17. Newtonsche Sterne

Newtonscher Grenzfall der TOV-Gleichung (15.22) ist Gl. (14.11):

$$P' \equiv \frac{dP}{dr} = -\frac{G\mathcal{M}(r)}{r^2}\,\rho(r) \qquad (17.1)$$

$$\Leftrightarrow \quad \frac{d}{dr}\left[\frac{r^2}{\rho(r)}\,P'\right] = -G\mathcal{M}' \overset{(14.9)}{=} -4\pi G r^2 \rho(r) \qquad (17.2)$$

Polytrope Zustandsgleichung (15.26): $\quad P(r) = K\rho^{\gamma}$

$$\Rightarrow \quad \gamma K \frac{d}{dr}\left(r^2 \rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G r^2 \rho \qquad (17.3)$$

Suche Lösung, die bei $r = 0$ endlich ist, $\quad \rho(0) \equiv \rho_0 < \infty$

$\Rightarrow$ Gl. (17.3) für $r \to 0$: $\quad \frac{d}{dr}\left(r^2\rho'\right) \sim r^2 \quad \Rightarrow \quad r^2\rho' \sim r^3$

$$\Rightarrow \quad \rho' \sim r \quad \Rightarrow \quad \rho'(0) = 0 \qquad (17.4)$$

Definiere
$$x = \left[\frac{4\pi G(\gamma-1)}{K\gamma}\right]^{1/2} \rho_0^{1-\frac{\gamma}{2}}\, r \tag{17.5}$$

$$\Rightarrow \quad K\gamma \frac{d}{dx}\left(x^2 \rho^{\gamma-2}\frac{d\rho}{dx}\right) = -\frac{4\pi G K\gamma}{4\pi G(\gamma-1)}\rho_0^{\gamma-2}x^2\rho$$

$$\Leftrightarrow \quad (\gamma-1)\frac{d}{dx}\left(x^2\rho^{\gamma-2}\frac{d\rho}{dx}\right) = -\rho_0^{\gamma-2}x^2\rho \tag{17.6}$$

Definiere Lane-Emden-Funktion $\quad \Theta(x) \equiv \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ (17.7)

$$\Rightarrow \quad \boxed{\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\Theta}{dx}\right) + \Theta^n = 0} \quad , \; n \equiv \frac{1}{\gamma-1} \tag{17.8}$$

Randbedingungen: $\quad \Theta(0) \overset{(17.7)}{=} 1 \quad , \quad \frac{d\Theta}{dx} \overset{(17.4)}{=} 0$ (17.9)

Analytische Lösungen der Dgl. (17.8) existieren für

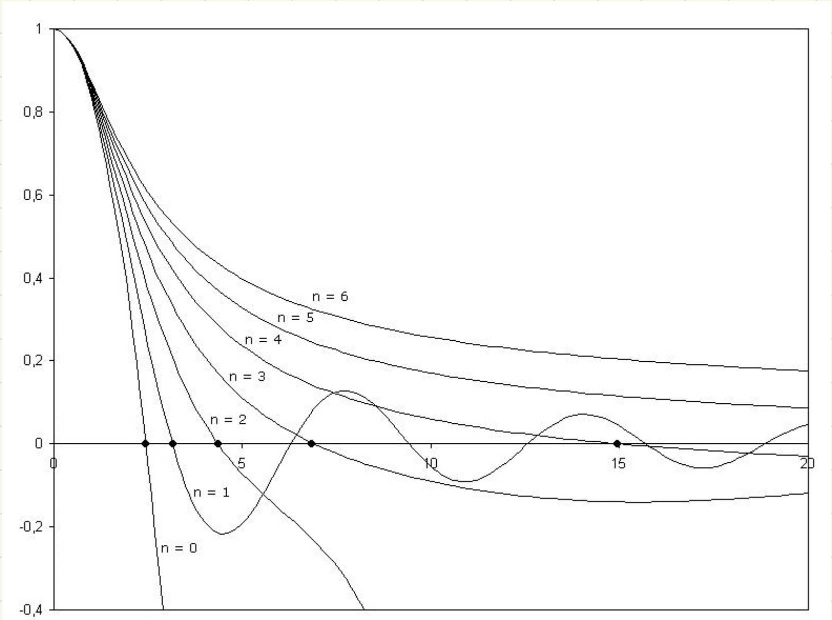
(i) $n=0 \Leftrightarrow \gamma=\infty : \quad \Theta(x) = 1 - \frac{x^2}{6}$ (17.10)

(ii) $n=1 \Leftrightarrow \gamma=2 : \quad \Theta(x) = \frac{\sin x}{x}$ (17.11)

(iii) $n=5 \Leftrightarrow \gamma=\frac{6}{5} : \quad \Theta(x) = \left(1+\frac{x^2}{3}\right)^{-1/2}$ (17.12)

Für $x << 1$: $$\Theta(x) = 1 - \frac{x^2}{6} + \frac{n}{120} x^4 - \frac{8n^2-5n}{15120} x^6 + \dots \qquad (17.13)$$

$\Theta(x)$



x

Für $n < 5$ hat $\Theta(x)$ eine oder mehrere Nullstellen

An der ersten Nullstelle x_1 wird $\rho = 0$

$\Rightarrow$ definiert Sternradius R

$$\boxed{R = \left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{1/2} \rho_0^{\frac{\gamma}{2}-1} x_1} \qquad (17.14)$$

