15. Innere Schwarzschild-Metrik & Tolman-Oppenheimer-Volkoff (TOV)-Gleichung

Lösung der Einsteinschen Feldgleichungen (0.2),

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right), \tag{15.1}$$

mit dem Ansatz (7.2) für die Metrik (statischer, sphärisch symmetrischer Fall),

$$(g_{\mu\nu}) = \operatorname{diag}\left(B(r), -A(r), -r^2, -r^2 \sin^2\Theta\right), \tag{15.2}$$

für Materieansammlung mit

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu - P g_{\mu\nu} \tag{15.3}$$

Statisch, sphärisch symmetrisch

$$\Rightarrow \quad \rho(t,\vec{r}) = \rho(r), \quad P(t,\vec{r}) = P(r), \quad u^i(t,\vec{r}) = 0 \tag{15.4}$$

$$\Rightarrow \quad c^2 = g_{\mu\nu} u^\mu u^\nu = g_{00} (u^0)^2 \overset{(15.2)}{=} B (u^0)^2$$

$$\Rightarrow \quad u^0 = \frac{c}{\sqrt{B}}, \quad u_0 = g_{0\mu} u^\mu = g_{00} u^0 = c\sqrt{B} \tag{15.5}$$

$$\Rightarrow \quad \boxed{(T_{\mu\nu})} = \mathrm{diag}\left(\left(\varrho + \frac{P}{c^2}\right) c^2 B - PB,\ PA,\ Pr^2,\ Pr^2 \sin^2\theta \right)$$

$$= \boxed{\mathrm{diag}\left(\varrho c^2 B,\ PA,\ Pr^2,\ Pr^2 \sin^2\theta \right)} \qquad (15.6)$$

Gl. (15.2) $\Rightarrow$ $$(g^{\mu\nu}) = \mathrm{diag}\left(\frac{1}{B},\ -\frac{1}{A},\ -\frac{1}{r^2},\ -\frac{1}{r^2 \sin^2\theta} \right) \qquad (15.7)$$

$$\Rightarrow \quad \boxed{T} = T^{\mu}{}_{\mu} = g^{\mu\nu} T_{\mu\nu} = \boxed{\varrho c^2 - 3P} \qquad (15.8)$$

$\Rightarrow$ Gl. (15.1) mit Glgen. (6.12) - (6.16), (15.6), (15.8):

$$\boxed{\begin{aligned} R_{00} &= -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} = -\frac{4\pi G}{c^4}\left(\varrho c^2 + 3P\right) B && (15.9) \\ R_{11} &= \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} = -\frac{4\pi G}{c^4}\left(\varrho c^2 - P\right) A && (15.10) \\ R_{22} &= -1 - \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{A} = -\frac{4\pi G}{c^4}\left(\varrho c^2 - P\right) r^2 && (15.11) \end{aligned}}$$

$$R_{33} = R_{22} \sin^2\theta = \left[-\frac{4\pi G}{c^4}\left(\varrho c^2 - P\right) r^2 \right] \sin^2\theta \qquad (15.12)$$

$\hookrightarrow$ durch Gl. (15.11) erfüllt

$$R_{\mu\nu} = 0 \quad , \mu \neq \nu \qquad (15.13)$$

$\hookrightarrow$ trivial erfüllt

$\boxed{17.6.20}$

$$\Rightarrow \quad \frac{R_{00}}{2B} + \frac{R_{11}}{2A} + \frac{R_{22}}{r^2}$$

$$= -\frac{B''}{4AB} + \frac{B'}{8AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{2rAB} + \frac{B''}{4AB} - \frac{B'}{8AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{2rA^2}$$

$$- \frac{1}{r^2} - \frac{1}{2Ar}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{r^2 A}$$

$$= -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{r^2 A} \underset{\uparrow}{=} -\frac{4\pi G}{c^4}\left(\frac{\rho c^2 + 3P}{2} + \frac{\rho c^2 - P}{2} + \rho c^2 - P\right)$$

(15.9) – (15.11)

$$= -\frac{8\pi G}{c^2}\rho \qquad (15.14)$$

$$\Rightarrow \quad -\frac{rA'}{A^2} + \frac{1}{A} = \frac{d}{dr}\frac{r}{A} = 1 - \frac{8\pi G}{c^2}\rho r^2 \qquad (15.15)$$

Für $A(0) < \infty \Rightarrow \left.\frac{r}{A}\right|_{r=0} = 0$

(14.9)

$$\Rightarrow \quad \frac{r}{A(r)} = \int_0^r dr' \left[1 - \frac{8\pi G}{c^2}\rho(r')r'^2\right] = r - \frac{2G\mathcal{M}(r)}{c^2} \qquad (15.16)$$

$$\Rightarrow \quad \boxed{A(r) = \left[1 - \frac{2G\mathcal{M}(r)}{c^2 r}\right]^{-1}} \qquad (15.17)$$

