Path Integrals in Quantum Mechanics
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Frankfurt Digital Summer School 2021

Exercises

Exercise 1: Euler–Lagrange Equations, Canonical Equations

In this exercise, some well-known facts from Classical Mechanics are reviewed.

(a) Using Hamilton's principle and variational calculus, derive the Euler–Lagrange equations.

(b) Using the modified Hamilton's principle and variational calculus, derive the canonical equations of motion.

Exercise 2: Schrödinger Equation in an Electromagnetic Field

In Eq. (3.50) of the script, the arithmetic mean $\frac{\mathfrak{q}}{2}\left[\vec{A}(\vec{x},\bar{t})+\vec{A}(\vec{y},\bar{t})\right]$ was used to derive the correct Schrödinger equation for a particle in an electromagnetic field. It was also discussed what happens if $\mathfrak{q}\vec{A}(\vec{x},\bar{t})$ is used instead. Repeat this analysis using $\mathfrak{q}\vec{A}(\vec{y},\bar{t})$ instead of the arithmetic mean. What happens to the Schrödinger equation?

Exercise 3: Diffraction on a Single Slit

Derive the diffraction pattern for particles going through a single slit.

Hint: It is helpful to first look up the analogous calculation for diffraction of light through a single slit. Repeat this calculation and then point out where the corresponding calculation for diffraction of particles differs.

Exercise 4: Classical Actions and Propagators

This exercise is meant to elucidate the connection between various propagators: for free particles, particles in the semiclassical approximation, and the harmonic oscillator.

(i) Derive the classical action of a free (i.e., non-interacting particle) on a trajectory from $\vec{q}_a=\vec{q}(t_a)$ to $\vec{q}_b=\vec{q}(t_b)$ and express it in terms of $t_a, t_b, \vec{q}_a, \vec{q}_b$.

(ii) Use the result of (i) to rewrite the propagator of a free particle.

(iii) Show how the propagator in the semiclassical approximation can be related to the free propagator. What is the function $f(t_b,t_a)$ in this case?

(iv) Show that there is a limit where the propagator for the harmonic oscillator coincides with the propagator for a free particle.

Exercise 5: Scattering Matrix in the Non-interacting Case

Equation (5.18) of the script claims that

$$\int \mathrm{d}^3\vec{q}_a\, G_0(\vec{q}_b,t_b;\vec{q}_a,t_a)\,\psi_{\mathrm{in}}(\vec{q}_a,t_a) \equiv \psi_{\mathrm{in}}(\vec{q}_b,t_b) = \frac{1}{\sqrt{2\pi\hbar}^3}\,\exp\left[-\frac{i}{\hbar}\left(E_a t_b - \vec{p}_a\cdot\vec{q}_b\right)\right]\,, \tag{1}$$

i.e., a plane wave which propagates without interaction from $(\vec{q}_a,t_a)$ to $(\vec{q}_b,t_b)$ is still a plane wave. Prove this relation using the explicit expression

$$G_0(\vec{q}_b,t_b;\vec{q}_a,t_a) = \sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}}^{\,3}\,\exp\left[\frac{i\,m}{2\hbar(t_b-t_a)}\,(\vec{q}_b-\vec{q}_a)^2\right]\,\Theta(t_b-t_a)\,. \tag{2}$$

Exercise 6: Feynman Rules in Momentum Space

(i) Use the integral representation of the Heaviside function,

$$\Theta(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-i\omega t}}{\omega + i\epsilon} \; , \tag{3}$$

to prove

$$\tilde{G}_0(\vec{p}_2, E_2; \vec{p}_1, E_1) = (2\pi\hbar)^4 \delta^{(4)}(P_2 - P_1) \, \frac{i\hbar}{E_2 - \frac{\vec{p}_2^{\,2}}{2m} + i\epsilon} \; . \tag{4}$$

(ii) Fourier-transform the first-order term of the Born series,

$$G_1(\vec{q}_b, t_b; \vec{q}_a, t_a) = -\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3\vec{x} \, G_0(\vec{q}_b, t_b; \vec{q}, t) \, V(\vec{q}, t) \, G_0(\vec{q}, t; \vec{q}_a, t_a) \; , \tag{5}$$

to derive the Feynman rules in momentum space.

Exercise 7: Heisenberg Equation of Motion

The equation of motion for the position operator $\hat{q}(t)$ (in the Heisenberg picture) reads

$$\dot{\hat{q}} = \frac{i}{\hbar}\left[\hat{H}, \hat{q}\right] \; . \tag{6}$$

Derive the equivalent relation in the path-integral formalism, using the discretized version

$$\left\langle \frac{q_{\ell+1} - q_\ell}{\tau} \right\rangle_S \tag{7}$$

for the average velocity.
Hint: The definition of the average $\langle \cdot \rangle_S$ implies that

$$\langle q_\ell \rangle_S = \langle q(t_\ell) \rangle_S \sim \langle q_b, t_b | \hat{q}(t_\ell) | q_a, t_a \rangle \; , \tag{8}$$

i.e., the average of $q(t_\ell)$ is proportional to the one-point correlation function of the position operator (in the Heisenberg picture). Use the evolution equation for Heisenberg operators,

$$\hat{q}(t) = \hat{U}(t_0, t) \hat{q} \, \hat{U}(t, t_0) \; , \tag{9}$$

where $\hat{q}$ is the position operator in the Schrödinger picture, which fulfills the eigenvalue equation $\hat{q}|q_k\rangle = q_k|q_k\rangle$. Then use the discretized path-integral expression for the one-point correlation function to derive the desired relation.

Exercise 8: Shifted and Generalized Gauss Integral in N Dimensions

Following similar arguments which lead to Eq. (4.43) in the script, prove that for a symmetric, positive definite, non-singular $(N \times N)$ matrix A and for N dimensional vectors $\vec{x}$, $\vec{b}$ one has the identity

$$\int \mathrm{d}^N\vec{r} \, \exp\left(-\frac{1}{2}\,\vec{r}^{\,T} \mathrm{A} \, \vec{r} + \vec{b}^{\,T}\vec{r}\right) = (2\pi)^{N/2} (\det\mathrm{A})^{-1/2} \, \exp\left(\frac{1}{2}\,\vec{b}^{\,T} \mathrm{A}^{-1} \vec{b}\right) \; . \tag{10}$$