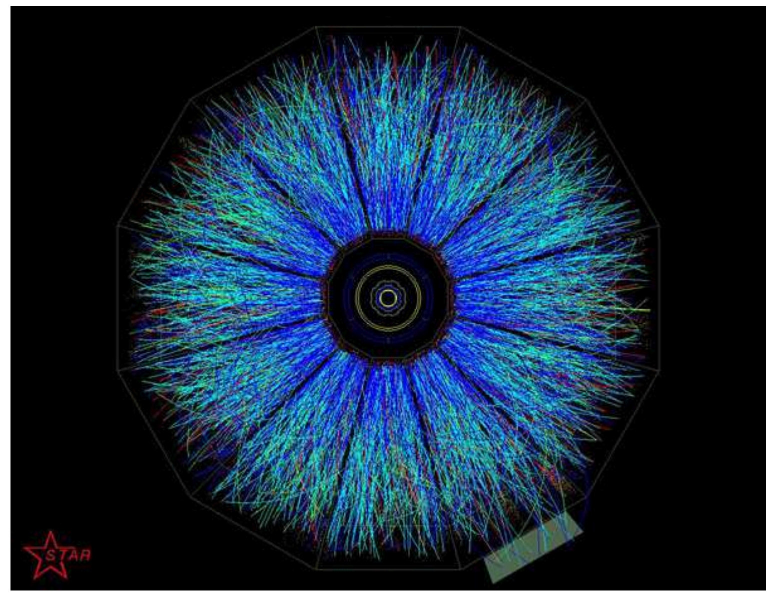
$\Longrightarrow$ Study phase transitions (in particular, deconfinement and chiral transitions) in fundamental theory of nature (QCD) in the laboratory!

$\Longrightarrow$ Study early-universe matter, as it existed $\sim 10^{-6}$ sec after the big bang!

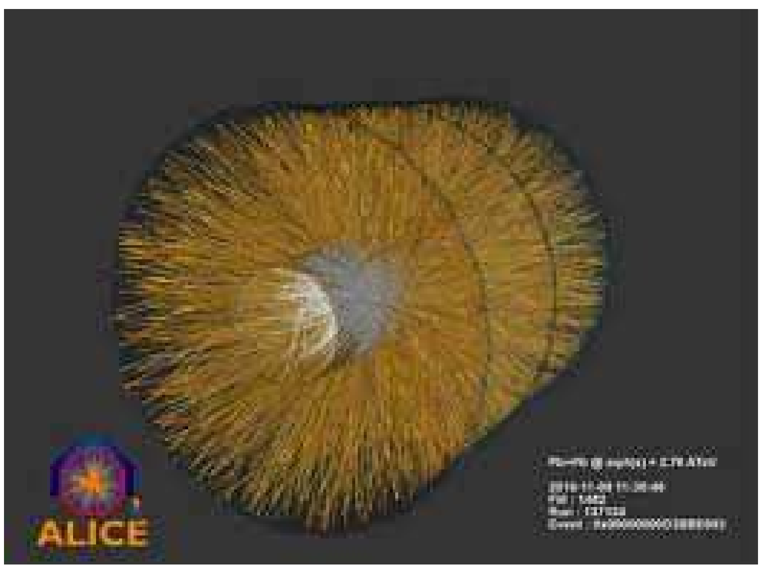
Heavy-ion collisions: the experimentalist's view



Pb+Pb collision at $\sqrt{s_{\rm NN}} = 17.3$ GeV
NA49 experiment @ CERN-SPS

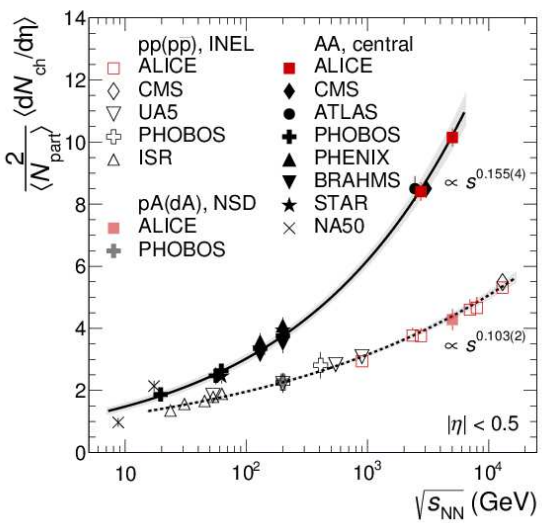


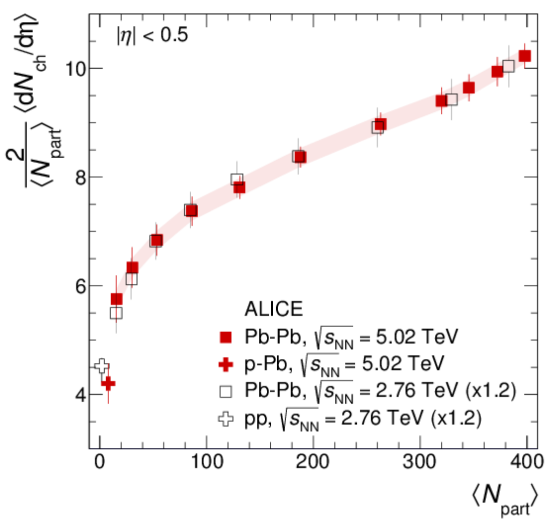
Au+Au collision at $\sqrt{s_{\rm NN}} = 200$ GeV
STAR experiment @ BNL-RHIC



Pb+Pb collision at $\sqrt{s_{\rm NN}} = 2.76$ TeV
ALICE experiment @ CERN-LHC

Heavy-ion collisions: creating early-universe matter





Central PbPb collisions at $\sqrt{s} = 5.02$ AGeV:

How many particles are created?

in cone of opening angle $\simeq 60^o$ transverse to beam axis:

$$\Longrightarrow \quad N = \frac{3}{2} N_{ch} \sim \frac{3}{2} \times 10 \times 200 \times \frac{1}{6} = 500 \text{ particles!}$$

$$\Longrightarrow \quad \text{average energy:} \quad \frac{dE_T}{d\eta} \Big/ \frac{dN_{ch}}{d\eta} \sim 1 \text{ GeV!}$$

What is the energy density at time τ_0?

J.D. Bjorken, PRD 27 (1983) 140

$$\Longrightarrow \quad \epsilon \sim \frac{1}{A_\perp \tau_0} \frac{dE_T}{d\eta} \sim \frac{2000\,\text{GeV}}{100\,\text{fm}^2 \tau_0} \sim 20 \frac{\text{GeV}}{\text{fm}^3} \text{ at } \tau_0 \sim 1 \frac{\text{fm}}{c}!$$

$\Longrightarrow$ **high-(energy-)density environment!**

$\Longrightarrow$ **in QGP phase of QCD matter!**

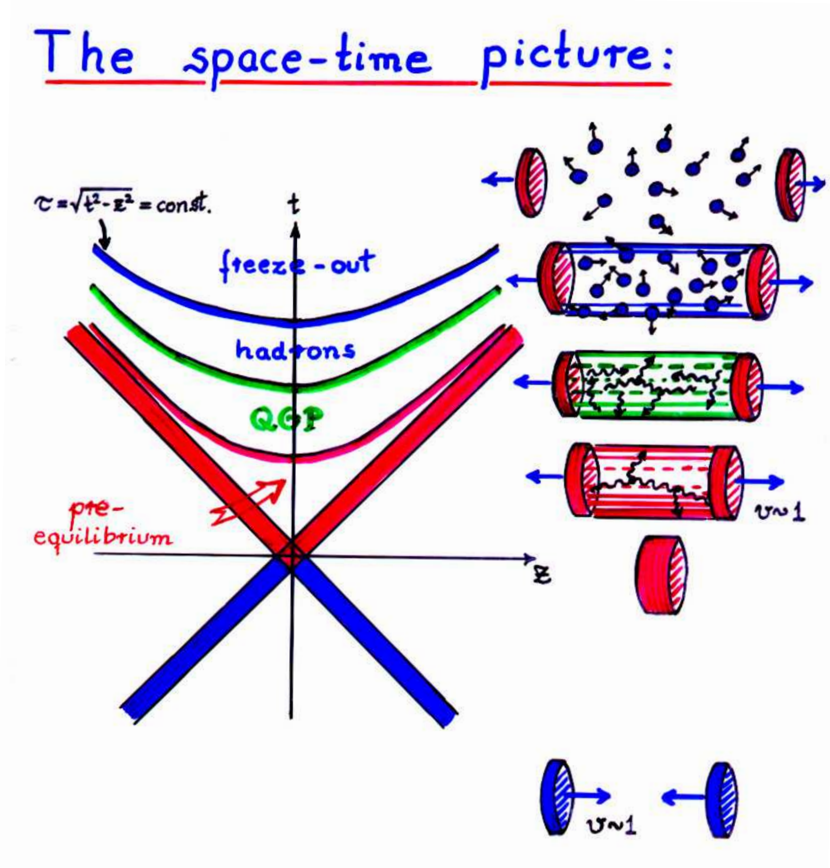
$\Longrightarrow$ **interparticle distance $\lambda \sim 1/T \lesssim 0.5$ fm,**
while system size $L \sim 10$ fm

$\Longrightarrow$ **Kn $\sim \lambda/L \sim 0.05 \ll 1$!**

$\Longrightarrow$ **Fluid dynamics may be applicable!**

Heavy-ion collisions: the theorist's view

Bjorken's picture: J.D. Bjorken, PRD 27 (1983) 140



Below certain temperature T_f hadrons freeze-out

⇑

Below certain temperature T_c QGP hadronizes

⇑

... QGP

⇑

Leave in their wake highly excited quark-gluon matter which thermalizes (Kn $\sim \lambda/L \ll 1$!) and becomes a ...

