Symmetries in Quantum Mechanics and Particle Physics
Elina Rischke
Frankfurt Digital Summer School 2021

Exercises

Exercise 1: Symmetries in Quantum Mechanics

In this exercise, some systems well-known in quantum mechanics are analyzed with respect to their symmetries.

(a) Show that a free particle, described by the Hamilton operator

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} , \tag{1}$$

where $\hat{\vec{p}} \equiv -i\hbar\vec{\nabla}$ is the momentum operator, is invariant under space-time translations. What are suitable eigenfunctions $|\psi\rangle$ of $\hat{H}_0$?

(b) Show that the hydrogen atom, described by the Hamilton operator

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \frac{e^2}{r} , \tag{2}$$

is invariant under rotations.
Hint: Use the representation of the Laplacian Δ in terms of spherical coordinates and express the angular part in terms of the angular momentum operator $\hat{\vec{L}} \equiv \vec{r} \times \hat{\vec{p}}$.
How many quantum numbers classify the eigenfunctions of $\hat{H}$? What are suitable eigenfunctions of $\hat{H}$? Is $\hat{H}$ invariant under space translations?

Exercise 2: Matrix Groups

So-called matrix groups play an important role in mathematics and physics. Therefore, some of these groups will be investigated in more detail in this exercise.

(a) The set of regular (means non-vanishing determinant) real-valued $(N \times N)$ matrices is denoted as $GL(N, \mathbb{R})$. Show that this set together with the standard matrix multiplication forms a group.

(b) The set of unimodular (means determinant equal 1) real-valued $(N \times N)$ matrices is denoted as $SL(N, \mathbb{R})$. Show that this set together with the standard matrix multiplication forms a group. Show in addition that it is a subgroup of $GL(N, \mathbb{R})$.

Exercise 3: The Groups $U(N)$ and $SU(N)$

The Lie groups $U(N)$ and $SU(N)$ are very important for physics. Here, $U(N)$ denotes the group of unitary $(N \times N)$ matrices and $SU(N)$ the group of unitary $(N \times N)$ matrices with determinant 1. It is intuitively clear that $SU(N)$ is a Lie subgroup of $U(N)$. The exact relation between both Lie groups is

$$U(N) = SU(N) \times U(1)$$

so that each $g \in U(N)$ can be written as

$$D(g) = D(h) \exp\left(-\frac{i}{\hbar}\alpha_0 \hat{T}_0\right) ,$$

where $\hat{T}_0 \propto \mathbb{1}_{N\times N}$ and $h \in SU(N)$. Here, $D(g)$ and $D(h)$ are $(N \times N)$ matrix representations of elements $g \in U(N)$ and $h \in SU(N)$, respectively. The exponential in the above equation is an $(N \times N)$ matrix representation of an element of $U(1)$.

Since $U(N)$ is a Lie group, its elements can be also written as

$$\hat{U}(\alpha_0, \alpha_1, \ldots, \alpha_{n(N)}) = \exp\left(-\frac{i}{\hbar}\sum_{a=0}^{n(N)} \alpha_a \hat{T}_a\right) ,$$

where $\hat{T}_a$, $a = 1, \ldots, n(N)$ denote the generators of $SU(N)$ and $\hat{T}_0 \propto \mathbb{1}_{N\times N}$ is the additional generator of $U(1)$.

The Lie algebra of $SU(N)$ is given by

$$[\hat{T}_a, \hat{T}_b] = i\hbar \sum_{c=1}^{n(N)} f_{abc}\hat{T}_c , \tag{3}$$

where the structure constants f_{abc} are completely antisymmetric in a basis where $\mathrm{Tr}(\hat{T}_a\hat{T}_b) \propto \delta_{ab}$. The anti-commutator of two $SU(N)$ generators is given by

$$\left\{\hat{T}_a, \hat{T}_b\right\} = \frac{\hbar^2}{N}\delta_{ab}\mathbb{1}_{N\times N} + \hbar \sum_{c=1}^{n(N)} d_{abc}\hat{T}_c . \tag{4}$$

Here, d_{abc} are the completely symmetric structure constants of $SU(N)$.

(i) Determine the number $n(N)$ of generators of $SU(N)$ as function of N.

(ii) Determine $\hat{T}_0$ such that the orthogonality relation $\mathrm{Tr}(\hat{T}_a\hat{T}_b) = \frac{\hbar^2}{2}\delta_{ab}$, $a, b = 0, 1, \ldots, n(N)$ is fulfilled.

