Insights into the QCD phase diagram
in/from sign-problem free setups

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STRONGLY INTERACTING MATTER AT NONZERO TEMPERATURE AND DENSITY

[Symmetries and states of QCD matter, nonzero temperature and density from the lattice]
QCD - Strong interactions within the Standard Model

1\textsuperscript{st} generation
- standard matter
  - up ($u$)
  - charm ($c$)
  - top ($t$)

2\textsuperscript{nd} generation
- unstable matter
  - down ($d$)
  - strange ($s$)
  - charm ($c$)

3\textsuperscript{rd} generation
- unstable matter
  - up ($u$)
  - charm ($c$)
  - top ($t$)

Goldstone bosons
- Higgs ($H$)

6 fermions
(+6 anti-fermions)
- standard matter
- unstable matter
- force carriers
- Goldstone bosons

Increasing mass

6 quarks (+6 anti-quarks)
- up ($u$)
- down ($d$)
- charm ($c$)
- strange ($s$)
- top ($t$)
- bottom ($b$)

6 leptons (+6 anti-leptons)
- electron ($e$)
- muon ($\mu$)
- tau ($\tau$)
- electron neutrino ($\nu_e$)
- muon neutrino ($\nu_\mu$)
- tau neutrino ($\nu_\tau$)

5 bosons
- photon ($\gamma$)
- $W^\pm$ and $Z$
- gluon ($g$)
- Higgs ($H$)

Charge
- anti-charged
- spin

Colors
- anti-color

Mass
- increasing mass

Spin

Standard Matter
- unstable matter
- force carriers
- Goldstone bosons

Goldstone Bosons
- 1\textsuperscript{st}
- 2\textsuperscript{nd}
- 3\textsuperscript{rd}

Weak Nuclear Force (Weak Isospin)

Electromagnetic Force (Charge)

Strong Nuclear Force (Color)

Electron ($e$)
- anti-electron ($\bar{e}$)
- mass less than 2 eV
- spin $1/2$

Neutrino ($\nu_e$)
- mass less than 511 keV
- spin $1/2$

Muon ($\mu$)
- anti-muon ($\bar{\mu}$)
- mass 105.7 MeV
- spin $1/2$

Neutrino ($\nu_\mu$)
- mass less than 190 keV
- spin $1/2$

Electron Neutrino ($\nu_e$)
- mass less than 2.3 MeV
- spin $1/2$

Muon Neutrino ($\nu_\mu$)
- mass less than 4.8 MeV
- spin $1/2$

Electron Anti-Neutrino ($\bar{\nu}_e$)
- mass less than 1.28 GeV
- spin $1/2$

Muon Anti-Neutrino ($\bar{\nu}_\mu$)
- mass less than 173.2 GeV
- spin $1/2$

Electron Anti-Neutrino ($\bar{\nu}_e$)
- mass less than 1.77 GeV
- spin $1/2$

Muon Anti-Neutrino ($\bar{\nu}_\mu$)
- mass less than 419 keV
- spin $1/2$

Electron Anti-Neutrino ($\bar{\nu}_e$)
- mass less than 18.2 MeV
- spin $1/2$

Muon Anti-Neutrino ($\bar{\nu}_\mu$)
- mass less than 180.4 GeV
- spin $1/2$

Electron Anti-Neutrino ($\bar{\nu}_e$)
- mass less than 1.8 GeV
- spin $1/2$

Muon Anti-Neutrino ($\bar{\nu}_\mu$)
- mass less than 191.2 GeV
- spin $1/2$

Electron Anti-Neutrino ($\bar{\nu}_e$)
- mass less than 1.8 GeV
- spin $1/2$

Muon Anti-Neutrino ($\bar{\nu}_\mu$)
- mass less than 191.2 GeV
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Electron Anti-Neutrino ($\bar{\nu}_e$)
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Muon Anti-Neutrino ($\bar{\nu}_\mu$)
- mass less than 191.2 GeV
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Electron Anti-Neutrino ($\bar{\nu}_e$)
- mass less than 1.8 GeV
- spin $1/2$

Muon Anti-Neutrino ($\bar{\nu}_\mu$)
- mass less than 191.2 GeV
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Quantum Chromodynamics

- Quantum field theory - non-abelian gauge theory with symmetry group $SU(3)$
- Interactions among color-charged quarks and (self-interacting) color-charged gluons acting as force carriers

Some symmetries of QCD

- Global $Z(3)$ center symmetry of $\mathcal{L}_{\text{gauge}}$ when temporal direction is compact
**Quantum Chromodynamics**

- Quantum field theory - non-abelian gauge theory with symmetry group $SU(3)$
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**Some symmetries of QCD** (considering $N_f$ mass-degenerate quark flavors)

- Global $Z(3)$ center symmetry of $\mathcal{L}_{\text{gauge}}$ when temporal direction is compact
- Global flavor $SU(N_f)_V \times U(1)_V$ chiral symmetry, remnant of the symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ in the corresponding massless theory
QCD matter under extreme conditions