⇑

Highly Lorentz-contracted nuclei pass through each other

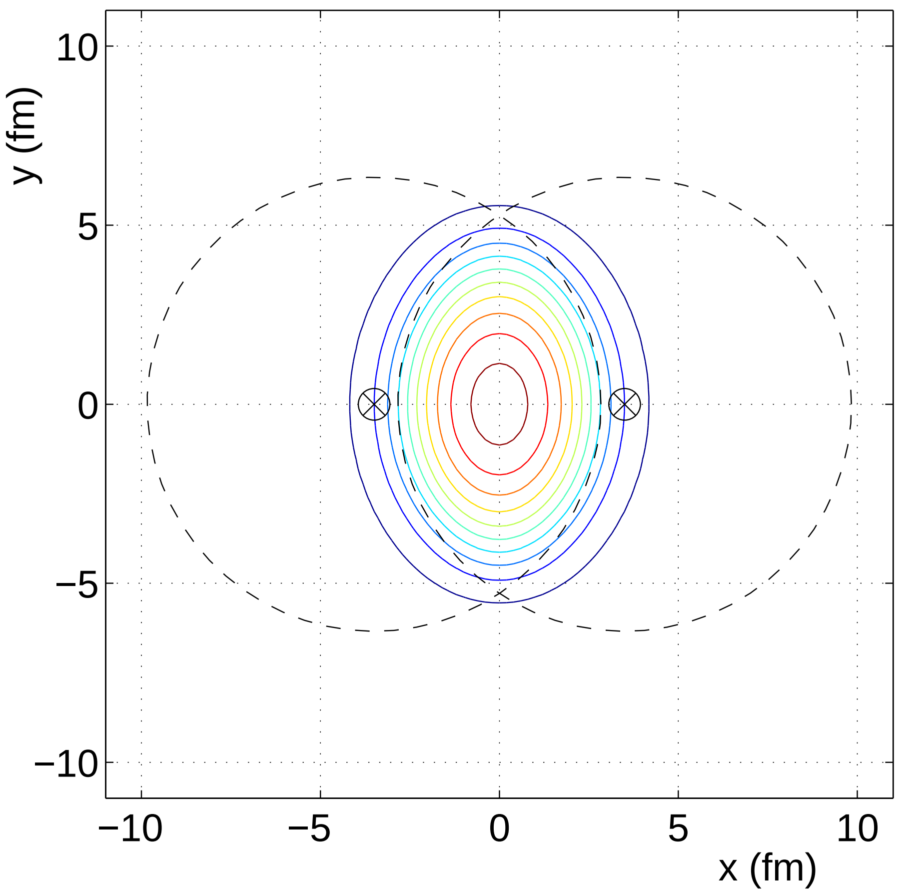
Note: Bjorken assumed that evolution of QGP and hadronic phase prior to freeze-out can be described by ideal fluid dynamics (1+1-d boost-invariant scaling solution)
$\Longrightarrow$ physics constant on proper-time hypersurfaces $\tau = \sqrt{t^2 - z^2} = const.$

How can we decide whether fluid dynamics applies?

Fluid dynamics implies collective flow:

non-central heavy-ion collision

in transverse $(x - y)$ plane $(z = 0)$:



If particles do not interact with each other, they stream freely towards the detector

$\Longrightarrow$ **single-inclusive particle spectrum:**

$$E\,\frac{dN}{d^3\vec{p}} \equiv E\,\frac{dN}{dp_z\, d^2\vec{p}_\perp}\,, \quad p_\perp = \sqrt{p_x^2 + p_y^2}$$

transverse momentum

$$\equiv E\,\frac{dN}{dp_z\, p_\perp dp_\perp\, d\varphi} \equiv \frac{dN}{dy\, p_\perp dp_\perp d\varphi}\,,$$

$\tanh y \equiv \frac{p_z}{E}$, $\quad y$: longitudinal rapidity

is independent of azimuthal angle φ

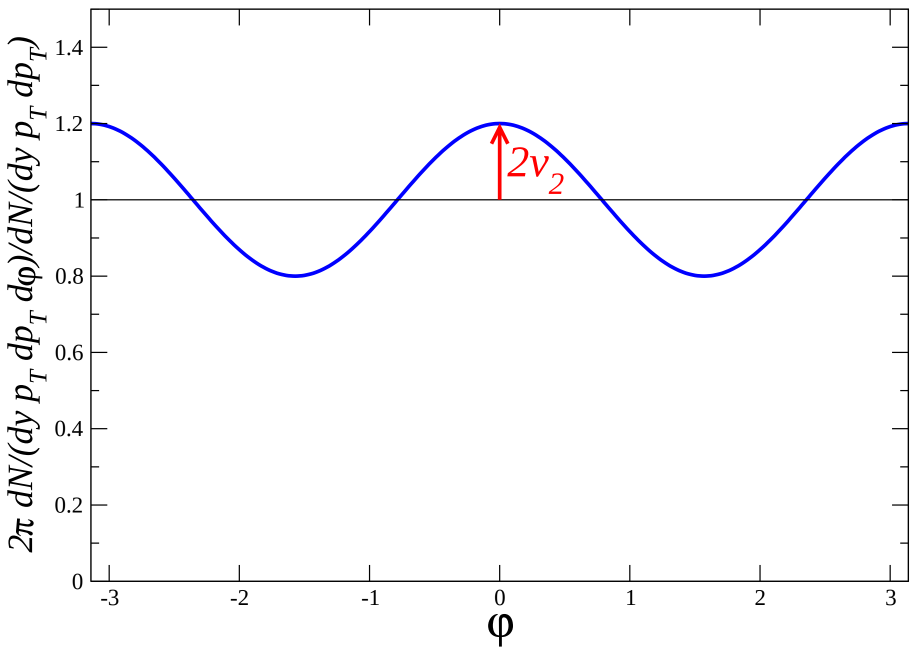
$\Longrightarrow$ **information on initial geometry is lost**

But: If particles interact strongly (like in a fluid), collective flow develops

$\Longrightarrow$ **initial spatial asymmetry is, by difference in pressure gradients, converted to final momentum anisotropy**

Characterization of collective flow

Event-averaged single-inclusive particle spectrum at $y = 0$ as function of φ:



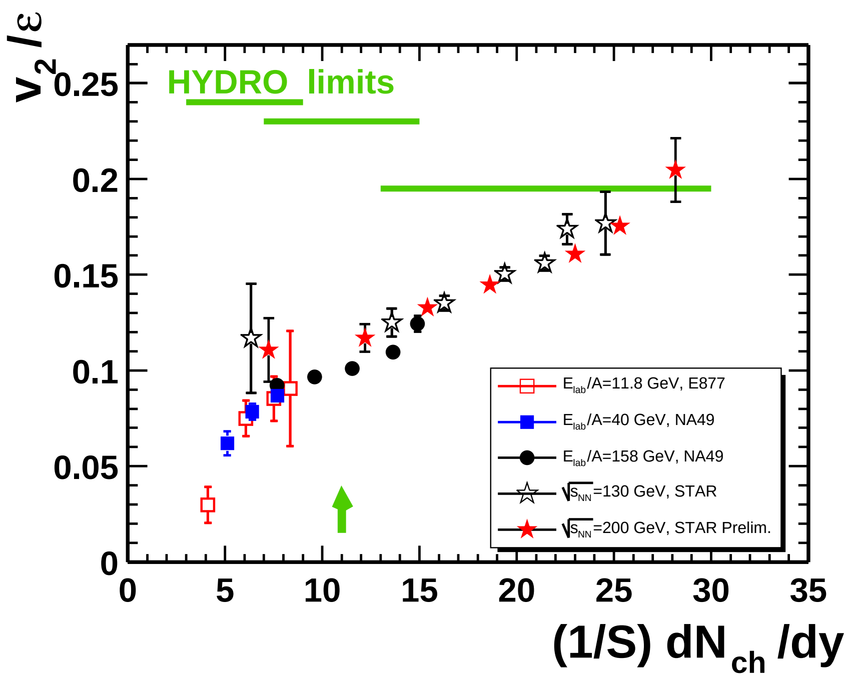
$\Longrightarrow$ **preferential emission of particles in the reaction $(x - z)$ plane**

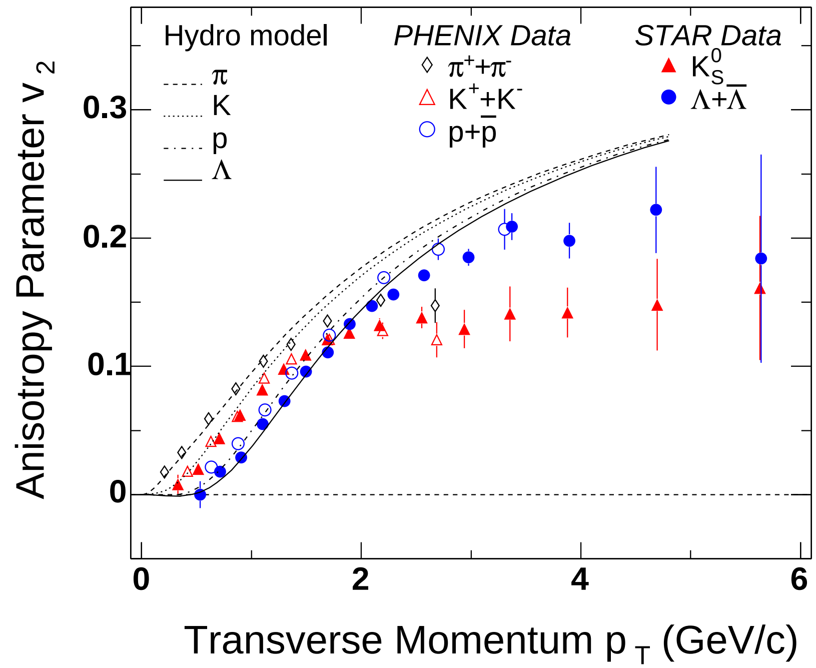
$\Longrightarrow$ **Fourier decomposition of single-inclusive particle spectrum:**

$$E\,\frac{dN}{d^3\vec{p}} \equiv \frac{dN}{dy\,p_\perp dp_\perp d\varphi} \equiv \frac{1}{2\pi}\,\frac{dN}{dy\,p_\perp dp_\perp}\left[1 + 2\sum_{n=1}^{\infty} v_n(y, p_\perp)\,\cos(n\varphi)\right]$$

v_1 : **directed flow**, v_2 : **elliptic flow** v_3 : **triangular flow** , **etc.**

Data confronts theory





$\Longrightarrow$ approach to fluid-dynamics with increasing centrality and beam energy!

