Hyperscaling relation for the conformal window

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15.2.12, strong-BSM workshop  Bad Honnef
Overview

• Intro: Motivation & introduction conformal window studies [4 slides]

• Part I: mass-deformed conformal gauge theories (observables) [8 slides]
  - hyperscaling laws of hadronic observables e.g. $f[0^{++}] \sim m^{\eta(Y_u)}$

• Part II: the quark condensate -- various approaches [4 slides]

  lattice material (Del Debbio’s talks)
  walking technicolour (Shrock, Sannino, others)

  Del Debbio & RZ
  PRD’10 & PLB’11
types of gauge theories

- Adjustable: gauge group $SU(N_c)$ -- $N_f$ (massless) fermions -- fermion irrep

- Focus on asymptotically free theories (not many representations)
  2) well-defined on lattice  2) chance for unification in TC

TC-models:

- **QCD-type**
  - Susskind-Weinberg '79

- **walking-type**
  - Holdom '84

- **IR-conformal**
  - Dietrich-Sannino'04
  - Luty-Okui'04

- **confinement & chiral symmetry breaking** $SU_L(n_F) \times SU_R(n_F) \rightarrow SU_V(n_F)$
  - the latter is dynamical electroweak symmetry breaking $M_W = g f_{\pi}^{(TC)}$
Conformal window (the picture)

N=1 SUSY

“SU(N)”

non-SUSY

just below pert. BZ/BM fixed pt:

lower line BZ/BM fixed pt “electromagnetic dual”

assume in between conformal use \( \beta_{NSVZ}(\gamma^*) = 0 \) to get \( \gamma^* \)

\( \gamma^* |_{\text{strong}} = 1 \) (unitarity bound QQ state)

\[ \beta_0 \text{ tuned small } \frac{\alpha_s^*}{2\pi} = \frac{\beta_0}{-\beta_1} \ll 1 \]

lower line Dyson-Schwinger eqs predict chiral symmetry breaking (lattice results later ...)

\( \gamma^* |_{\text{strong}} \approx 1 \) DS eqs ladder
anomalous dimension

canonical dimension \(d\): (classical)

e.g. fermion field \(q\):
\[
d_q = \frac{D-1}{2}
\]

composite:
\[
d_{\bar{q}q} = D - 1
\]

anomalous dimension \(\gamma\): (quantum corrections)

physical

scheme-dependent

\[
\gamma = \gamma_* + \gamma_0 (g - g_*) + \mathcal{O}((g - g_*)^2)...
\]

- significance: change of renormalization scale \(\mu \rightarrow \mu'\):

scaling

leading correction

\[
\mathcal{O}(\mu) = \mathcal{O}(\mu') \left( \frac{\mu}{\mu'} \right)^{\gamma_*} \left( 1 + \mathcal{O}(\ln \mu, /\mu') \right)
\]

- QCD: UV-fixed point (asymptotic freedom)
  - \(\gamma_*(g_* = 0) = 0\) (trivial)
  - correction RGE (logarithmic)

- our interest: IR fixed point non-trivial
  - \(\gamma_* \neq 0\) (large?)
Scaling dimension

\[ \Delta_{\text{scaling}} = d_{\text{canonical}} + \gamma_{\text{anomalous}} \]

\[ \mathcal{L} = \sum_i C_i(\mu) \mathcal{O}_i(\mu) \]

- Value of \( \Delta_0 \) is a dynamical problem (= \( d_0 \) at trivial fixed-point, \( g^*=0 \))
  - Unitarity bounds \( \Delta_{\text{Scalar}} \geq 1 \) etc, Mack’77
  - \( \Delta_{00'} \neq \Delta_0 \Delta_0' \) generally (except SUSY and large-\( N_c \))

- Gauge theory: expect \( \bar{q}q \) to be most relevant operator
  \[ \Delta_{\bar{q}q} = 3 + \gamma_{\bar{q}q} = 3 - \gamma_m \] (\( \gamma_m = \gamma^* \) at fixed point)
  \( \gamma^* \) is a very important parameter for model building
Part I:

Observables for Monte Carlo (lattice) for conformal gauge theories
-- parametric control --
Part I: Observables in a CFT?