Quark gluon plasma
Thermal QCD from the lattice

Quantum mechanical statistical system in heatbath

\[ Z(T) = \text{Tr} \left[ e^{-\hat{H}/T} \right] \]

Euclidean quantum field theory

\[ Z(T) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_E[A,\psi,\bar{\psi}]} \]

\[ S_E[A,\psi,\bar{\psi}] = \frac{1}{T} \int_0^T d\tau \int_{\mathcal{V}} d^3x J[A,\psi,\bar{\psi}] \]

- Bosonic (fermionic) fields periodic (anti-periodic) in the finite time direction to ensure Bose-Einstein (Fermi-Dirac) statistics
- Continuum limit at fixed \( T \): \( a \to 0 \iff N_\tau \to \infty \)

\[ T = \frac{1}{a(\beta)N_\tau} \]

\[ a \ll \xi \ll aN_\sigma \]

\[ a \ll aN_\tau \ll aN_\sigma \]
Consider QCD with three flavors of fermions in the grand canonical ensemble

\[
\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q, \quad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q, \quad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S
\]

- Nonzero baryon chemical potential setup

\[\mu_B \neq 0, \quad \mu_S = 0, \quad \mu_Q = 0\]

- Nonzero isospin chemical potential with \(\mu_I = \mu_u = -\mu_d \neq 0\) and \(\mu_s = 0\)
  - it corresponds to \(\mu_B = \mu_S = -\mu_I\) and \(\mu_Q = 2\mu_I\)
  - one can then define a pion chemical potential \(\mu_\pi = \mu_u - \mu_d = 2\mu_I\) to which corresponds the isospin density \(n_I = n_u - n_d\)
A nonzero isospin density \( n_I = n_u - n_d \) describes an asymmetry between the densities of up and down quarks

- hence between the densities of protons and neutrons
- hence between the densities of \( \pi^+ \) and \( \pi^- \)

A nonzero baryon density \( n_B = \frac{n_u + n_d}{3} \) measures the overall excess of strongly interacting matter over antimatter

- hence the asymmetry between the densities of e.g. protons and anti-protons
Consider \( N_f \) flavors of fermions \( \psi \), with \( L_F = \bar{\psi} M(\phi) \psi \), and bosons \( \phi \). In the Euclidean formulation and after fermion fields are integrated out:

\[
\langle O \rangle = \int \mathcal{D}\phi \ O[\phi] \ \frac{(\det M(\phi))^{N_f} \ e^{-S_B(\phi)}}{\int \mathcal{D}\phi \ (\det M(\phi))^{N_f} \ e^{-S_B(\phi)}}
\]

Once the theory is discretized on a lattice, one would like to estimate \( \langle O \rangle \) by employing importance sampling techniques.
Consider $N_f$ flavors of fermions $\psi$, with $\mathcal{L}_F = \bar{\psi} M(\phi) \psi$, and bosons $\phi$. In the Euclidean formulation and after fermion fields are integrated out

$$
\langle O \rangle = \int \mathcal{D}\phi \mathcal{O}[\phi] \frac{(\det M(\phi))^{N_f} e^{-S_B(\phi)}}{\int \mathcal{D}\phi \ (\det M(\phi))^{N_f} e^{-S_B(\phi)}}
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The sign problem forces us to restrict ourselves to cases where $(\det M)^{N_f} \in \mathbb{R}^+$!!
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POSITIVITY $\leftarrow$ even number of flavors each with $\det M \in \mathbb{R}$

REALITY $\leftarrow$ $\exists P$ invertible, such that $M^\dagger = PMP^{-1}$

☞ Alford, Kapustin, Wilczek (1999)
Dense QCD - reality & positivity of the measure

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In two-flavor QCD with finite density of isospin and $L(\mu)$, Dirac operator for one flavor with chemical potential $\mu$, satisfying $L(\mu)^\dagger = \gamma_5 L(-\mu) \gamma_5$

$$
M(\mu) = \begin{pmatrix} L(\mu) & 0 \\ 0 & L(-\mu) \end{pmatrix}, \quad \det M(\mu) = |\det L(\mu)|^2 \geq 0, \quad P = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}
$$

$\phi$ Alford, Kapustin, Wilczek (1999)
(SOME OF) THE MANY PHASES OF QCD
Some of the many phases of QCD

Courtesy of G. Endrödi
Some of the many phases of QCD

\[ (T, \mu_B): \]
- CEP?
- CSC phase?
Some of the many phases of QCD

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(\(T, \mu_I\)):
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Some of the many phases of QCD

- $(T, \mu_B)$:
  - CEP?
  - CSC phase?

- $(T, \mu_I)$:
  - BCS phase?

- $(T, B)$:
  - CEP
Some of the many phases of QCD
Some of the many phases of QCD

• BCS phase?

• CEP

• Better known
• Periodic

Sign problem!
Some of the many phases of QCD

Courtesy of G. Endrödi
Where is QCD under extreme conditions to be found?