Energie-Impulserhaltung:

$$0 = T^{\mu\nu}{}_{||\nu} = \underbrace{T^{\mu\nu}{}_{|\nu}}_{=\frac{\partial T^{\mu\lambda}}{\partial x^\lambda}} + \Gamma^\mu_{\nu\lambda} T^{\lambda\nu} + \underbrace{\Gamma^\nu_{\nu\lambda}}_{=\frac{\partial \ln\sqrt{|g|}}{\partial x^\lambda}} T^{\mu\lambda} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{|g|}\, T^{\mu\lambda}\right) + \Gamma^\mu_{\nu\lambda} T^{\lambda\nu} \qquad (15.18)$$

$\mu = 1$:

$$0 \overset{(15.3)}{=} \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\lambda}\left\{\sqrt{|g|}\left[\left(\rho+\frac{P}{c^2}\right)\underbrace{u^1}_{\underset{(15.4)}{=}0} u^\lambda - P g^{1\lambda}\right]\right\} + \Gamma^1_{\nu\lambda}\left[\left(\rho+\frac{P}{c^2}\right)\underbrace{u^\nu}_{\underset{(15.4)}{=} u^0\delta^{0\nu} \underset{(15.5)}{=} \frac{c}{\sqrt{B}}\delta^{0\nu}} u^\lambda - P g^{\nu\lambda}\right]$$

$$= \frac{\partial}{\partial x^\lambda}\left(-P g^{1\lambda}\right) + \Gamma^\nu_{\nu\lambda}\left(-P g^{1\lambda}\right) + \Gamma^1_{\nu\lambda}\left(-P g^{\nu\lambda}\right) + \Gamma^1_{00}\left(\rho+\frac{P}{c^2}\right)\frac{c^2}{B}$$

$$\overset{(6.9)}{=} \left(-P g^{1\lambda}\right)_{||\lambda} + \frac{B'}{2AB}\left(\rho c^2 + P\right)$$

$$= \underbrace{-g^{1\lambda} P_{|\lambda}}_{\underset{(15.7)}{=}\frac{P'}{A}} - P \underbrace{g^{1\lambda}{}_{||\lambda}}_{=0} + \frac{B'}{2AB}\left(\rho c^2+P\right) = \frac{1}{A}\left[P' + \frac{B'}{2B}\left(\rho c^2+P\right)\right] \qquad (15.19)$$

$$\Leftrightarrow \quad \boxed{\frac{B'}{B} = -\frac{2P'}{\rho c^2+P}} \qquad (15.20)$$

nichtrelativistisch: $\quad B \simeq 1 + \frac{2\Phi}{c^2} \quad , \quad \frac{B'}{B} \simeq 2\frac{\Phi'}{c^2} \quad , \quad P \ll \rho c^2$

$$\overset{(15.20)}{\Longrightarrow} \quad \Phi' \simeq -\frac{P'}{\rho} \quad \Longleftrightarrow \quad P' = -\rho\,\Phi' \qquad \text{Gl. (14.6)!}$$

Gl. (15.14): $\quad \boxed{-\frac{A'}{A^2}} = \frac{1}{r}\underbrace{\left(1 - \frac{1}{A}\right)} - \frac{8\pi G}{c^2}\, r\rho$

$$\overset{(15.17)}{=} \frac{1}{r}\,\frac{2GM}{c^2 r} - \frac{8\pi G}{c^2}\, r\rho = \boxed{\frac{2G}{c^2 r^2}\left(M - 4\pi r^3 \rho\right)} \qquad (15.21)$$

Gl. (15.11):

$$-\frac{4\pi G}{c^4}\left(\rho c^2 - P\right) r^2 = -1 + \underbrace{\frac{r}{2}\left(-\frac{A'}{A^2}\right)} + \frac{r}{2A}\,\frac{B'}{B} + \frac{1}{A}$$

$$\underset{(15.21)}{\overset{(15.17)\,(15.20)}{=}} -1 + \frac{G}{c^2 r}\left(M - 4\pi r^3 \rho\right) + \left(1 - \frac{2GM}{c^2 r}\right)\left(1 - \frac{r P'}{\rho c^2 + P}\right)$$

$$= -1 + \frac{GM}{c^2 r} - \frac{4\pi G}{c^2}\rho r^2 + 1 - \frac{2GM}{c^2 r} - \left(1 - \frac{2GM}{c^2 r}\right)\frac{r P'}{\rho c^2 + P}$$

$$\Longleftrightarrow \quad \frac{4\pi G}{c^2} P r^2 = -\frac{GM}{r} - \left(1 - \frac{2GM}{c^2 r}\right)\left(\rho + \frac{P}{c^2}\right)^{-1} r P'$$

$$\Longleftrightarrow \quad r P' \left(1 - \frac{2GM}{c^2 r}\right) = -\frac{GM}{c^2 r}\left(1 + \frac{4\pi r^3 P}{M c^2}\right)\left(\rho c^2 + P\right)$$

$$\Longleftrightarrow \quad \boxed{\frac{dP}{d\ln r} = -\left(1-\frac{2G\mathcal{M}}{c^2 r}\right)^{-1}\frac{G\mathcal{M}}{c^2 r}\left(1+\frac{4\pi r^3 P}{\mathcal{M}c^2}\right)\left(\rho c^2+P\right)} \qquad (15.22)$$