**pure-CFT:**
Vanishing β-function & correlators (form 2 & 3pt correlators known)

\[ \langle O(x)O(0) \rangle \sim (x^2)^{-\Delta} \]

e.g.

**deformed-CFT:**
Lattice: quarks massive / finite volume
⇒ consider mass-deformed conformal gauge theories (mCGT)*

\[ \mathcal{L} = \mathcal{L}_{\text{CGT}} - m\bar{q}q \]

* hardly related to 2D CFT mass deformation a part of algebra and ‘therefore’ integrability is maintained
• if mass-deformation relevant \( \Delta_{qq} = 3 - \gamma^* < 4 \)

theory flows away from fixed-point (likely)

**physical picture:** (Miransky ‘98) finite \( m_q \); quarks decouple \( \Rightarrow \) pure YM confines

(string tension confirmed lattice) \( \Rightarrow \) hadronic spectrum

**signature:** hadronic observables (masses, decay constants)

**hypothesis:** hadronic observables \( \rightarrow 0 \) as \( m_q \rightarrow 0 \) (conformality restored)

\[ \mathcal{O} \sim m^{\eta_{\mathcal{O}}}(1 + ..), \quad \eta_{\mathcal{O}}(\gamma^*) > 0 \]

If fct \( \eta_{\mathcal{O}} \) known:

a) way to measure \( \gamma^* \)

b) consistency test through many observable
Mass scaling from trace anomaly & Feynman-Hellman thm

**trace/scale anomaly:**

\[ \theta_\alpha^{\alpha}|_{\text{on-shell}} = \frac{1}{2} \beta G^2 + N_f m(1 + \gamma_m) \bar{q}q \]

\[ \beta = 0 \quad \& \quad \langle H(p)|H(k)\rangle = 2E_p \delta^{(3)}(p-k) \Rightarrow \]

\[ 2M_h^2 = N_f (1 + \gamma_*) m \langle H|\bar{q}q|H\rangle \]

**Feynman-Hellman thm:**

\[ \frac{\partial \langle \psi(\lambda) | \frac{\partial \hat{H}(\lambda)}{\partial \lambda} | \psi(\lambda) \rangle}{\partial \lambda} = \langle \psi(\lambda) | \frac{\partial \hat{H}(\lambda)}{\partial \lambda} | \psi(\lambda) \rangle \]

idea: \[ \frac{\partial \langle \psi(\lambda) | \psi(\lambda) \rangle}{\partial \lambda} = 0 \]

* applied to our case (\( \lambda \gg m \)):

\[ m \frac{\partial M_h^2}{\partial m} = N_f m \langle H|\bar{q}q|H\rangle \]

* combined with GMOR-like:

\[ m \frac{\partial M_H}{\partial m} = \frac{1}{1+\gamma_*} M_H \]

\[ M_H \sim m^{\frac{1}{1+\gamma_*}} \]
Brief comparison with QCD

**QCD-spectrum**

- \( m_q = O(\Lambda_{\text{QCD}}) + m_q \)
- \( m_B = m_b + O(\Lambda_{\text{QCD}}) \)
- \( m_\pi = O\left( (m_q \Lambda_{\text{QCD}})^{1/2} \right) \)

**mCGT-spectrum**

- \( m_{\text{ALL}} = m^{1/(1+\gamma^*)} \Lambda_{\text{ETC}} \gamma^*/(1+\gamma^*) \)

- Breaking global flavour symmetry: \( \text{SU}_L(n_F) \times \text{SU}_R(n_F) \rightarrow \text{SU}_V(n_F) \) (chiral symmetry)

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<th></th>
<th>QCD:</th>
<th>mCGT:</th>
<th>CGT:</th>
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<tbody>
<tr>
<td>spontaneous</td>
<td>yes*</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>explicit</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
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<tr>
<td>(mass term)</td>
<td></td>
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<tr>
<td>confinement</td>
<td>yes</td>
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<td>no</td>
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⇒ no chiral perturbation theory in mCGT

(pion not singled out -- Weingarten-inequality still applies)

* \( F_\pi \neq 0 \) \( m \rightarrow 0 \) order parameter
Hyperscaling laws from RG