The $n_B > 0$, $n_I < 0$ and $B > 0$ cases are relevant for

- Phenomenology of heavy ion collisions
- Structure of cold neutron stars
- Evolution of the early Universe
- Relic density of primordial gravitational waves
- Mass distribution of primordial black holes
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Actual systems in none of the considered planes!

⚠️ Sign problem for the relevant parameters triplet

❓ Some effect might dominate over the others...
“Coping” with the sign problem

Constrain sign-problem-affected regions in the phase diagram on the basis of results obtained in calculable domains

1. Depart from point/region where QCD matter undergoes smooth crossover
   - Taylor expansion or analytic continuation
   - talk by J. Günther  Borsanyi et al. (2020)

2. Follow critical boundaries in the “QCD” phase diagram provided that we know the location of a critical point in some parameters setup
   - Extrapolation according to known critical exponents
From what we have discussed so far...

Nonzero temperature can be varied at will

- Continuum limit at fixed temperature requires large $N_\tau$ (way larger $N_\sigma$)

Nonzero density is accessible, but only on condition that $\det M \in \mathbb{R}^+$!

- High nonzero real baryonic chemical potential “out of reach”

Simulation cost grows for decreasing quark masses

→ what if we changed the quark masses too!?

- Cost benefits going to larger masses

Phase diagram “more critical” in the chiral limit

→ possibilities for controlled extrapolations!
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Applications

1. Controlled chiral extrapolation in $\mu = 0$ QCD phase diagram

2. The BCS phase in the $(T, \mu_I)$ QCD phase diagram

3. QCD at nonzero isospin density, beyond the phase diagram
LET SYMMETRIES & UNIVERSALITY DRIVE EXPLORATIONS WITHIN A “BROADER” QCD PHASE DIAGRAM


Standard \((m_s, m_{u,d})\) Columbia plot

Dependence of the order of the QCD thermal phase transition on the quark masses
Standard \((m_s, m_{u,d})\) Columbia plot

Dependence of the order of the QCD thermal phase transition on the quark masses

- Broken global \(Z(3)\) for \(m_{u,d}, m_s \to \infty\) \(\bowtie\) Svetitsky, Yaffe (1982)
- Restored global \(SU_L(N_f) \times SU_R(N_f)\) for \(m_{u,d}, m_s \to 0\) \(\bowtie\) Pisarski, Wilczek (1984)
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- Continuum extrapolated results @ $m_{\text{phys}}$
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- chiral $Z_2$ boundary: strong cut-off and discretization dependence

\( \triangleright \) Aoki et al. (2006)
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\[ \begin{align*}
&\text{Continuum extrapolated results @}\, m_{\text{phys}} \\
&m_{u,d} \text{ very small. Transition affected by remnants of the chiral universality class?}
\end{align*} \]

- Chiral \(Z_2\) boundary: strong cut-off and discretization dependence
- Chiral 1\(^{\text{st}}\) order region wider for larger \(N_f\), until \(N_f = 4\) \(\bowtie\) de Forcrand, D’Elia (2017)

\(\bowtie\) de Forcrand, Philipsen (2007)
\(\bowtie\) Bazavov et al. (2017)
(\(m_s, m_{u,d}\)) Columbia plot in the continuum

Columbia plot from the “unimproved viewpoint”, for \(N_f = 2\), \(m_{u,d} \to 0, \infty\)

- Light/heavy 1\(^{st}\) order region does shrink/enlarge as \(a \to 0\)
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At least two possible scenarios for the nature of $N_f = 2$ chiral transition

- **Physical point**

• Relevance of the strength of the $U(1)_A$ anomaly at $T_c$

Pisarski, Wilczek (1984)
At least two possible scenarios for the nature of $N_f = 2$ chiral transition

- **"Direct approach":** $\mu = 0$, $N_f = 2$ and $m_{u,d} \to 0$ proved to be too expensive
- **"Indirect approaches":** tricritical scaling laws for extrapolations to $m_{u,d} \to 0$
Our phase transition of interest is the thermal phase transition in QCD
State of system is defined by the set of parameters \((m_s, m_u, m_d, \beta)\)
\(\beta\) tuned to be at the transition and plot its order

**Crossover**

“Order parameter” - \(m_u, m_d, m_s \to \infty\)

\[
P(x) = \frac{1}{N} \prod_{\tau_0}^{N_{\tau}} U_4(\tau, x)
\]

\[
\langle P(x)P(y)^\dagger \rangle = e^{-F_{q\bar{q}}(r)} \quad r \to \infty \quad |\langle P \rangle|^2
\]

“Order parameter” - \(m_u, m_d, m_s \to 0\)

\[
\langle \bar{\psi}\psi \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial m}
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"Order parameter" - \(m_{u,d}, m_s \to \infty\)

\[
P(x) = \frac{1}{N} \prod_{\tau=0}^{N-1} U_4(\tau, x)
\]

\[
\langle P(x) P(y) \dagger \rangle = e^{-\frac{F_{q\bar{q}}(r)}{T}} \quad r \to \infty \quad |\langle P \rangle|^2
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\[
P(x) = \frac{1}{N} \prod_{\tau=0}^{N_{\tau}-1} U_4(\tau, x)
\]

\[
\langle P(x) P(y) \rangle = e^{\frac{-F_{q\bar{q}}(r)}{T}} \quad \xrightarrow{r \to \infty} \quad |\langle P \rangle|^2
\]