$\Longrightarrow$ prediction of fluid dynamics: mass ordering of $v_2(p_T)$!

Success story no. 1: quantitative description of elliptic flow at RHIC within ideal fluid dynamics

$\Longrightarrow$ no dissipative effects! $\Longrightarrow$ "RHIC scientists serve up the perfect fluid"

Two problems (I)

1. **There is no real ideal fluid!**

shear viscosity $\eta \sim \frac{T}{\langle\sigma\rangle} \to 0 \quad \Longleftrightarrow \quad$ **average scattering cross section** $\langle\sigma\rangle \to \infty$

minimal value for shear viscosity to entropy density ratio:

(i) from uncertainty principle ("quantum limit"): $\quad \frac{\eta}{s} \simeq \frac{1}{12}$

P. Danielewicz, M. Gyulassy, PRD 31 (1985) 53

(ii) from AdS/CFT correspondence: conjectured lower bound $\quad \frac{\eta}{s} = \frac{1}{4\pi}$

P. Kovtun, D.T. Son, A. Starinets, PRL 94 (2005) 111601

$\Longrightarrow$ **What is** $\frac{\eta}{s}$ **of hot and dense hadronic matter?**

If $\frac{\eta}{s} \ll 1 \quad \Longrightarrow$ **matter is strongly interacting!**

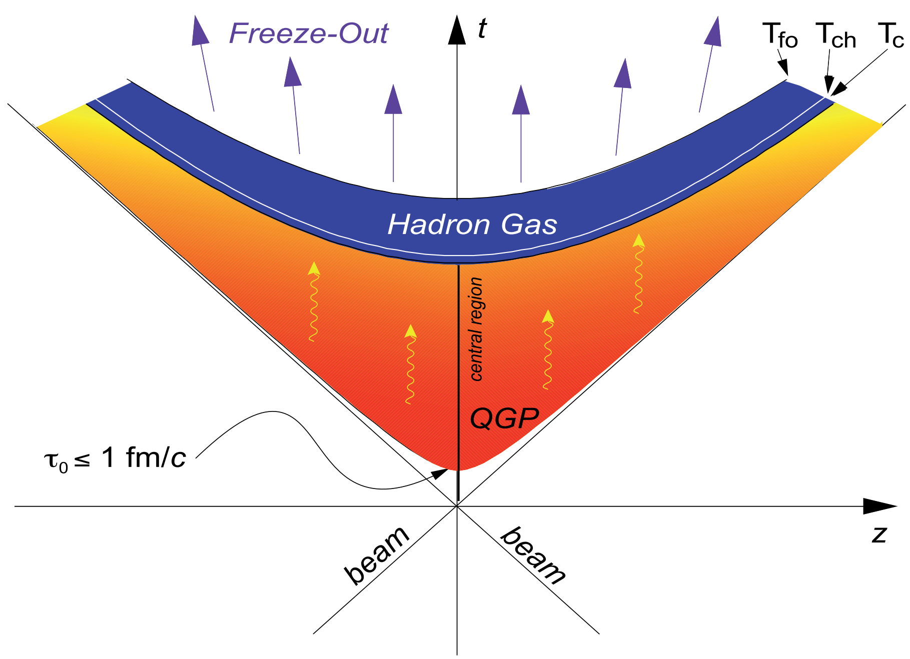
$\Longrightarrow$ **"strongly coupled quark-gluon plasma" (sQGP)**

Two problems (II)

2. **Fluid-dynamical equations of motion:** $\partial_\mu T^{\mu\nu} = 0$

$\Longrightarrow$ **partial differential equations**

$\Longrightarrow$ **require initial conditions on a space-time hypersurface**



energy-momentum tensor $T^{\mu\nu}(\tau_0, \vec{x})$
on initial space-time hypersurface
$\tau \equiv \sqrt{t^2 - z^2} \equiv \tau_0 = const.$

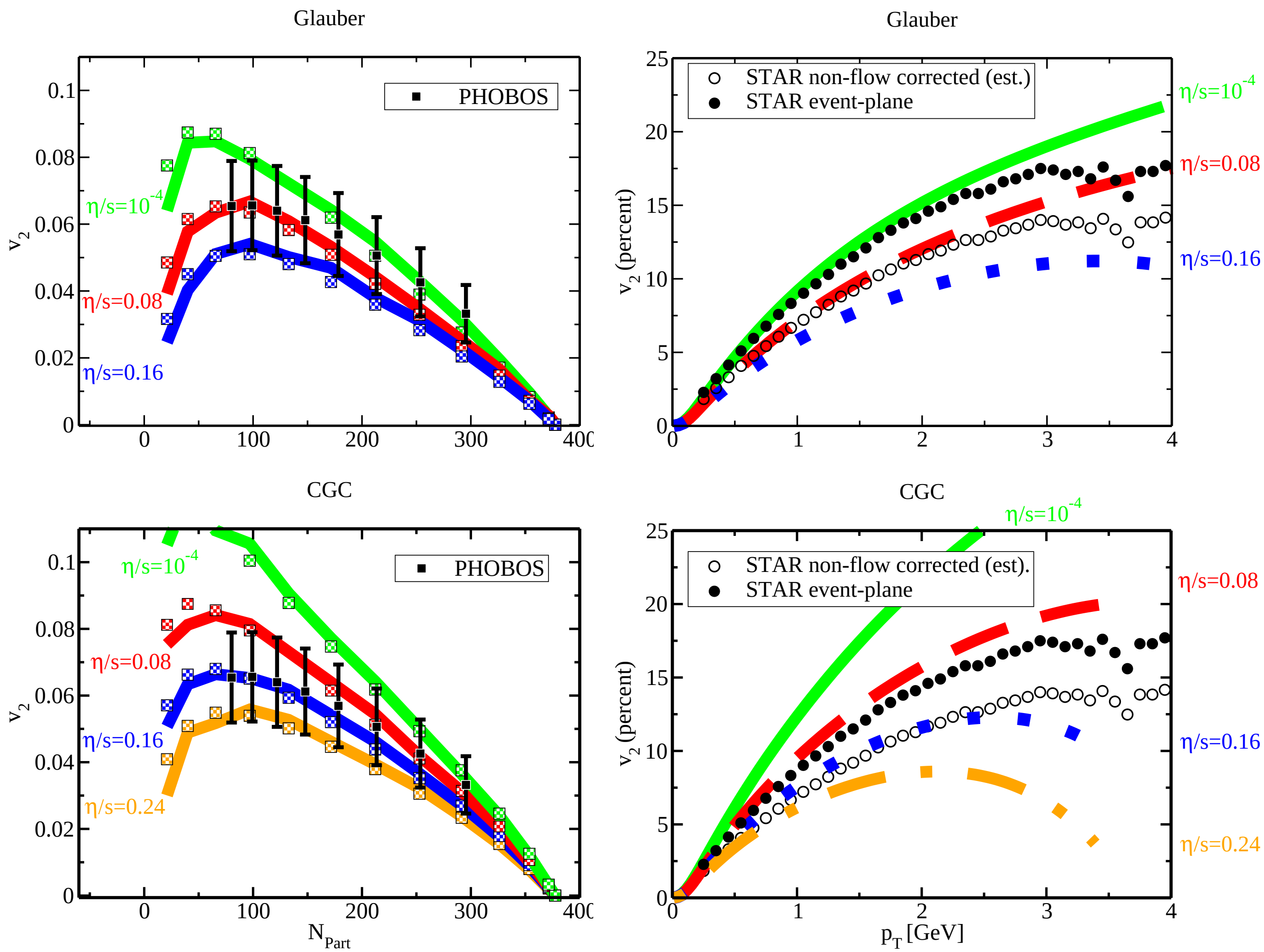
$\Longrightarrow$ **continuum of parameters to fit to experimental data**

$\Longrightarrow$ **experimental data may allow for non-zero viscosity!**

$\Longrightarrow$ **need calculations within dissipative fluid dynamics and with realistic initial conditions!**