- local matrix element: \[ \mathcal{O}_{12}(g, \hat{m} \equiv \frac{m}{\mu}, \mu) = \langle \varphi_2 | \mathcal{O} | \varphi_1 \rangle \]

1. \[ \mathcal{O}_{12}(g, \hat{m}, \mu) = b^{-\gamma \sigma} \mathcal{O}_{12}(g', \hat{m}', \mu') , \]
   \[ g' = b^{0+\gamma_g} g \quad \hat{m}' = b^{1+\gamma_*} \hat{m} \quad y_m = 1 + \gamma_* \quad \gamma_g < 0 \text{ (irrelevant)} \]

2. \[ \mathcal{O}_{12}(\hat{m}', \mu') = b^{-(d\sigma + d\varphi_1 + d\varphi_2)} \mathcal{O}_{12}(\hat{m}', \mu) \]

3. Choose \( b \) s.t. \( \hat{m}' = 1 \Rightarrow \text{trade } b \text{ for } m \)

\[ \mathcal{O}_{12}(\hat{m}, \mu) \sim (\hat{m})^{(\Delta \sigma + d\varphi_1 + d\varphi_2)/(1+\gamma_*)} \]

* From Weinberg-like RNG eqs on correlation functions (widely used in critical phenomena)
Applications:

- master formula (local matrix element): \[ \langle \varphi_1 | O | \varphi_2 \rangle \sim m^{\Delta_O + d_{\varphi_1} + d_{\varphi_2}}/(1 + \gamma_*) \]

1. hadronic masses: 
\[ 2M_h^2 = N_f (1 + \gamma_*) m \langle H | \bar{q}q | H \rangle \sim m^{\frac{2}{1 + \gamma_*}} \]

2. vacuum condensates: 
\[ \langle G^2 \rangle \sim m^{\frac{4}{1 + \gamma_*}}, \quad \langle \bar{q}q \rangle \sim m^{\frac{3 - \gamma_*}{1 + \gamma_*}} \]

3. decay constants:
\[ |\phi\rangle = |H(\text{adronic})\rangle \]
N.B. (\( \Delta_H = d_H = -1 \) choice)

| \( O \) | \( \text{def} \) | \( \langle 0 | O | J^{(C)}(p) \rangle \) | \( J^{(C)} \) | \( \Delta_O \) | \( \eta_{G[F]} \) |
|---|---|---|---|---|---|
| \( S \) | \( \bar{q}q \) | \( G_S \) | 0++ | 3 - \( \gamma_* \) | (2 - \( \gamma_* \))/ym |
| \( S^a \) | \( \bar{q}\lambda^a q \) | \( G_{S^a} \) | 0+ | 3 - \( \gamma_* \) | (2 - \( \gamma_* \))/ym |
| \( P^a \) | \( \bar{q}\gamma_5 q \) | \( G_{P^a} \) | 0- | 3 - \( \gamma_* \) | (2 - \( \gamma_* \))/ym |
| \( V \) | \( \bar{q}\gamma_\mu q \) | \( \epsilon_{\mu}(p) M_V F_V \) | 1-- | 3 | 1/ym |
| \( V^a \) | \( \bar{q}\gamma_\mu \lambda^a q \) | \( \epsilon_{\mu}(p) M_V F_{V^a} \) | 1-- | 3 | 1/ym |
| \( A^a \) | \( \bar{q}\gamma_\mu \gamma_5 \lambda^a q \) | \( \epsilon_{\mu}(p) M_A F_{A^a} \) | 1+ | 3 | 1/ym |
| \( A^a \) | \( \bar{q}\gamma_\mu \gamma_5 \lambda^a q \) | \( i\epsilon_{\mu}(p) F_{P^a} \) | 0- | 3 | 1/ym |
Remarks S-parameter: \[ S = 4\pi \Pi_{V-A}(0) - [\text{pion - pole}] \]

\[ (q^\mu q^\nu - q^2 g^{\mu\nu}) \delta_{ab} \Pi_{V-A}(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T(V_a^\mu(x)V_b^\nu(0) - (V \leftrightarrow A)) | 0 \rangle \]

hadronic representation: \[ \Pi_{V-A}(q^2) \simeq \frac{f_V^2}{m_V^2 - q^2} - \frac{f_A^2}{m_A^2 - q^2} - \frac{f_P^2}{m_P^2 - q^2} + \ldots \]