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**“Order parameter”** - \(m_{u,d}, m_s \to 0\)

\[
\langle \bar{\psi}\psi \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial m}
\]
Towards real-$\mu$ Columbia plot

- 2nd order CEP real $\mu \equiv \mu_B$ depending on the bending of the $Z_2$ surfaces?
FROM LATTICE SIMULATIONS TO MAPPING OUT PHASE DIAGRAMS
Sample the order parameter $\mathcal{O}$ and extract central moments of the distribution, which is shifted and deformed while varying $\beta$

$$B_n(\beta, m, N_\sigma) = \frac{\langle (\mathcal{O} - \langle \mathcal{O} \rangle)^n \rangle}{\langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle^{n/2}}$$

- On phase boundaries, i.e. at $\beta_c$ and in the thermodynamic limit:

  $$B_3(\beta_c, m) = 0; \quad B_4(\beta_c, m) = \begin{cases} 1, & 1^{st} \text{ order} \\ 1.5, & 1^{st} \text{ order triple} \\ 1.604, & 2^{nd} \text{ order } Z_2 \\ 2, & \text{tricritical} \\ 3, & \text{crossover} \end{cases}$$

- Around $\beta_c$, with $t \equiv (T - T_c)/T_c$,

  $$B_2 = N_\sigma^{\gamma/\nu} f(tN_\sigma^{1/\nu}); \quad B_4 = g(tN_\sigma^{1/\nu})$$
Sample the order parameter $\mathcal{O}$ and extract central moments of the distribution, which is shifted and deformed while varying $\beta$. 

![Graphs showing $\mu(x)$, $\sigma^2(x)$, $P(x)$, and $B_3(x)$, $B_4(x)$ for different $\beta$ values.](image)
Sample the order parameter $\mathcal{O}$ and extract central moments of the distribution, which is shifted and deformed while varying $\beta$. 

- $\mu(x)$
- $\sigma^2(x)$
- $B_3(x)$
- $B_4(x)$
Phase transitions - How to locate them & extract their order

Sample the order parameter $O$ and extract central moments of the distribution, which is shifted and deformed while varying $\beta$

$B_3(\beta_c) = 0$ pinpoints $\beta_c$ & $B_4(\beta_c)$ reveals the order of the transition
Phase transitions - How to locate them & extract their order

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Sample the order parameter $\mathcal{O}$ and extract central moments of the distribution, which is shifted and deformed while varying $\beta$

$$B_3(\beta_c) = 0 \text{ pinpoints } \beta_c \text{ & } B_4(\beta_c) \text{ reveals the order of the transition}$$
Finite Size Scaling (FSS) analysis

\[ x \equiv (X - X_c) N_{\sigma}^{1/\nu} \]

\[ X = m, \kappa \]

\[ B_4(X, N_{\sigma}) = \left[ B_4(X_c, \infty) + a_1 x + O(x^2) \right] \]
Finite Size Scaling (FSS) analysis

\[ x \equiv (X - X_c) N_\sigma^{1/\nu} \]

\[ X = m, \kappa \]

\[ B_4(X, N_\sigma) = \left[ B_4(X_c, \infty) + a_1 x + O(x^2) \right] \cdot \left[ 1 + B N_\sigma^{(\alpha-\gamma)/2\nu} \right] \]

Leading finite volume correction
Results for $m_{\text{light}}^{Z_2}$ - cutoff & discretization effects

- Physical point

- Crossover
Results for $m_{Z_2}^{Z_2}$ - cutoff & discretization effects

### Staggered fermion discretization

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>Action</th>
<th>$N_\tau$</th>
<th>$m_{\pi}^{Z_2}$ [MeV] at $\mu = 0$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>std</td>
<td>4</td>
<td></td>
<td><a href="#">plot</a></td>
<td>Karsch, Laermann, Schmidt (2001)</td>
</tr>
<tr>
<td>3</td>
<td>std</td>
<td>6</td>
<td><a href="#">plot</a></td>
<td>de Forcrand, Kim, Philipsen (2007)</td>
</tr>
<tr>
<td>p4</td>
<td>4</td>
<td><a href="#">plot</a></td>
<td></td>
<td>Karsch et al. (2004)</td>
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<tr>
<td>HISQ</td>
<td>6</td>
<td><a href="#">plot</a></td>
<td></td>
<td>Bazavov et al. (2017)</td>
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<tr>
<td>2</td>
<td>std-$\mu_i$</td>
<td>4</td>
<td><a href="#">plot</a></td>
<td>Bonati et al. (2014)</td>
</tr>
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</table>