Interplay between dissipation and initial conditions

M. Luzum, P. Romatschke, PRC78 (2008) 034915, Erratum C79 (2009) 039903



Glauber initial conditions:

$$\eta/s \simeq 0.08$$

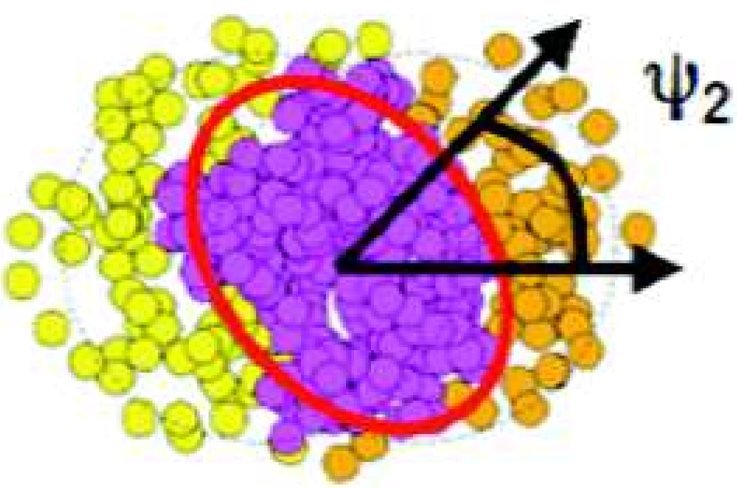
Color-Glass Condensate (CGC) initial conditions:

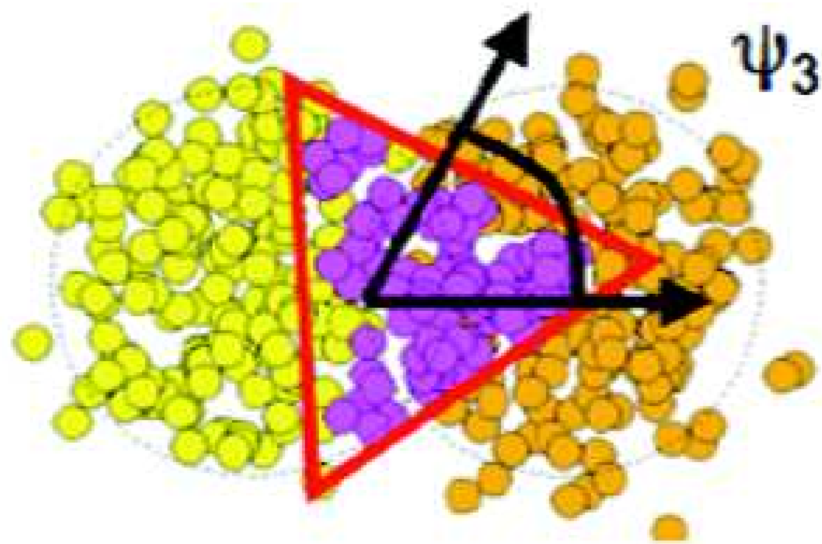
$$\eta/s \simeq 0.16$$

Event-by-event fluctuations

event by event: fluctuations of initial geometry

$\Longrightarrow$ **rotate participant plane vs. reaction plane $\psi_2 \neq 0$**





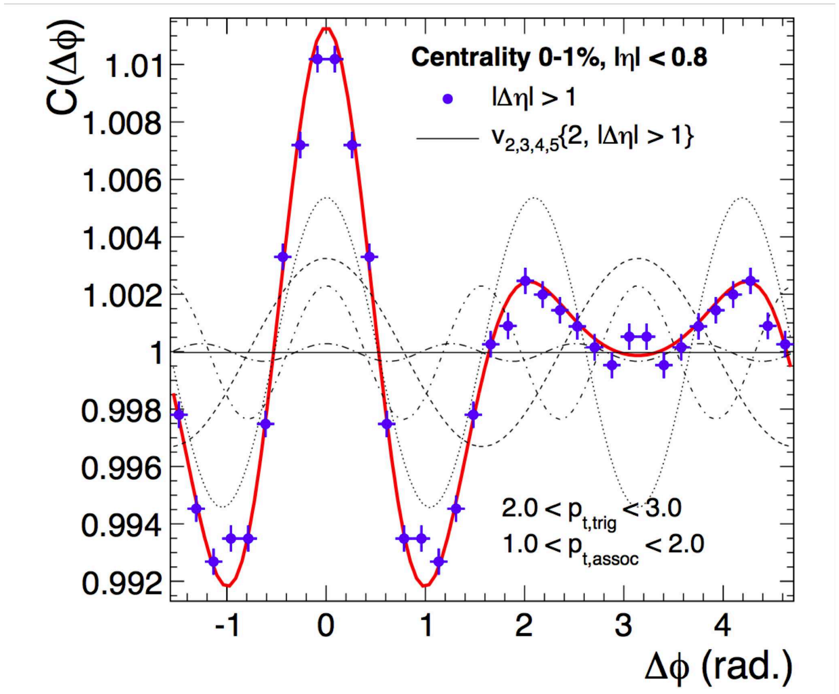
$\Longrightarrow$ **(i) induce higher flow harmonics! $v_n \neq 0\,,\ n = 3, 4, \ldots$**

$\Longrightarrow$ **(ii) provide additional constraint on η/s and initial conditions!**

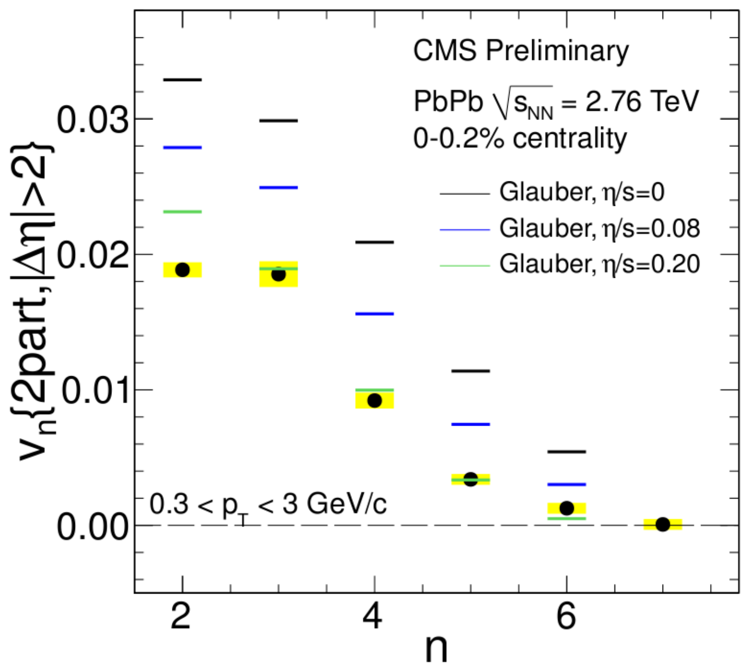
Event-by-event higher flow harmonics

$\Longrightarrow$ **two-particle correlation functions as superposition of higher flow harmonics**

B. Alver, G. Roland, PRC 81 (2010) 054905



ALICE Collaboration, PRL 107 (2011) 032301

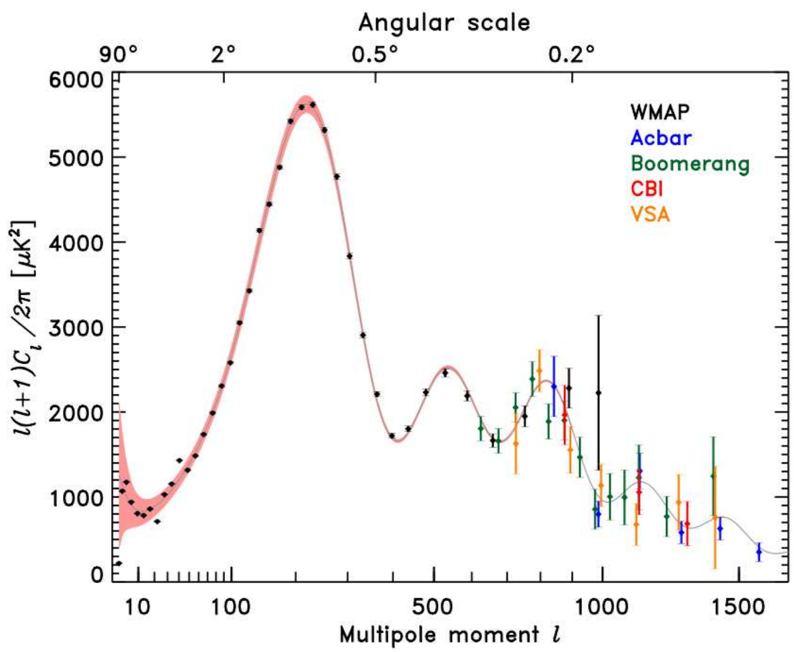


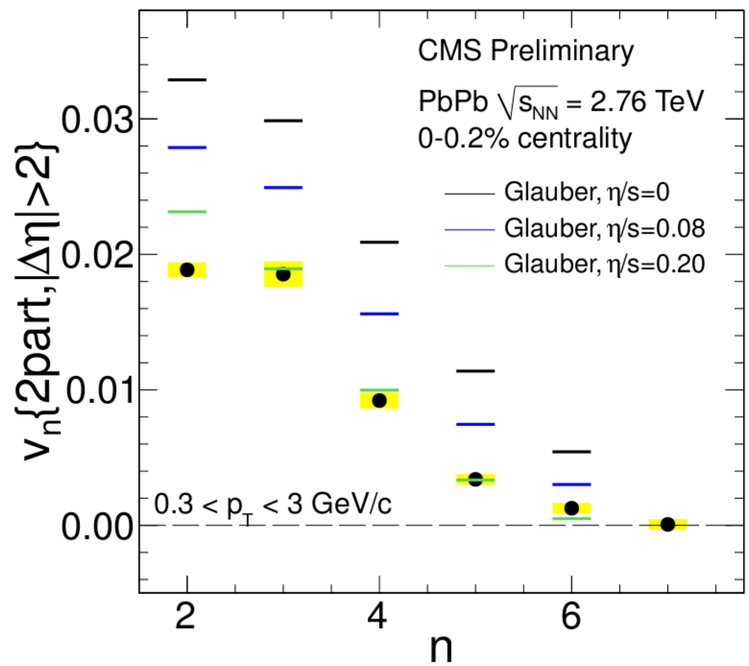
G. Roland for the CMS collaboration, presentation at QM 2012

