Exercises

Exercise 1: Matrix groups

So-called matrix groups play an important role in mathematics and physics. Therefore, some of these groups will be investigated in more detail in this exercise.

(a) The set of regular (means non-vanishing determinant) real-valued $(N \times N)$ matrices is denoted as $GL(N, \mathbb{R})$. Show that this set together with the standard matrix multiplication forms a group.

(b) The set of unimodular (means determinant equal 1) real-valued $(N \times N)$ matrices is denoted as $SL(N, \mathbb{R})$. Show that this set together with the standard matrix multiplication forms a group. Show in addition that it is a subgroup of $GL(N, \mathbb{R})$.

(c) We define the following $(2N \times 2N)$ matrix

$$J = \begin{pmatrix} 0_{N\times N} & \mathbb{1}_{N\times N} \\ -\mathbb{1}_{N\times N} & 0_{N\times N} \end{pmatrix} ,$$

where $0_{N\times N}$ denotes the N-dimensional null matrix and $\mathbb{1}_{N\times N}$ the N-dimensional unit matrix. The $2N \times 2N$ matrices M with the property

$$M^T J M = J ,$$

form the set of all real-valued symplectic $(2N \times 2N)$ matrices, denoted as $Sp(2N, \mathbb{R})$. Show that the above relation implies $\det(M) = \pm 1$. Then show that $Sp(2N, \mathbb{R})$ together with the standard matrix multiplication forms a group. Is $Sp(2N, \mathbb{R})$ a subgroup of $GL(2N, \mathbb{R})$?

Exercise 2: The groups $U(N)$ and $SU(N)$

The Lie groups $U(N)$ and $SU(N)$ are very important for physics. Here, $U(N)$ denotes the group of unitary $(N \times N)$ matrices and $SU(N)$ the group of unitary $(N \times N)$ matrices with determinant 1. It is intuitively clear that $SU(N)$ is a Lie subgroup of $U(N)$. The exact relation between both Lie groups is

$$U(N) = SU(N) \times U(1)$$

so that each $g \in U(N)$ can be written as

$$D(g) = D(h) \exp\left(-\frac{i}{\hbar}\alpha_0 \hat{T}_0\right) ,$$

where $\hat{T}_0 \propto \mathbb{1}_{N\times N}$ and $h \in SU(N)$. Here, $D(g)$ and $D(h)$ are $(N \times N)$ matrix representations of elements $g \in U(N)$ and $h \in SU(N)$, respectively. The exponential in the above equation is an $(N \times N)$ matrix representation of an element of $U(1)$.

Since $U(N)$ is a Lie group, its elements can be also written as

$$\hat{U}(\alpha_0, \alpha_1, \ldots, \alpha_{n(N)}) = \exp\left(-\frac{i}{\hbar}\sum_{a=0}^{n(N)} \alpha_a \hat{T}_a\right) ,$$

where $\hat{T}_a$, $a = 1, \ldots, n(N)$ denote the generators of $SU(N)$ and $\hat{T}_0 \propto \mathbb{1}_{N\times N}$ is the additional generator of $U(1)$.

The Lie algebra of $SU(N)$ is given by

$$[\hat{T}_a, \hat{T}_b] = i\hbar \sum_{c=1}^{n(N)} f_{abc}\hat{T}_c \ ,$$

where the structure constants f_{abc} are completely antisymmetric in a basis where $\mathrm{Tr}(\hat{T}_a\hat{T}_b) \propto \delta_{ab}$. The anti-commutator of two $SU(N)$ generators is given by

$$\left\{\hat{T}_a, \hat{T}_b\right\} = \frac{\hbar^2}{N}\delta_{ab}\mathbb{1}_{N\times N} + \hbar \sum_{c=1}^{n(N)} d_{abc}\hat{T}_c \ .$$

Here, d_{abc} are the completely symmetric structure constants of $SU(N)$.

(i) Determine the number $n(N)$ of generators of $SU(N)$ as function of N.

(ii) Determine $\hat{T}_0$ such that the orthogonality relation $\mathrm{Tr}(\hat{T}_a\hat{T}_b) = \frac{\hbar^2}{2}\delta_{ab}$, $a, b = 0, 1, \ldots, n(N)$ is fulfilled.

(iii) Show that

$$f_{abc} = \frac{2}{i\hbar^3}\mathrm{Tr}\left\{\left[\hat{T}_a, \hat{T}_b\right]\hat{T}_c\right\} \ , \qquad d_{abc} = \frac{2}{\hbar^3}\mathrm{Tr}\left(\left\{\hat{T}_a, \hat{T}_b\right\}\hat{T}_c\right) \ .$$

(iv) Prove the Jacobi identity

$$\sum_{n=1}^{n(N)} \left(f_{abn}d_{ncd} + f_{dbn}d_{nca} + f_{cbn}d_{nad}\right) = 0 \ .$$

(v) Show that

$$\hat{C}_1 = \sum_{a,b,c=1}^{n(N)} f_{abc}\hat{T}_a\hat{T}_b\hat{T}_c \ , \qquad \hat{C}_2 = \sum_{a,b,c=1}^{n(N)} d_{abc}\hat{T}_a\hat{T}_b\hat{T}_c$$

are Casimir operators of the Lie group $SU(N)$ and compute $\hat{C}_1$ explicitly for $SU(2)$.

(vi) Generalize the above commutation and anti-commutation relations to the generators of $U(N)$ and determine the additional structure constants f_{abc} and d_{abc}.