### Wilson fermion discretization

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>Action</th>
<th>$N_\tau$</th>
<th>$m_{\pi}^{Z_2}$ [MeV] at $\mu = 0$</th>
<th>Ref.</th>
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</thead>
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<tr>
<td>3</td>
<td>clover</td>
<td>6-8</td>
<td><a href="#">plot</a></td>
<td>Jin et al. (2015)</td>
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<tr>
<td></td>
<td>clover</td>
<td>8-10</td>
<td><a href="#">plot</a></td>
<td>Jin et al. (2017)</td>
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<tr>
<td>2</td>
<td>std</td>
<td>4</td>
<td><a href="#">plot</a></td>
<td>Philipsen, Pinke (2016)</td>
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<tr>
<td></td>
<td>tm</td>
<td>12</td>
<td><a href="#">plot</a></td>
<td>tftM coll. (2013)</td>
</tr>
<tr>
<td></td>
<td>clover</td>
<td>16</td>
<td><a href="#">plot</a></td>
<td>Brandt et al. (2017)</td>
</tr>
</tbody>
</table>
Results for $\kappa_{\text{heavy}}^Z$

\[ N_f = 2 \]

$m_s$

$N_f = 1$

$N_f = 3$

$1^{st}$

$Z_2$

$\text{Physical point}$

$Crossover$

Determined critical hopping parameter $\kappa_c$ for $N_f = 2$ standard Wilson fermions

- $N_\tau \in \{6, 8, 10\}$ corresponding to $a \in \{0.12, 0.09, 0.07\}$ fm
- Analysis of the growing statistics requirements towards the continuum
  - growing $\tau_{\text{int}}$ of the skewness of the approximate order parameter $|L|$
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Determined critical hopping parameter \( \kappa_c \) for \( N_f = 2 \) standard Wilson fermions

- \( N_\tau \in \{6, 8, 10\} \) corresponding to \( a \in \{0.12, 0.09, 0.07\} \) fm
- Analysis of the growing statistics requirements towards the continuum
  - growing \( \tau_{\text{int}} \) of the skewness of the approximate order parameter \( |L| \)
- Kurtosis fit requires leading finite size correction for largest \( N_\tau \)
- Significant cut-off effects, first-order region grows as the lattice gets finer

Enabled comparison with results from LO/NLO hopping expanded fermion determinant

\( \therefore \) WHOT-QCD Collaboration (2020)
YET ANOTHER CONTROLLED
(BY UNIVERSALITY ARGUMENTS)
EXTRAPOLATION TO THE CHIRAL LIMIT

Yet another “indirect approach”: promoting $N_f$ to continuous real parameter

$$Z_{N_f=2} = \int \mathcal{D}U \left[ \det M(U, m) \right]^{2} \ e^{-S_G}$$

1st order for $N_f \geq 3 \implies$ 2nd order for $N_f = 2$ requires $N_f^{tric} \in (2, 3)$
Results for the tricritical scaling region at $N_T = 4$

![Graph showing linear and tricritical scaling fits. The tricritical scaling fit is $m \propto (N_f - N^{tric}_f)^{5/2}$, where $N_f \in [2.0, 2.2]$. The linear fit is $N_f \in [2.4, 5]$.](image-url)
Results for the tricritical scaling region at $N_\tau = 4$

- Linear fit: $N_f \in [2.4, 5]$
- Tricritical scaling fit: $N_f \in [2.0, 2.2]$
- $m \propto (N_f - N_{\text{tric} f})^{5/2}$
- $N_{\tau} = 4$

![Graph showing results for tricritical scaling region](image-url)
$Z_2$ boundary linear in some $N_f$ range?

- If it is reasonable to expect both linearity within some range in $N_f$ and tricritical scaling closer to the chiral limit
  - linear extrapolation to $m = 0$ to get an upper bound for $N_f^{\text{tricr}}$

- If $N_f^{\text{lin}} < 2$, while one simulates at larger and larger $N_\tau$ towards the continuum limit
  - Transition in the $N_f = 2$ chiral limit keeps being a first order one
Results for the linear scaling region at $N_\tau = 4, 6$

$N_f^{\text{lin}} \sim 2 \ @ \ N_\tau = 4,$

$N_f^{\text{lin}} \lesi 3 \ @ \ N_\tau = 6$

- First order scenario more and more contrived with larger and larger $N_\tau$ values.
- Discretization effects?!?
- Is “our linearity” related to the linearity of $T_c$ as a function of $N_f$ in the chiral limit from FRG?

$N_f = 3, \ N_\tau = 6$ by de Forcrand et al. (2007)

Braun, Gies (2006)
THE QCD PHASE DIAGRAM WITH NONZERO ISOSPIN CHEMICAL POTENTIAL


[Symmetries, Pion BEC, signatures of BCS ]
QCD at Finite Isospin Density

D. T. Son$^{1,3}$ and M. A. Stephanov$^{2,3}$

$^1$Physics Department, Columbia University, New York, New York 10027
$^2$Department of Physics, University of Illinois, Chicago, Illinois 60607-7059
$^3$RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973

(Received 31 May 2000)

QCD at finite isospin chemical potential $\mu_I$ has no fermion sign problem and can be studied on the lattice. We solve this theory analytically in two limits: at low $\mu_I$, where chiral perturbation theory is applicable, and at asymptotically high $\mu_I$, where perturbative QCD works. At low isospin density the ground state is a pion condensate, whereas at high density it is a Fermi liquid with Cooper pairing. The pairs carry the same quantum numbers as the pion. This leads us to conjecture that the transition from hadron to quark matter is smooth, which passes several tests. Our results imply a nontrivial phase diagram in the space of temperature and chemical potentials of isospin and baryon number.