Event-by-event higher flow harmonics

⟹ **two-particle correlation functions as superposition of higher flow harmonics**

B. Alver, G. Roland, PRC 81 (2010) 054905





WMAP multipole power spectrum

⟹ **Big bang vs. "little bang"**

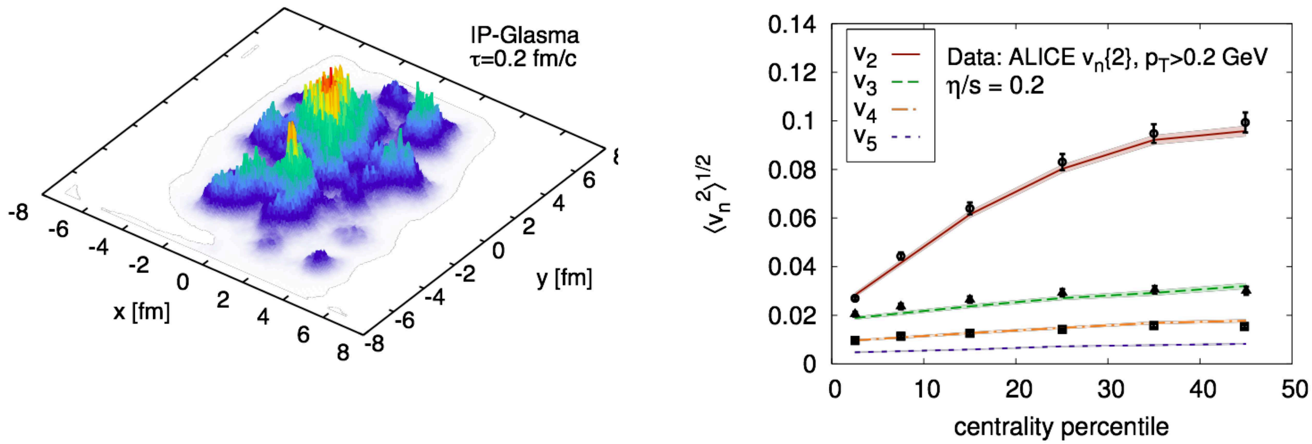
G. Roland for the CMS collaboration, presentation at QM 2012

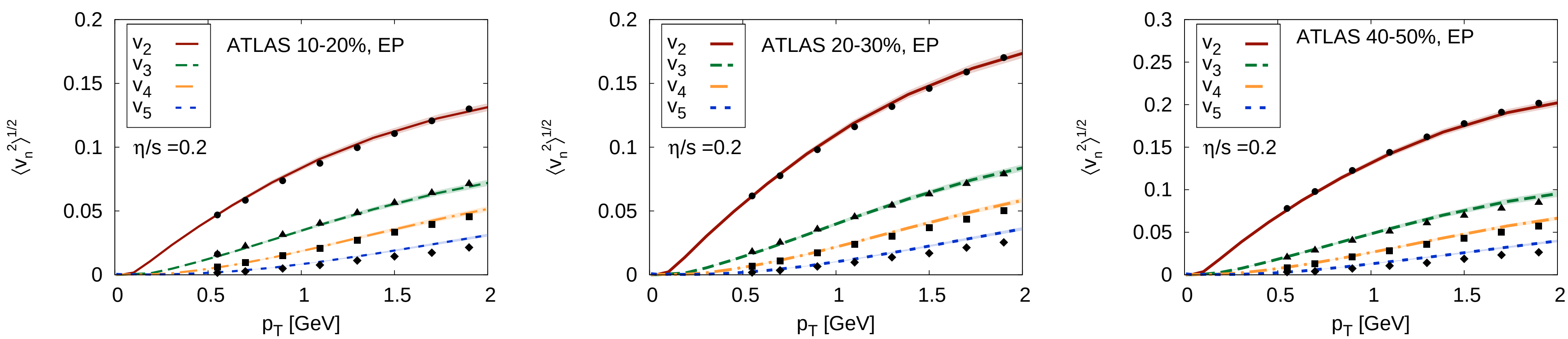
Additional constraint on η/s and initial conditions

initial conditions:

IP glasma

$\eta/s = 0.2 = const.$





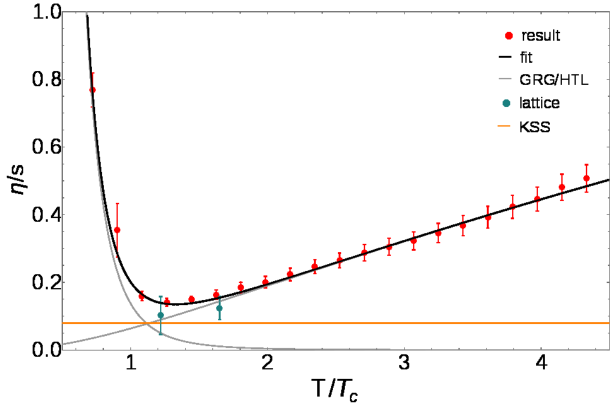
C. Gale, S. Jeon, B. Schenke, P. Tribedy, R. Venugopalan, PRL 110 (2013) 1, 012302

$\Longrightarrow$ **Success story no. 2: Quantitative description of collective flow by dissipative fluid dynamics, for all centralities and event by event!**
(with IP glasma initial conditions, $\eta/s = const.$)

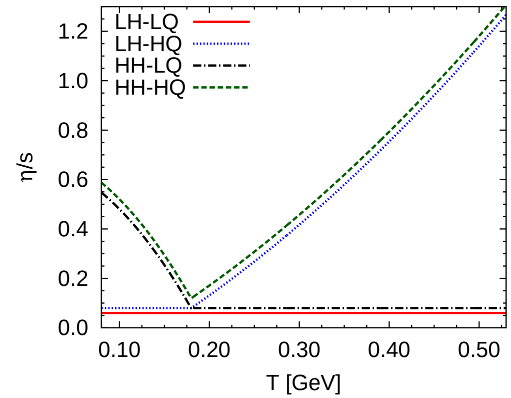
$\eta/s = const.$ vs. $\eta/s(T)$ (I)

η/s is not constant but a function of T (and μ)!

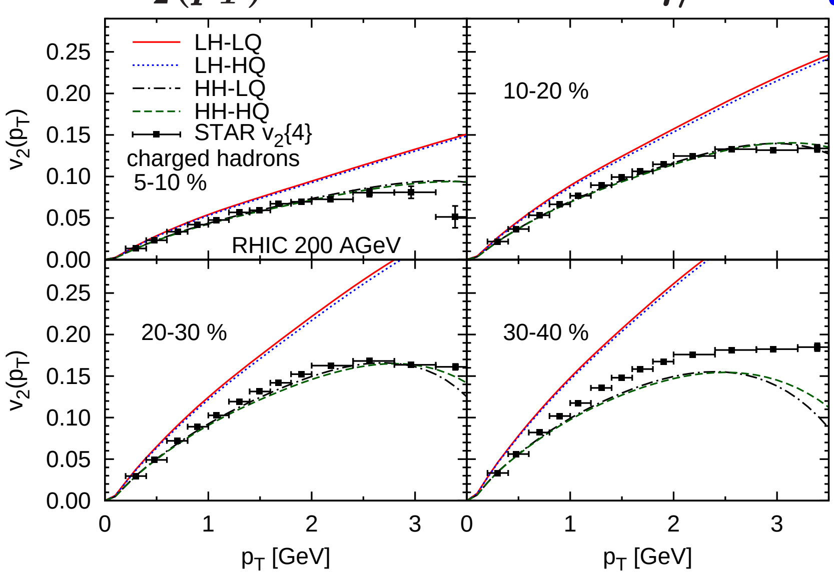
N. Christiansen, M. Haas, J.M. Pawlowski, N. Strodthoff, PRL 115 (2015) 11, 112002



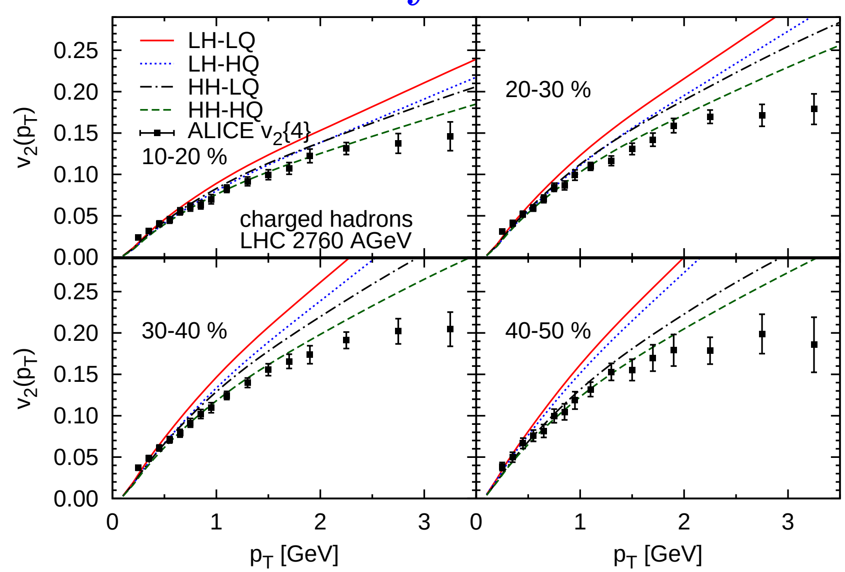
H. Niemi, G.S. Denicol, P. Huovinen, E. Molnár, DHR, PRL 106 (2011) 212302; PRC86 (2012) 014909



RHIC: $v_2(p_T)$ not sensitive to η/s in QGP!