(iii) Show that

$$f_{abc} = \frac{2}{i\hbar^3}\mathrm{Tr}\left\{\left[\hat{T}_a, \hat{T}_b\right]\hat{T}_c\right\} , \qquad d_{abc} = \frac{2}{\hbar^3}\mathrm{Tr}\left(\left\{\hat{T}_a, \hat{T}_b\right\}\hat{T}_c\right) .$$

Exercise 4: The Groups $U(N)$ and $SU(N)$ (continued)

(iv) Prove the Jacobi identity

$$\sum_{n=1}^{n(N)} (f_{abn}d_{ncd} + f_{dbn}d_{nca} + f_{cbn}d_{nad}) = 0 .$$

(v) Show that

$$\hat{C}_1 = \sum_{a,b,c=1}^{n(N)} f_{abc}\hat{T}_a\hat{T}_b\hat{T}_c , \qquad \hat{C}_2 = \sum_{a,b,c=1}^{n(N)} d_{abc}\hat{T}_a\hat{T}_b\hat{T}_c$$

are Casimir operators of the Lie group $SU(N)$ and compute $\hat{C}_1$ explicitly for $SU(2)$.

(vi) Generalize the commutation and anti-commutation relations (3) and (4) to the generators of $U(N)$ and determine the additional structure constants f_{abc} and d_{abc}.

Exercise 5: Ladder Operators

Prove the commutation relations (4.22) – (4.37) of the lecture script.

Exercise 6: Fundamental and Antifundamental Representation

In the lecture it was shown that the triplet [3] corresponds to the so-called fundamental representation of the group $SU(3)$. Furthermore, it was shown that there exists a so-called antifundamental representation in the form of the anti-triplet $[\bar{3}]$. The latter is also known as conjugate representation. Physical objects that live in the fundamental representation of $SU(3)$ transform (in natural units, where $\hbar = 1$) with the group element (in matrix representation)

$$\hat{U} = \exp\left(-i\sum_{a=1}^{8}\alpha_a\hat{T}_a\right) \ , \tag{5}$$

with $\hat{T}_a = \hat{\lambda}_a/2$, where $\hat{\lambda}_a$, $a = 1,\ldots,8$, are the Gell-Mann matrices. The corresponding group element $\hat{\bar{U}}$ which transforms objects in the conjugate representation can be obtained from the matrix representation (5) of U by complex conjugation.

(a) Show that the generators in the conjugate representation are given by $\hat{\bar{T}}_a = -\hat{T}_a^*$, $a = 1,\ldots,8$.

(b) Show that $\hat{\bar{T}}_a = -\hat{T}_a^*$, $a = 1,\ldots,8$, fulfill the Lie algebra of the group $SU(3)$.

(c) Two representations of a group are called equivalent, if they are related by a similarity transformation. Show that equivalence of $\hat{U}$ und $\hat{\bar{U}}$ implies that

$$\hat{\bar{T}}_a = S\hat{T}_aS^{-1} \ , \qquad a = 1,\ldots,8 \ . \tag{6}$$

(d) Show that a similarity transformation of the form $\hat{\bar{T}}_a = S\hat{T}_aS^{-1}$, $a = 1,\ldots,8$, exists, iff the eigenvalues of the generators $\hat{T}_a$ appear exclusively in pairs $\{\pm\lambda_a\}$.

Hint: For the case that the generators are matrices of uneven dimension, there exists also null eigenvalues of the generators.

(e) Show that a similarity transformation of the form (6) requires that the completely symmetric structure constants d_{abc} of the group vanish.

(f) Show that a similarity transformation of the form (6) exists for the group $SU(2)$.

Exercise 7: Z_2 as Center of $SU(2)$

(i) Center of a group:

The center $Z(G)$ of a group G is defined as all group elements $z \in G$ which commute with all other group elements,

$$Z(G) = \{z \in G | z \otimes g = g \otimes z \ \forall \ g \in G\} \ .$$

Show that $(Z(G), \otimes)$ is an Abelian subgroup of $(G, \otimes)$.

(ii) Cyclic group Z_2:

The cyclic group of order 2 is denoted as Z_2. It is defined by the following link table

$\circ$	e	a
e	e	a
a	a	e

.

(a) Show that Z_2 is an Abelian group and find a one-dimensional representation of Z_2.

(b) Find a two-dimensional representation of Z_2 and show that $Z(SU(2)) = Z_2$.