Son, Stephanov (2001)

Non trivial phase diagram drawn on the basis of analytical computations in

- the $n_I \to 0$ limit $\leftarrow$ Chiral Perturbation Theory
- the $n_I \to \infty$ limit $\leftarrow$ Perturbative QCD
QCD at finite isospin density - “Analytical phase diagram”

\[ \langle \pi^- \rangle \neq 0 \]

In the limit \( n_I \to 0 \), i.e. \( |\mu_I| \ll m_\rho \) \( \chi \)PT applies

- \( \pi^\pm \) lightest hadrons coupling to \( \mu_I \): \( \chi \)PT describes their effective dynamics
- At \( T = 0 \), \( \mu_I \geq \mu_{I,c} = m_\pi/2 \), sufficient energy to create \( \pi^\pm \)
- A Bose-Einstein condensate (BEC) is formed
- Hadronic/BEC phase transition predicted, by \( \chi \)PT, to be second order (\( O(2) \) universality class)
In the limit $n_I \rightarrow \infty$, i.e. $|\mu_I| \gg \Lambda_{QCD}$, p-QCD applies

- Attractive gluon interaction forms pseudoscalar $u - \bar{d}$ Cooper-pairs
- BEC/BCS phase transition expected to be analytic crossover (same symmetry breaking pattern)
- At asymptotically large $\mu_I$, decoupling of the gluonic sector and first-order deconfinement phase transition
$SU_\nu(2) \times U_\nu(1)$ flavor symmetry group for QCD with light quark matrix

$$\mathcal{M}_{ud}|_{\mu_i=\lambda=0} = \gamma_\mu (\partial_\mu + iA_\mu) \mathbb{1} + m_{ud} \mathbb{1}, \quad \psi = (u, d)^T$$

At $\mu_1 \neq 0$ \quad $\rightarrow$ \quad $\mathcal{M}_{ud} = \mathcal{M}_{ud}|_{\mu_i=\lambda=0} + \mu_1 \gamma_4 \tau_3$

$SU_\nu(2) \times U_\nu(1) \rightarrow U_{\tau_3}(1) \times U_\nu(1)$

**Spontaneous breaking** with pion condensate $\langle \bar{\psi} \gamma_5 \tau_{1,2} \psi \rangle$

$\rightarrow$ Appearance of Goldstone mode
\( n_l \)-QCD on the lattice - Symmetry breaking patterns

- \( SU_V(2) \times U_V(1) \) flavor symmetry group for QCD with light quark matrix

\[
\mathcal{M}_{ud}|_{\mu_i=\lambda=0} = \gamma_\mu (\partial_\mu + iA_\mu) 1 + m_{ud} 1, \quad \psi = (u, d)^T
\]

- At \( \mu_1 \neq 0, \lambda \neq 0 \), \( \mathcal{M}_{ud} = \mathcal{M}_{ud}|_{\mu_i=\lambda=0} + \mu_1 \gamma_4 \tau_3 + i\lambda \gamma_5 \tau_2 \)

\[
SU_V(2) \times U_V(1) \longrightarrow U_{\tau_3}(1) \times U_V(1) \longrightarrow \emptyset \times U_V(1)
\]

- **Spontaneous breaking** with pion condensate \( \langle \bar{\psi} \gamma_5 \tau_{1,2} \psi \rangle \)

\[ \longrightarrow \text{Appearance of Goldstone mode} \]

- **Explicit breaking** via pionic source \( \lambda \),

\[ \longrightarrow \text{pseudo-Goldstone boson} \]

(\( \lambda \) necessary trigger for spontaneous breaking and I.R. regulator)
$n_l$-QCD results - The $(T, \mu_l)$ phase diagram

$N_f = 2 + 1$ improved dynamical staggered quarks with physical quark masses

- BEC boundary at $\mu_{l,c} = m_\pi/2$ up to $T \approx 140$ MeV, very flat at larger $\mu_l$
QCD at finite isospin density - “Numerical phase diagram”

- Prediction of a superfluid state of $u$ and $\bar{d}$ Cooper pairs (BCS phase) at very high-$\mu_I$ and $T = 0$, plausibly connected via crossover to the BEC phase at $\mu_I \geq m_\pi/2$  
  Son, Stephanov (2001)