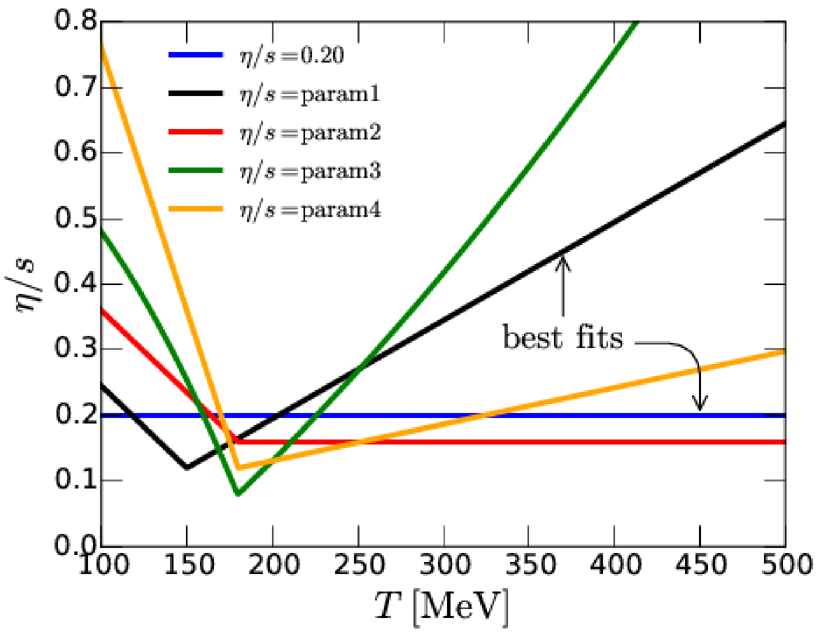


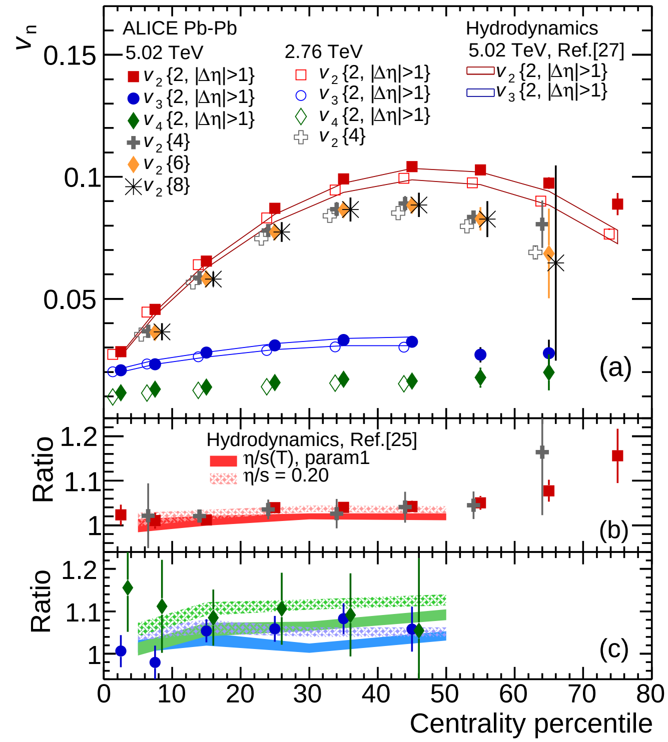
LHC: sensitivity increases!



$\eta/s = const.$ vs. $\eta/s(T)$ (II)

Prediction for highest LHC energies:
EbyE, NLO-pQCD, initial-state saturation + dissipative fluid dynamics (EKRT) model H. Niemi, K.J. Eskola, R. Paatelainen, K. Tuominen, PRC 93 (2016) 014912





ALICE coll., arXiv:1602.01119 [nucl-ex]

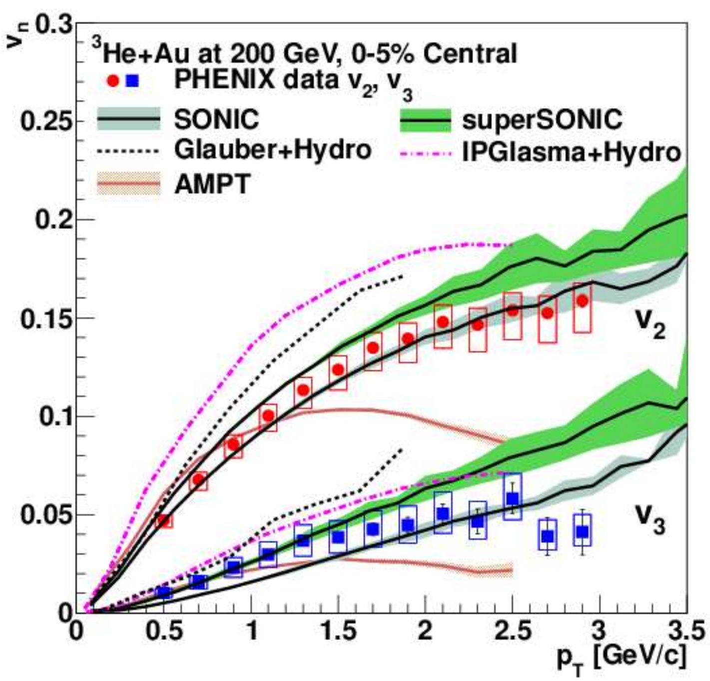
Success story no. 3: initial state and subsequent evolution sufficiently well understood to make quantitative predictions!

But: available range of collision energies not yet sufficient to determine $\eta/s(T)$!

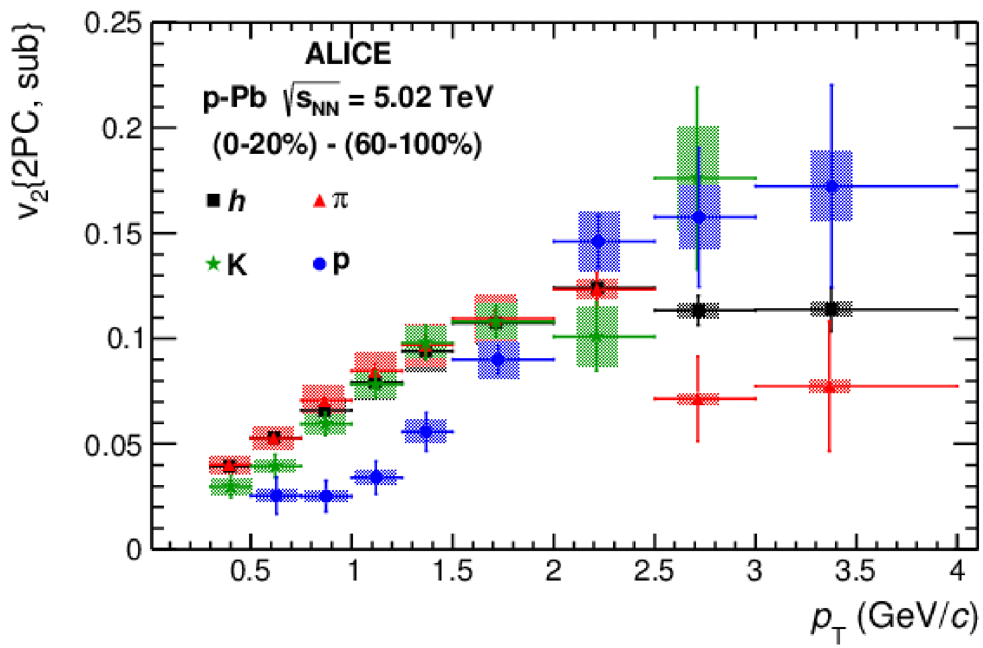
Collective flow in small systems: how small is too small?

Considerable flow seen in He^{3}Au-collisions at RHIC!

Considerable flow and mass ordering seen in pPb-collisions at LHC!



PHENIX coll., PRL 115 (2015) 142301



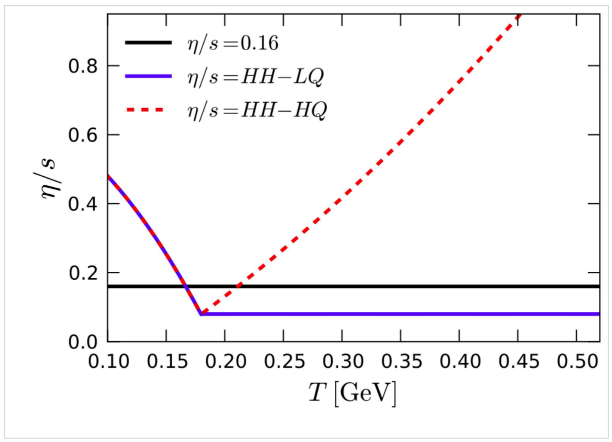
ALICE coll., PLB 726 (2013) 164

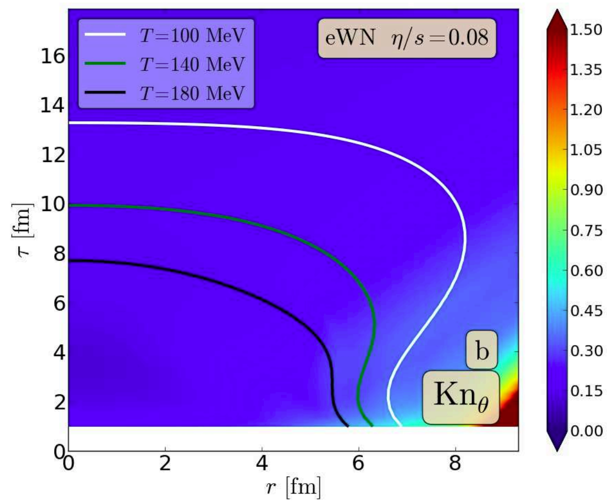
Collective flow has also been seen in pp-collisions at LHC!