Quelle: https://de.wikipedia.org/wiki/Lane-Emden-Gleichung

benötige x_1 für

(i) $\gamma = \frac{5}{3} \Leftrightarrow n = 1{,}5 \Rightarrow x_1 = 3.654$ (17.15)

(ii) $\gamma = \frac{4}{3} \Leftrightarrow n = 3 \Rightarrow x_1 = 6{,}897$

Masse des Sterns:

$$\boxed{M} \overset{(14.9)}{=} 4\pi \int_0^R dr\, r^2 \varrho(r) \underset{(17.7)}{\overset{(17.5)}{=}} 4\pi \left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{3/2} \varrho_0^{\frac{3\gamma}{2}-2} \int_0^{x_1} dx\, x^2 \Theta^n$$

$$\overset{(17.8)}{=} -4\pi \left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{3/2} \varrho_0^{\frac{3\gamma}{2}-2} \underbrace{\int_0^{x_1} dx \frac{d}{dx}\left(x^2 \frac{d\Theta}{dx}\right)}_{\underset{(17.9)}{=} x_1^2 \underbrace{\frac{d\Theta}{dx}}_{<0}(x_1)}$$

$$\boxed{= 4\pi \left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{3/2} \varrho_0^{\frac{3\gamma}{2}-2} x_1^2 \left|\frac{d\Theta}{dx}(x_1)\right|} \qquad (17.16)$$

$\boxed{26.6.20}$

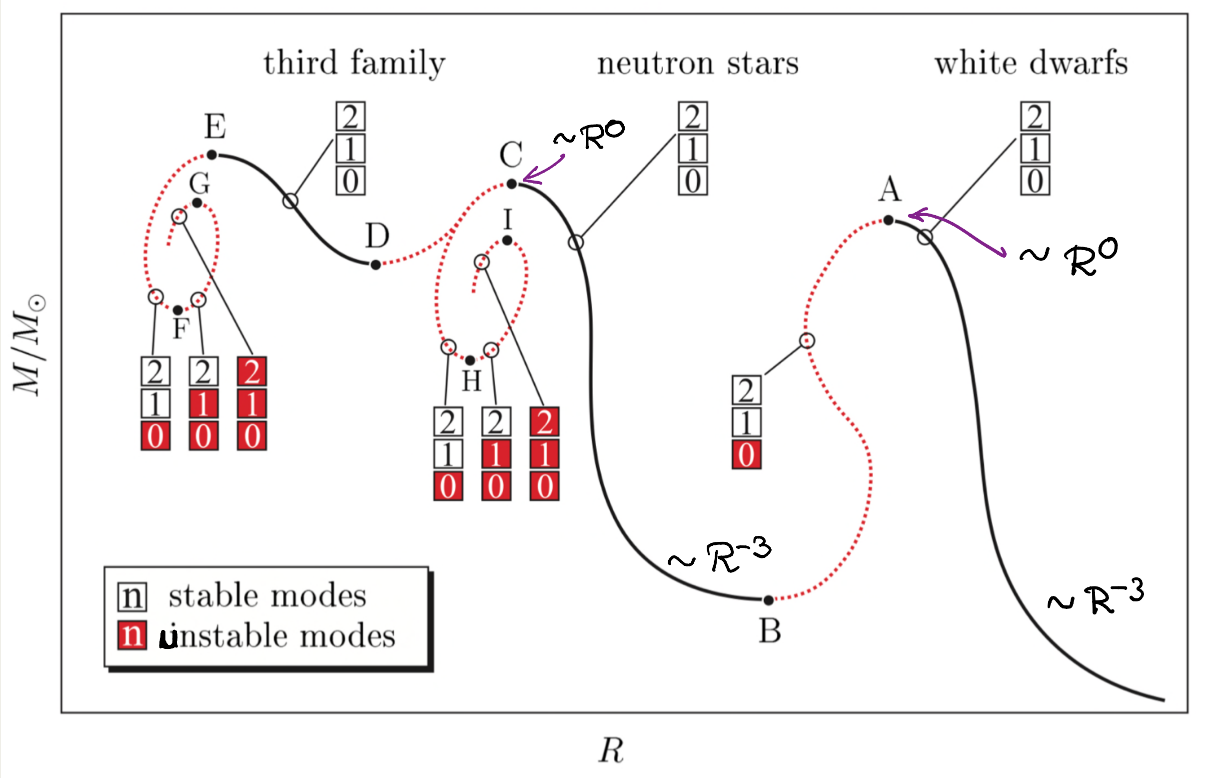
$\Rightarrow$ Glgen. (17.15), (17.16): $R = C_R(\gamma)\, \varrho_0^{\frac{\gamma}{2}-1} \Leftrightarrow \varrho_0 = \left[\frac{R}{C_R(\gamma)}\right]^{\frac{2}{\gamma-2}}$ (17.17)

$$M = C_M(\gamma)\, \varrho_0^{\frac{3\gamma}{2}-2} \qquad (17.18)$$

$$\Rightarrow \boxed{M(R)} = C_M(\gamma) \left[\frac{R}{C_R(\gamma)}\right]^{\frac{3\gamma-4}{\gamma-2}} = \boxed{\bar{C}_M(\gamma)\, R^{\frac{2}{\gamma-2}+3}} \qquad (17.19)$$

$$\Rightarrow \quad \gamma = \frac{5}{3}: \quad M(R) \sim R^{-3}$$

$$\gamma = \frac{4}{3}: \quad M(R) \sim R^{0} \qquad (17.20)$$



K. Schertler, C. Greiner, J. Schaffner-Bielich, M.H. Thoma, Nucl. Phys. A 677 (2000) 463