- Deconfinement transition connecting continuously to BEC-BCS crossover in the $(T, \mu_I)$ phase diagram
  - Large Polyakov loops $P_r$ within BEC phase, i.e. $\langle \bar{u} \gamma_5 d \rangle \neq 0$
  - $T_{c_{\text{deconf}}}^\mu_I$ insensitive to BEC boundary
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\begin{align*}
  &T \\
  &T_d \\
  &m_\pi/2 \\
  &|\mu_I|
\end{align*}

\begin{align*}
  &\text{Quark-gluon plasma phase} \\
  &\text{Hadronic phase} \\
  &\text{BEC phase} \\
  &\text{BCS phase}
\end{align*}

\begin{align*}
  &U_{\tau_3}(1) \times U_V(1) \\
  &SU_V(2) \text{ explicitly broken} \\
  &U_V(1) \text{ spontaneously broken} \\
  &\langle \pi^- \rangle \neq 0
\end{align*}
Signatures of BCS in closely related literature

- At $T = 0$, BEC-BCS crossover from the LO $\chi_{PT}$ Lagrangian at nonzero isospin defined by the conformality of the system
  ↪ Carignano et al. (2017)

- In a Polyakov-loop quark-meson model, BEC-BCS crossover defined by the “shift”, with $\mu_I$, of the minimum of the quark energies
  ↪ Adhikari, Andersen, Kneschke (2018)

- In $Q_2\bar{C}D$, BCS phase identified by the number density of Cooper pairs (diquark condensate) being proportional to the Fermi-surface area $\langle qq \rangle \sim \mu^2$
  ↪ Boz et al. (2020)
Why looking at the Dirac spectrum?

- Banks-Casher relation for QCD at $\mu = 0$ \cite{BanksCasher1980}:
  \[ |\langle \bar{\psi}\psi \rangle| = \pi \rho(0) \]

- $\Sigma B\chi S$ signaled by the accumulation of near-zero Dirac eigenvalues

- At $\mu_B \neq 0$, non positive-definiteness of the fermionic measure
  - renders $\rho(0)$ undefined
  \cite{LeutwylerSmilga1992}
  - complicates the connection between $\langle \bar{\psi}\psi \rangle$ and the complex Dirac spectrum
  \cite{OsbornSplittorffVerbaarschot2005}

- At only $\mu_I \neq 0$ though, the fermionic measure is still positive definite...
  - Generalization of the Banks-Casher relation is possible!
Signatures of the BCS phase at large $\mu_I$ from $\mathcal{D}(\mu_I)$ spectrum

- Banks-Casher relation extensible to the case of complex Dirac eigenvalues for QCD at $T=0$, $\mu_I \neq 0$
- For $|\mu_I| \gg \Lambda_{QCD}$... Kanazawa, Wettig, Yamamoto (2013)

![Image of the BCS gap formula and spectral density]

$$\Delta^2 = \frac{2\pi^3}{3N_C} \rho(0)$$
Complex spectrum of $\mathcal{D}(\mu I)$ - Results, qualitatively

- Simulations are carried out away from the chiral limit → extract $\rho(m_{ud})$

- $\mu I < m_\pi/2$: eigenvalues clustered along imaginary axis → $\rho(m_{ud}) = 0$

- $\mu I / m_\pi = 0.51$

- $\mu I < m_\pi/2$: eigenvalues clustered along imaginary axis → $\rho(m_{ud}) = 0$
Complex spectrum of $\mathcal{D}(\mu_I)$ - Results, qualitatively

- Simulations are carried out away from the chiral limit $\rightarrow$ extract $\rho(m_{ud})$

- $\mu_I < m_\pi/2$: eigenvalues clustered along imaginary axis $\rightarrow \rho(m_{ud}) = 0$

- $\mu_I > m_\pi/2$: spectrum ‘wide’ enough to include $m + i0 \rightarrow \rho(m_{ud}) \neq 0$
Complex spectrum of $\mathcal{D}(\mu_I)$ - Results, qualitatively

- Simulations are carried out away from the chiral limit $\rightarrow$ extract $\rho(m_{ud})$

- $\mu_I < m_\pi/2$: eigenvalues clustered along imaginary axis $\rightarrow \rho(m_{ud}) = 0$

- $\mu_I > m_\pi/2$: spectrum ‘wide’ enough to include $m + i0$ $\rightarrow \rho(m_{ud}) \neq 0$

- Higher-$\mu_I$: eigenvalues drifting away from the real axis $\rightarrow \rho(m_{ud}) \rightarrow 0$
Complex spectrum of $\mathcal{D}(\mu_I)$ - Results, quantitatively

$\mu_I$-scans at fixed $T$

$T_{1} > T_{0}$

$U_{V}(1) \times U_{V}(1)$ explicitly broken

$\langle \pi^- \rangle \neq 0$

$S_{V}(2) \times U_{V}(1)$

$H$adronic phase

$B$CS phase

$B$EC phase

$\langle \pi^- \rangle \neq 0$

$T \sim 113$ MeV

$T \sim 148$ MeV

$\rho(m_{ud}) [\text{GeV}^2]$

$\mu_I/m_{\pi}$
Complex spectrum of $\Phi(\mu_I)$ - Results, quantitatively

$\mu_I$-scans at fixed $T$

$T_0 < T_1 < T_2 < T_3$

$\langle \pi^- \rangle \neq 0$

$T \sim 113 \text{ MeV}$
$T \sim 148 \text{ MeV}$
$T \sim 155 \text{ MeV}$
$T \sim 162 \text{ MeV}$
Complex spectrum of $\phi(\mu_I)$ - Results, quantitatively

Can we read

- $\mu_I^{\text{Hadr.}}/\text{BEC}$
- $\mu_I^{\text{BEC}}/\text{BCS}$

off our $\rho(m_{ud})$?