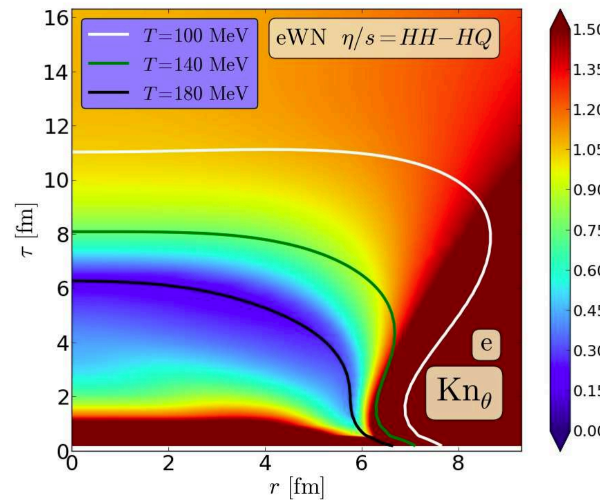
But: is fluid dynamics applicable to describe such small systems???

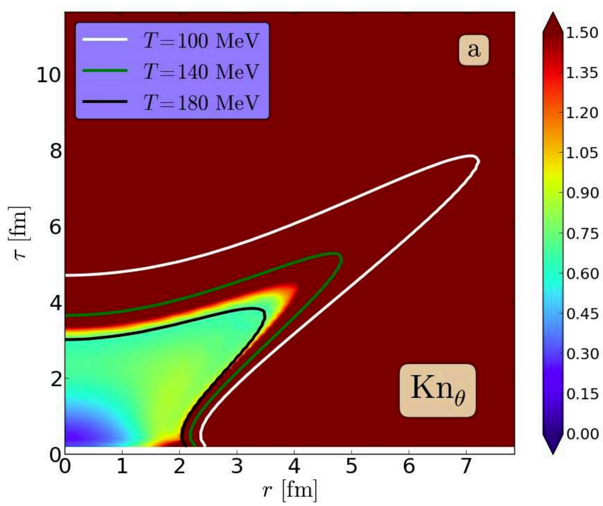
Consistency check

Compute $\mathrm{Kn} \equiv \frac{\ell}{L} \sim \frac{|\nabla \vec{v}|}{\langle \sigma \rangle n} \sim \frac{\eta}{s}\frac{|\nabla \vec{v}|}{T}$ **H. Niemi, G.S. Denicol, arXiv:1404.7327 [nucl-th]**









- **AA collisions, event-averaged initial conditions:**
  If $\eta/s \ll 1$, fluid dynamics applicable
- **pA collisions:**
  Even for $\eta/s \ll 1$, fluid dynamics barely applicable

$\Longrightarrow$ **Small systems behave collectively, but cannot be reliably described by (standard) fluid dynamics!**

$\Longrightarrow$ **Improve theory of fluid dynamics by including higher-order corrections in Kn!**

Fluid dynamics: degrees of freedom

1. **Net charge (e.g., baryon number, strangeness, etc.) current:** $\boxed{N^\mu = n\, u^\mu + n^\mu}$

u^μ	fluid 4-velocity, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$
	$g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ (West coast!!) metric tensor
$n \equiv u^\mu N_\mu$	net charge density in fluid rest frame
$n^\mu \equiv \Delta^{\mu\nu} N_\nu \equiv N^{<\mu>}$	diffusion current (flow of net charge relative to u^μ), $n^\mu u_\mu = 0$
	$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ projector onto 3-space orthogonal to u^μ, $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:** $\boxed{T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\, \Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}}$

$\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$	energy density in fluid rest frame
p	pressure in fluid rest frame
Π	bulk viscous pressure, $p + \Pi \equiv -\frac{1}{3}\, \Delta^{\mu\nu} T_{\mu\nu}$
$q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$	heat flux current (flow of energy relative to u^μ), $q^\mu u_\mu = 0$
$\pi^{\mu\nu} \equiv T^{<\mu\nu>}$	shear stress tensor, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0\,,\ \pi^\mu_{\ \mu} = 0$
	$a^{(\mu\nu)} \equiv \frac{1}{2}\,(a^{\mu\nu} + a^{\nu\mu})$ symmetrized tensor
	$a^{<\mu\nu>} \equiv \left(\Delta_\alpha^{\ (\mu} \Delta^{\nu)}_{\ \ \beta} - \frac{1}{3}\, \Delta^{\mu\nu} \Delta_{\alpha\beta}\right) a^{\alpha\beta}$ symmetrized, traceless spatial projection

Fluid dynamics: equations of motion

1. **Net charge conservation:**

$$\boxed{\partial_\mu N^\mu = 0} \iff \boxed{\dot{n} + n\,\theta + \partial \cdot n = 0}$$

$\dot{n} \equiv u^\mu \partial_\mu n$ convective (comoving) derivative
(time derivative in fluid rest frame, $\dot{n}_{\mathrm{RF}} \equiv \partial_t n$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$\boxed{\partial_\mu T^{\mu\nu} = 0} \iff$ energy conservation:

$$\boxed{u_\nu\,\partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0}$$

acceleration equation:

$$\boxed{\begin{array}{l} \Delta^{\mu\nu}\,\partial^\lambda T_{\nu\lambda} = 0 \quad \iff \\ (\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q \cdot \partial u^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda} \end{array}}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient (spatial gradient in fluid rest frame)

Solvability

Problem:

5 equations, but 15 unknowns (for given u^μ): $\epsilon\,,\ p\,,\ n\,,\ \Pi\,,\ n^\mu\,(3)\,,\ q^\mu\,(3)\,,\ \pi^{\mu\nu}\,(5)$

Solution:

1. **clever choice of frame (Eckart, Landau,...): eliminate n^μ or q^μ**
   $\Longrightarrow$ **does not help! Promotes u^μ to dynamical variable!**
2. **ideal fluid limit: all dissipative terms vanish, $\Pi = n^\mu = q^\mu = \pi^{\mu\nu} = 0$**
   $\Longrightarrow$ **6 unknowns: $\epsilon\,,\ p\,,\ n\,,\ u^\mu\,(3)$ (not quite there yet...)**
   $\Longrightarrow$ **fluid is in local thermodynamical equilibrium**
   $\Longrightarrow$ **provide equation of state (EOS) $p(\epsilon, n)$ to close system of equations**
3. **provide additional equations for dissipative quantities**
   $\Longrightarrow$ **relativistic dissipative fluid dynamics**
   (a) **First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Landau, Lifshitz)**
   (b) **Second-order theories: e.g. Israel-Stewart (IS) equations**

Navier-Stokes equations

Navier-Stokes (NS) equations: first-order relativistic dissipative fluid dynamics

1. **bulk viscous pressure:** $\boxed{\Pi_{\rm NS} = -\zeta\,\theta}$

   ζ **bulk viscosity**

2. **diffusion current:** $\boxed{n^\mu_{\rm NS} = \kappa_n\,\nabla^\mu \alpha}$

   $\beta \equiv 1/T$ **inverse temperature,**

   $\alpha \equiv \beta\,\mu,$ μ **chemical potential,**

   κ_n **net-charge diffusion coefficient**

3. **shear stress tensor:** $\boxed{\pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}}$

   η **shear viscosity,**

   $\sigma^{\mu\nu} = \nabla^{<\mu} u^{\nu>}$ **shear tensor**

$\Longrightarrow$ **algebraic expressions in terms of thermodynamic and fluid variables**

$\Longrightarrow$ **simple... but: unstable and acausal equations of motion!!**

W.A. Hiscock, L. Lindblom, PRD 31 (1985) 725

Israel-Stewart equations

Israel-Stewart (IS) equations: second-order relativistic dissipative fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" version:

$$\boxed{\begin{array}{c}\tau_\Pi\,\dot{\Pi} + \Pi = \Pi_{\rm NS} \\ \tau_n\,\dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS} \\ \tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS}\end{array}}$$

cf. also T. Koide, G.S. Denicol, Ph. Mota, T. Kodama, PRC 75 (2007) 034909

$\Longrightarrow$ dynamical (instead of algebraic) equations for dissipative terms!

solution: e.g. bulk viscous pressure $\boxed{\Pi(t) = \Pi_{\rm NS}\left(1 - e^{-t/\tau_\Pi}\right) + \Pi(0)\,e^{-t/\tau_\Pi}}$

$\Longrightarrow$ dissipative quantities $\Pi\,,\ n^\mu\,,\ \pi^{\mu\nu}$ relax to their respective NS values $\Pi_{\rm NS}\,,\ n^\mu_{\rm NS}\,,\ \pi^{\mu\nu}_{\rm NS}$ on time scales $\tau_\Pi\,,\ \tau_n\,,\ \tau_\pi$

$\Longrightarrow$ stable and causal fluid dynamical equations of motion!

see, e.g., S. Pu, T. Koide, DHR, PRD 81 (2010) 114039

However: Simplified IS equations do not contain all possible second-order terms!