- difficulties: a) non-local  b) difference (density not positive definite)

\[ \Pi_{V-A}^{W-TC}(0) \sim O(m^{-1}) \]
\[ \Pi_{V-A}^{mCGT}(0) \sim O(m^0) \]
\[ \Pi_{V-A}^{mCGT}(q^2) \sim \frac{m^2/y_m}{q^2} \]

for \(-q^2 \gg (\Lambda_{ETC})^2\)

Sannino'10 free theory

• (conspiracy) cancellations improve on non-perturbative computations (lattice, FRG, DSE...)

modulo (conspiracy) cancellations
Summary - Transition

- low energy (hadronic) observables carry *memory of scaling phase*

\[ m_H \sim m \frac{1}{1+\gamma_*} \]
\[ f_{H(0^-)} \sim m \frac{2-\gamma_*}{1+\gamma_*} \]
\[ \langle \bar{q}q \rangle \sim m \frac{3-\gamma_*}{1+\gamma_*} \]

- “all” quantities scale with one parameter -- witness relations between the zoo critical exponents \( \alpha, \beta, \gamma, \nu \ldots = \text{hyperscaling} \)

- clarify: heavy quark phase and mCGT are parametrically from

  similar: \( m_{B(0^-)} \sim m_b \neq m_{H(0^-)} \sim m \frac{1}{1+\gamma_*} \)

  distinct: \( f_{B(0^-)} \sim m^{-1/2} \neq f_{H(0^-)} \sim m \frac{2-\gamma_*}{1+\gamma_*} \)
Part 2:

story quark condensate $<qq>$
the most relevant operator

• important for conformal TC/ partially gauged TC models
How does CFT react to a perturbation

**Unparticle area**

I. couple CFT Higgs-sector:  
\[ \mathcal{L}_{\text{eff}} \sim C \mathcal{O}|H|^2 \xrightarrow{\text{VEV}} C \mathcal{O}v^2 \]

criteria breaking (NDA):  
\[ \Lambda_B^4 \sim C v^2 \Lambda_B^{\Delta_\mathcal{O}} \quad \Rightarrow \quad \Lambda_B \sim (C v^2)^{\frac{1}{4-\Delta_\mathcal{O}}} \]

Fox, Rajaraman, Shirman ’07

II. Heuristics: **deconstruct** the continuous spectrum of a 2-function.  
Infinite sum of adjusted particles can mimick continuous spectrum.

\[ \mathcal{O}(x) \sim \sum_n f_n \varphi_n(x) ; \quad \langle \varphi_n | \mathcal{O} | 0 \rangle \sim f_n , \quad \begin{cases} f_n^2 = \delta^2 (M_n^2)^{\Delta - 2} \\ M_n^2 = n\delta^2 \end{cases} \]

⇒ tadpole & mass term as potential ⇒ find new minimum

\[ V_{\text{eff}} = -m \sum_n f_n \varphi_n - 1/2 \sum_n M_n^2 \varphi_n^2 \]

Delgado, Espinosa, Quiros’07
minimise - solve - reinsert:

\[ \delta \varphi_n V_{\text{eff}} = 0 \Rightarrow m f_n + M_n^2 \varphi_n = 0 \Rightarrow \langle \varphi_n \rangle = -m f_n / M_n^2 \]

\[ \langle O \rangle \sim \sum_n f_n \langle \varphi_n \rangle - m \sum_n \frac{f_n^2}{M_n^2} \delta \to 0 \to -m \int_{\Lambda_{\text{IR}}^2}^{\Lambda_{\text{UV}}^2} s \Delta O^{-3} ds \]

result depends on IR and UV physics ⇨ need model(s)

### III. Within conformal gauge theory

Sannino RZ ’08

generic 0++-operator: \( O \to \bar{q}q \) within gauge theories

N.B. 4D only ”known” CFT gauge theories. why?