Tolman-Oppenheimer-Volkoff (TOV) - Gleichung

nichtrelativistisch: $\quad \frac{G\mathcal{M}}{c^2 r} \ll 1\,, \quad P \ll \frac{\mathcal{M}c^2}{4\pi r^3}\,, \quad P \ll \rho c^2$

$$\Longrightarrow \quad \frac{dP}{dr} = -\frac{G\mathcal{M}}{r^2}\rho$$

Gl. (14.11)!

Bestimme *B(r)*: Gl. (15.20) mit Gl. (15.22)

$$\frac{B'}{B} = \frac{d}{dr}\ln B = \frac{2G\mathcal{M}}{c^2 r^2}\left(1+\frac{4\pi r^3 P}{\mathcal{M}c^2}\right)\left(1-\frac{2G\mathcal{M}}{c^2 r}\right)^{-1} \qquad (15.23)$$

$$\Longrightarrow \quad \frac{B(\infty)}{B(r)} = \exp\left\{\frac{2G}{c^2}\int_r^\infty \frac{dr'}{r'^2}\,\frac{\mathcal{M}(r')+4\pi r'^3 P(r')/c^2}{1-2G\mathcal{M}(r')/(c^2 r')}\right\}$$

$$\overset{B(\infty)=1}{\Longleftrightarrow} \quad \boxed{B(r) = \exp\left\{-\frac{2G}{c^2}\int_r^\infty \frac{dr'}{r'^2}\,\frac{\mathcal{M}(r')+4\pi r'^3 P(r')/c^2}{1-2G\mathcal{M}(r')/(c^2 r')}\right\}} \qquad (15.24)$$

zusammen mit Gl. (15.17) bestimmt dies die innere Schwarzschild-Metrik (15.2).

Für $r > R$: $\quad \rho = P = 0\,, \quad \mathcal{M}(r) = \mathcal{M}(R) \equiv M$

$$\Rightarrow \quad B(r) = \exp\left\{ -\frac{2G}{c^2} \int_r^\infty \frac{dr'}{r'^2}\, M \Big(\underbrace{1 - \frac{2GM}{c^2 r'}}_{\equiv x} \Big)^{-1} \right\}$$

$$\equiv x\,, \quad dx = +\frac{2GM}{c^2 r'^2}\, dr'$$

$$= \exp\left\{ -\int_{1-\frac{2GM}{c^2 r}}^{1} \frac{dx}{x} \right\} = \exp\left\{ \int_1^{1-\frac{2GM}{c^2 r}} \frac{dx}{x} \right\}$$

$$= \exp \ln\left(1 - \frac{2GM}{c^2 r}\right) \equiv 1 - \frac{2GM}{c^2 r}$$

Gl. (15.17): $\quad A(r) = \left(1 - \dfrac{2GM}{c^2 r}\right)^{-1} \equiv \dfrac{1}{B(r)} \quad \Rightarrow \quad$ Schwarzschild-Lösung, Gl. (7.12)!

Beachte: bei Schwarzschild-Lösung ergab sich *M* durch Vergleich mit Newton

hier dagegen: Gl. (14.9)

$$M = \mathcal{M}(R) = 4\pi \int_0^R dr\, r^2 \rho(r)$$

Sterngleichgewicht:

Zustandsgleichung der Materie: $P(r) = P(\varrho(r)) \Rightarrow P'(r) = \frac{dP}{d\varrho}\varrho'(r)$ (15.25)

Beispiel: polytrope Zustandsgleichung $P = K\varrho^{\gamma}$ (15.26)

$$\Rightarrow \frac{dP}{d\varrho} = K\gamma\varrho^{\gamma-1}$$

nichtrelativistisch: $\gamma = \frac{5}{3}$, Gl. (14.17)

ultrarelativistisch: $\gamma = \frac{4}{3}$, Gl. (14.18)

$\Rightarrow$ Gl. (15.22) ist Differentialgleichung: $\varrho'(r) = \frac{d\varrho}{dP} F(\mathcal{M}, \varrho)$ (15.27)

Gl. (14.9) kann als Dgl. geschrieben werden: $\mathcal{M}'(r) = 4\pi r^2 \varrho(r) = G(\varrho)$ (15.28)

$\Rightarrow$ Glgen. (15.27), (15.28) sind zwei gewöhnliche Differentialgleichungen für Felder $\varrho, \mathcal{M}$

Falls Temperatur eine Rolle spielt: $P(r) = P(\varrho(r), T(r))$

$\Rightarrow$ benötige zusätzliche Zustandsgleichung, z.B. kalorische Zustandsgleichung

$$E = E(\varrho(r), T(r))$$