Power counting (I)

3 length scales: 2 microscopic, 1 macroscopic

- **interparticle distance (thermal wavelength)** $\lambda \sim \beta \equiv 1/T$
- **mean-free path** $\ell \sim (\langle\sigma\rangle n)^{-1}$
- **length scale over which macroscopic fluid fields vary** $L\,,\quad \partial_\mu \sim L^{-1}$

$$n,\, s \sim T^3 = \beta^{-3} \sim \lambda^{-3}\,, \qquad \eta \sim T/\langle\sigma\rangle = (\langle\sigma\rangle\lambda)^{-1} \quad \Longrightarrow \quad \boxed{\frac{\ell}{\lambda} \sim \frac{1}{\langle\sigma\rangle n}\,\frac{1}{\lambda} \sim \frac{1}{\langle\sigma\rangle\lambda}\,\frac{1}{n} \sim \frac{\eta}{s}}$$

$\Longrightarrow \dfrac{\eta}{s}$ **solely determined by 2 microscopic length scales! (similarly:** $\dfrac{\zeta}{s}\,,\ \dfrac{\kappa_n}{\beta\, s}$ **)**

3 regimes:

- **dilute-gas limit** $\dfrac{\ell}{\lambda} \sim \dfrac{\eta}{s} \gg 1 \iff \langle\sigma\rangle \ll \lambda^2 \implies$ **weak-coupling limit**
- **viscous fluids** $\dfrac{\ell}{\lambda} \sim \dfrac{\eta}{s} \sim 1 \iff \langle\sigma\rangle \sim \lambda^2$

  interactions happen on the scale λ $\implies$ moderate coupling
- **ideal-fluid limit** $\dfrac{\ell}{\lambda} \sim \dfrac{\eta}{s} \ll 1 \iff \langle\sigma\rangle \gg \lambda^2 \implies$ **strong-coupling limit**

Power counting (II)

Knudsen number: $\boxed{\mathrm{Kn} \equiv \frac{\ell}{L} \sim \ell\, \partial_\mu}$

$\Longrightarrow$ expansion in Knudsen number equivalent to gradient (derivative) expansion

$\Longrightarrow$ if microscopic particle dynamics (small scale $\sim \ell$) is well separated from macroscopic fluid dynamics (large scale $\sim L$), expansion in powers of $\mathrm{Kn} \ll 1$ expected to converge!

$\Longrightarrow$ Estimate Navier-Stokes terms: use: $\epsilon + p = Ts + \mu n \Longrightarrow \beta\, \epsilon \sim s$!

$\Longrightarrow$ $\frac{\Pi_{\rm NS}}{\epsilon} = -\frac{\zeta}{\beta\, \epsilon}\beta\, \theta \sim -\frac{\zeta}{s}\frac{\lambda}{\ell}\ell\, \theta \sim \ell\, \partial_\mu u^\mu \sim \mathrm{Kn}$ (similarly $\frac{n^\mu_{\rm NS}}{s}, \frac{\pi^{\mu\nu}_{\rm NS}}{\epsilon} \sim \mathrm{Kn}$)

But: in IS theory $\Pi, n^\mu, \pi^{\mu\nu}$ independent dynamical quantities!

$\Longrightarrow$ (inverse) Reynolds number(s): $\boxed{\mathrm{Re}^{-1} \sim \frac{\Pi}{\epsilon}, \frac{n^\mu}{s}, \frac{\pi^{\mu\nu}}{\epsilon}}$

$\Longrightarrow$ asymptotically, terms $\sim \mathrm{Re}^{-1} \sim \mathrm{Kn}$

$\Longrightarrow$ additional relaxation term in IS equation is of second order in Kn :

$$\frac{1}{\epsilon}\tau_\Pi\, \dot{\Pi} \sim \frac{1}{\epsilon} u^\mu\, \ell\, \partial_\mu \Pi \sim \mathrm{Re}^{-1}\mathrm{Kn} \sim \mathrm{Kn}^2$$

$\Longrightarrow$ to be consistent, have to include other second-order terms as well!

Israel-Stewart equations revisited

$$\tau_\Pi \, \dot{\Pi} + \Pi = \Pi_{\rm NS} + \mathcal{K} + \mathcal{J} + \mathcal{R}$$

$$\tau_n \, \dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS} + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu$$

$$\tau_\pi \, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}$$

Kn²:

$$\mathcal{K} = \zeta_1 \, \omega_{\mu\nu} \, \omega^{\mu\nu} + \zeta_2 \, \sigma^{\mu\nu} \, \sigma_{\mu\nu} + \zeta_3 \, \theta^2 + \zeta_4 \, (\nabla\alpha)^2 + \zeta_5 \, (\nabla p)^2 + \zeta_6 \, \nabla\alpha \cdot \nabla p + \zeta_7 \, \nabla^2\alpha + \zeta_8 \, \nabla^2 p \, ,$$

$$\mathcal{K}^\mu = \kappa_1 \, \sigma^{\mu\nu} \, \nabla_\nu \alpha + \kappa_2 \, \sigma^{\mu\nu} \, \nabla_\nu p + \kappa_3 \, \theta \, \nabla^\mu \alpha + \kappa_4 \, \theta \, \nabla^\mu p + \kappa_5 \, \omega^{\mu\nu} \, \nabla_\nu \alpha + \kappa_6 \, \Delta^{\mu\lambda} \partial^\nu \sigma_{\lambda\nu} + \kappa_7 \, \nabla^\mu \theta \, ,$$

$$\mathcal{K}^{\mu\nu} = \eta_1 \, \omega_\lambda{}^{<\mu} \, \omega^{\nu>\lambda} + \eta_2 \, \theta \, \sigma^{\mu\nu} + \eta_3 \, \sigma_\lambda{}^{<\mu} \, \sigma^{\nu>\lambda} + \eta_4 \, \sigma_\lambda{}^{<\mu} \, \omega^{\nu>\lambda} + \eta_5 \, \nabla^{<\mu}\alpha \, \nabla^{\nu>}\alpha$$
$$+ \, \eta_6 \, \nabla^{<\mu} p \, \nabla^{\nu>} p + \eta_7 \, \nabla^{<\mu}\alpha \, \nabla^{\nu>} p + \eta_8 \, \nabla^{<\mu}\nabla^{\nu>}\alpha + \eta_9 \, \nabla^{<\mu}\nabla^{\nu>} p$$

Re⁻¹Kn:

$$\mathcal{J} = -\ell_{\Pi n} \, \nabla \cdot n - \tau_{\Pi n} \, n \cdot \nabla p - \delta_{\Pi\Pi} \, \theta \, \Pi - \lambda_{\Pi n} \, n \cdot \nabla\alpha + \lambda_{\Pi\pi} \, \pi^{\mu\nu} \sigma_{\mu\nu}$$

$$\mathcal{J}^\mu = \omega^{\mu\nu} \, n_\nu - \delta_{nn} \, \theta \, n^\mu - \ell_{n\Pi} \, \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu} \, \nabla^\lambda \pi_{\nu\lambda} + \tau_{n\Pi} \, \Pi \, \nabla^\mu p - \tau_{n\pi} \, \pi^{\mu\nu} \, \nabla_\nu p - \lambda_{nn} \, \sigma^{\mu\nu} \, n_\nu$$
$$+ \, \lambda_{n\Pi} \, \Pi \, \nabla^\mu \alpha - \lambda_{n\pi} \, \pi^{\mu\nu} \, \nabla_\nu \alpha$$

$$\mathcal{J}^{\mu\nu} = 2 \, \pi_\lambda{}^{<\mu} \, \omega^{\nu>\lambda} - \delta_{\pi\pi} \, \theta \, \pi^{\mu\nu} - \tau_{\pi\pi} \, \pi_\lambda{}^{<\mu} \, \sigma^{\nu>\lambda} + \lambda_{\pi\Pi} \, \Pi \, \sigma^{\mu\nu} - \tau_{\pi n} \, n^{<\mu} \, \nabla^{\nu>} p + \ell_{\pi n} \, \nabla^{<\mu} n^{\nu>}$$
$$+ \, \lambda_{\pi n} \, n^{<\mu} \nabla^{\nu>} \alpha$$

where $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ vorticity

Re⁻²:

$$\mathcal{R} = \varphi_1 \, \Pi^2 + \varphi_2 \, n \cdot n + \varphi_3 \, \pi^{\mu\nu} \pi_{\mu\nu}$$

$$\mathcal{R}^\mu = \varphi_4 \, \pi^{\mu\nu} \, n_\nu + \varphi_5 \, \Pi \, n^\mu$$

$$\mathcal{R}^{\mu\nu} = \varphi_6 \, \Pi \, \pi^{\mu\nu} + \varphi_7 \, \pi_\lambda{}^{<\mu} \, \pi^{\nu>\lambda} + \varphi_8 \, n^{<\mu} \, n^{\nu>}$$

$\Longrightarrow$ **transport coefficients can be determined by matching to underlying theory, e.g. kinetic theory** G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD 85 (2012) 114047

Conclusions

1. **Phase transitions in a fundamental theory of nature (QCD) can be studied in the laboratory via heavy-ion collisions**
2. **Success stories nos. 1 – 3: fluid dynamics can quantitatively describe and predict collective flow phenomena in AA-collisions**
3. **Determining transport coefficients by comparison to experimental data requires thorough understanding of initial conditions**
4. **System created in pA- and pp-collisions exhibits collective flow but may be too small to apply (standard) fluid dynamics**

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5. **Second-order relativistic dissipative fluid dynamics has been systematically derived as long-distance, large-time limit of kinetic theory**
6. **Treatment allows for further systematic improvements**

$\Longrightarrow$ **Anisotropic relativistic dissipative fluid dynamics**
E. Molnár, H. Niemi, DHR, arXiv:1602.00573 [nucl-th]

$\Longrightarrow$ **Can a macroscopic approach be applied to a microscopic system?**
Yes, if one carefully monitors its range of applicability!