- \( \Lambda_{\text{UV}} \to \Lambda_{\text{ETC}} \) where \( \Delta_{qq} \to 3 \Rightarrow (\Lambda_{\text{ETC}})^2 \)-effect
- \( \Lambda_{\text{IR}} \): \( (m_{\text{constituent}})^{\Delta_{qq}} \sim <qq> \) generalising Politzer OPE
  Solve self-consistency condensate eqn

\[ \frac{1}{Aq-B} = \frac{1}{q} + \frac{4g^2}{q^4} \]

- OPE extension
- of pole mass

\[ m = \frac{B}{A} \]

- ’10: \( \Lambda_{\text{IR}}: = m_H m^{1/(1+\gamma^*)} O(\Lambda_{\text{ETC}} \gamma^*/(1+\gamma^*)) \) agrees depending on value \( \gamma^* \)

Del Debbio, RZ Sep’10
IV. Generalized Banks-Casher relation

- Banks & Casher ’80 à la Leutwyler & Smilga 92:

Green’s function: \( \langle q(x)\bar{q}(y) \rangle = \sum_n \frac{u_n(x)u_n^\dagger(y)}{m - i\lambda_n} \), where \( \mathcal{D}u_n = \lambda_n u_n \)

\[
\langle \bar{q}q \rangle_V = \int \frac{dx}{V} \langle \bar{q}(x)q(x) \rangle \lambda_n \rightarrow -\lambda_n - \frac{2m}{V} \sum_{\lambda_n > 0} \frac{1}{m^2 + \lambda_n^2} \quad V \rightarrow \infty - 2m \int_0^\infty \frac{d\lambda \rho(\lambda)}{m^2 + \lambda^2}
\]

\[
= -2m \left[ \int_0^{\mu_F} d\lambda \frac{\rho(\lambda)}{m^2 + \lambda^2} \right] - \left[ \int_{\mu_F}^\infty d\lambda \frac{\rho(\lambda)}{m^2 + \lambda^2} \right] + \gamma_1 m + \gamma_2 m^3
\]

- IR-part: change of variable: \( \rho(\lambda) \sim 0 \lambda \eta_{\bar{q}q} \quad \Leftrightarrow \quad \langle \bar{q}q \rangle \sim 0 m \eta_{\bar{q}q} \)

- QCD: \( \eta_{\bar{q}q} = 0 \Rightarrow \rho(0) = -\pi \langle \bar{q}q \rangle \)

- mCGT: another way to measure \( \gamma^* \)

- UV-part: known from perturbation theory (scheme dependent)
I. NDA \( \Lambda_{\text{IR}} \sim m^{1/(1+\gamma^*)} \) ok

II. Deconstruction: \( \langle \mathcal{O} \rangle \sim \int_{\Lambda_{\text{IR}}^2}^{\Lambda_{\text{UV}}^2} s^{\Delta \mathcal{O} - 3} ds \) model dependence

III. Model mCGT & deconstruction
   - \( \Lambda_{\text{UV}} \) properly identified thanks asymptotic freedom
   - \( \Lambda_{\text{IR}} \) \( m_{\text{constituent}} \) generalized QCD (Polizter OPE) ok dep value \( \gamma^* \)

IV. Model mCGT & generalized Banks-Casher
   - clean separation of IR & UV everything consistent
     e.g. can use \( m_H \sim m^{1/(1+\gamma^*)} \) in deconstruction as well
Epilogue

• Identified “universal” hyperscaling laws in mass deformation valid for any conformal theory in the vicinity of the fixed point (small mass)

• One thought:
  1) CGT likely phase diagram as compared to walking theory
  2) CGT instable m-deformation. Any quark that receives mass can be expected to decouple and finally drive the theory into a confining phase and the remaining quarks can undergo chiral symmetry breaking and thus dynamical electroweak symmetry breaking.

Contrasts: “Dynamical stability of local gauge symmetry...”