Exercise 8: The Groups $O(p,q), SO(p,q)$, and $SO^+(1,3)$

Let $p, q \in \mathbb{N}$ be positive natural numbers with the property $p+q=N$. Define the matrix $\boldsymbol{\eta}_p = (\eta_{ij})$ with

$$\eta_{ij} = \begin{cases} 1, & \text{for} \quad i=j \leq p \ , \\ -1, & \text{for} \quad i=j > p \ , \\ 0, & \text{for} \quad i \neq j \ . \end{cases}$$

(i) Show that the set of $(N \times N)$ matrices M with the property

$$M^T \boldsymbol{\eta}_p M = \boldsymbol{\eta}_p \ , \tag{7}$$

forms a group with respect to the standard matrix multiplication. To this end, first show that $\det M = \pm 1$ and check all group axioms.

This group is called pseudo-orthogonal group $O(p,q)$. If we demand in addition that $\det M = 1$, we obtain the so-called special pseudo-orthogonal group $SO(p,q)$. In the case $p=1, q=3$ the matrix $\boldsymbol{\eta}$ corresponds to the metric tensor of Minkowski space, $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. From the theory of special relativity one knows that Lorentz transformations Λ fulfill relation (7) and thus the Lorentz group corresponds to the group $O(1,3)$. If we demand in addition that $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$, we obtain a subgroup of the Lorentz group, the so-called proper orthochronous Lorentz group $SO^+(1,3)$, which contains Lorentz boosts and rotations in space. The latter can be written (in natural units, where $\hbar = 1$) as

$$R(\vec{\phi}) = \exp\left(-i\vec{\phi} \cdot \hat{\vec{L}}\right) \ , \tag{8}$$

where $\vec{\phi} = (\phi_x, \phi_y, \phi_z)^T$ is a rotation vector and $\hat{\vec{L}}$ is the angular momentum operator. Alternatively, they can be represented by orthogonal (3×3) matrices with determinant 1. These matrices form the group of special orthogonal (3×3) matrices $SO(3)$. From the lecture we know that these groups are Lie groups, since their group structure is that of a differentiable manifold. Let $G(\vec{\alpha})$ be an element of a Lie group with N parameters α_i, $i = 1, \ldots, N$, then the generators of this group can be determined by the relation

$$\hat{T}_i = i \left. \frac{\partial G(\vec{\alpha})}{\partial \alpha^i} \right|_{\vec{\alpha}=\vec{0}} \ .$$

(ii) Determine the generators $\hat{L}_i$ of $SO(3)$ in the representation (8) as well as in their representation as (3×3) matrices.

(iii) Determine the generators $\hat{K}_i$ of Lorentz boosts in their representation as (4×4) matrices.

(iv) State the commutation relations of the $\hat{L}_i$ (without proof) and compute $[\hat{K}_i, \hat{K}_j]$ und $[\hat{L}_i, \hat{K}_j]$ by extending the generators $\hat{L}_i$ to (4×4) matrix representation.

(v) Determine new generators $\hat{J}_i^\pm$ as linear combinations of $\hat{L}_i$ and $\hat{K}_i$, which fulfill the algebra

$$[\hat{J}_i^\pm, \hat{J}_j^\pm] = i \sum_{k=1}^{3} \epsilon_{ijk} \hat{J}_k^\pm \ .$$

Show in addition that

$$[\hat{J}_i^\pm, \hat{J}_j^\mp] = 0 \ .$$

Remark: Apparently, the generators $\left\{\hat{J}_i^\pm\right\}$ fulfill the Lie algebra of $SU(2) \times SU(2)$. But one has to be careful: there are several reasons why the groups $SO^+(1,3)$ and $SU(2) \times SU(2)$ cannot be isomorphic. For instance, $SU(2) \times SU(2)$ is a compact Lie group, while $SO^+(1,3)$ is a non-compact Lie group. Indeed, it can be shown that the proper orthochronous Lorentz group $SO^+(1,3)$ and the group of complex (2×2) matrices with $\det U = 1$, $SL(2,\mathbb{C})$ are in an analogous relationship as the groups $SO(3)$ and $SU(2)$. It holds that $SO^+(1,3) \cong SL(2,\mathbb{C})/Z_2$. The reason for the isomorphism of the Lie algebras from part (v) has its origin in the fact that the generators $\hat{J}_i^\pm$, $i = 1,2,3$, are a complexification of the Lie algebra $\mathfrak{so}^+(1,3)$. Then, $SO^+(1,3)_\mathbb{C} \cong (SU(2) \times SU(2))/Z_2$.