Forster, Nielsen & Ninomiya’80

Danke für die Aufmerksamkeit!
Backup slides ...
Some relevant/useful references

- Miransky hep-ph/9812350 spectrum (with mass) as signal of conformal window
  works with pole mass -- weak coupling regime $\Lambda_{YM} \approx m \exp[-1/b_{YM} \, \alpha^*]$
  - glueballs lighter than mesons
- Luty Okui JHEP'96 conformal technicolor propose spectrum as signal of cw
- Dietrich/Sannino PRD’07 conformal window SU(N) higher representation using
  Dyson-Schwinger techniques known from WTC
- Sannino/RZ PRD’08 <qq> done heuristically IR and UV effects understood 0905
- DelDebbio et al ArXiv 0907 Mass lowest state from RGE equation
- DeGrand scaling <qq> stated ArXiv 0910
- DelDebbio RZ ArXiv 0905 scaling of vacuum condensates, all lowest lying states
- DelDebbio RZ ArXiv 0909 scaling extended to entire spectrum and all local
  matrix elements
Mass & decay constant trajectory

At large-$N_c$ neglect width

$$\Delta(q^2) \sim \int_x e^{ixq} \langle 0 | \mathcal{O}(x) \mathcal{O}(0) | 0 \rangle = \sum_n \frac{|g_{H_n}|^2}{q^2 + M_{H_n}^2}$$

In limit $m \to 0$ (scale invariant correlator)

$$\Delta(q^2) = \int_0^\infty ds \frac{s^{1-\gamma_*}}{q^2 + s} + s.t \propto (q^2)^{1-\gamma_*}$$

Solution are given by:

$$M_{H_n}^2 \sim \alpha_n m^{\frac{2}{1+\gamma_*}}, \quad g_{H_n}^2 \sim \alpha_n'(\alpha_n)^{1-\gamma_*} m^{\frac{2(2-\gamma_*)}{1+\gamma_*}}$$

where $\alpha_n$ arbitrary function (corresponds freedom change of variables in $\int$)

QCD expect $\alpha_n \sim n$ (linear radial Regge-trajectory) (few more words)

For those who know: resembles deconstruction Stephanov’07

difference physical interpretation of spacing due to scaling spectrum
Addendum (bounds scaling dimension)

- Assume $\text{add } L = mqq$ (N.B. not a scalar under global flavour symmetry!)
- Using bootstrap (‘associative’ OPE on 4pt function) possible to obtain upper-bound on scaling dimension $\Delta$ of lowest operator in OPE

\[ \langle \phi \phi \phi \phi \rangle = \sum_{\Delta} \]

**Non-singlet**

Rattazzi, Rychkov Tonni & Vichi’08

**Singlet $\Delta \leq 4$**

allows for $\Delta_{qq}$ to be:

Rattazzi, Rychkov & Vichi ’10

<table>
<thead>
<tr>
<th>$G$</th>
<th>$U(1) \equiv SO(2)$</th>
<th>$SO(3)$</th>
<th>$SO(4)$</th>
<th>$SU(2)$</th>
<th>$SU(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_+^*$</td>
<td>1.063 $(k = 2)$</td>
<td>1.082 $(k = 2)$</td>
<td>1.017 $(k = 2)$</td>
<td>1.016 $(k = 2)$</td>
<td>1.003 $(k = 2)$</td>
</tr>
<tr>
<td>$d_-$</td>
<td>1.12 $(k = 4)$</td>
<td>1.08 $(k = 4)$</td>
<td>1.06 $(k = 4)$</td>
<td>1.016 $(k = 2)$</td>
<td>1.003 $(k = 2)$</td>
</tr>
</tbody>
</table>

Very close to unitarity bound!

1.35 still rather close to unitarity bound
QCD observable
Gell-Mann Oakes Renner:
\[ f_\pi^2 m_\pi^2 = -2m\langle \bar{q}q \rangle \]

mCGT not so directly observable
very important for Technicolour

\[ \mathcal{L}^{\text{eff}} = \alpha_{ab} \frac{\bar{Q}T^a Q\bar{T}^b \psi}{\Lambda_{ETC}^2} + \beta_{ab} \frac{\bar{Q}T^a Q\bar{T}^b Q}{\Lambda_{ETC}^2} + \gamma_{ab} \frac{\bar{\psi}T^a \psi \bar{T}^b \psi}{\Lambda_{ETC}^2} + \ldots \]

↑
fermion masses

↑
FCNC