$T_0 < T_1 < T_2 < T_3$

$T \sim 113$ MeV
$T \sim 148$ MeV
$T \sim 155$ MeV
$T \sim 162$ MeV
Complex spectrum of $\mathcal{D}(\mu_I)$ - outlook

![Graph 1](Image)

$\lambda/m_{ud} \sim 0.72$

$\lambda/m_{ud} \sim 0.29$

![Graph 2](Image)

$T \sim 113 \text{ MeV}$

$T \sim 124 \text{ MeV}$

$T \sim 136 \text{ MeV}$

$T \sim 142 \text{ MeV}$

$T \sim 148 \text{ MeV}$

$T \sim 155 \text{ MeV}$

$T \sim 162 \text{ MeV}$

![Graph 3](Image)

$N_\sigma : 16$

$N_\sigma : 24$

$N_\sigma : 32$

$a \rightarrow 0$
Complex spectrum of $\mathcal{D}(\mu_I)$ - outlook

\[ \mu_I/m_\pi \quad \rho(m_{ud}) \quad [\text{GeV}^2] \]

\[ \lambda/m_{ud} \sim 0.72 \quad \lambda/m_{ud} \sim 0.29 \]

\[ T \sim 113 \text{ MeV} \quad T \sim 124 \text{ MeV} \quad T \sim 136 \text{ MeV} \]
\[ T \sim 142 \text{ MeV} \quad T \sim 148 \text{ MeV} \quad T \sim 155 \text{ MeV} \]
\[ T \sim 162 \text{ MeV} \]

\[ V \rightarrow \infty \quad a \rightarrow 0 \]
Complex spectrum of $\mathcal{D}(\mu_I)$ - outlook

$$\lambda \rightarrow 0$$

$$V \rightarrow \infty$$

$$a \rightarrow 0$$
Complex spectrum of $\mathcal{D}(\mu_I)$ - outlook

$\lambda \rightarrow 0$  \hspace{1cm}  $T \rightarrow 0$

$V \rightarrow \infty$  \hspace{1cm}  $a \rightarrow 0$
QCD THERMODYNAMICS AT NONZERO ISOSPIN CHEMICAL POTENTIAL


[Pion condensation in the early Universe at nonvanishing lepton flavor asymmetry]
QCD EoS at nonzero isospin as input for phenomenology

Equation of state $\epsilon(p)$ at nonzero isospin measured in lattice simulations...

- The isospin density $n_I(T, \mu_I) = \frac{T}{V} \left( \frac{\partial \log Z}{\partial \mu_I} \right)$ is determined first.
- Employing the integral method, one can obtain:
  - Pressure differences
  - Trace anomaly i.e. interaction measure differences
Equation of state $\epsilon(p)$ at nonzero isospin measured in lattice simulations...

- @ $T=0$ (fixed $a$) Brandt, Endrödi, Fraga, Hippert, Schaffner-Bielich, Schmalzbauer ('18)
- onto a grid of nonzero $T$ values for $N_{\tau} \in \{10, 12\}$
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Results **calibrate & test** a variant of HRG model incorporating $\pi$-condensation!

$\rightarrow$ Model equation of state of cosmic matter fed to the conservation equations

$$\frac{n_B}{s} = b, \quad \frac{n_Q}{s} = 0, \quad \frac{n_e}{s} = l_e, \quad \frac{n_\mu}{s} = l_\mu, \quad \frac{n_\tau}{s} = l_\tau$$

defining the cosmic trajectory in the 6-d space of $T$, $\mu_B$, $\mu_Q$, $\mu_e$, $\mu_\mu$, and $\mu_\tau$
QCD EoS at nonzero isospin as input for phenomenology

Equation of state $\epsilon(p)$ at nonzero isospin measured in lattice simulations...

- @ $T=0$ (fixed $a$)  
  Brandt, Endrödi, Fraga, Hippert, Schaffner-Bielich, Schmalzbauer ('18)
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defining the cosmic trajectory in the 6-d space of $T, \mu_B, \mu_Q, \mu_e, \mu_\mu, \text{and } \mu_\tau$
Pion condensation in the early Universe

Working hypotheses

- Isentropic expansion of the Universe
- Cosmic QCD epoch: equation of state mainly determined by QCD matter
- $b = (8.60 \pm 0.06) \times 10^{-11}$ and $q = 0$
- Unequally distributed lepton asymmetries $l_\tau = -(l_e + l_\mu)$
  - in agreement with $|l_e + l_\mu + l_\tau| < 0.012$ Wygas et al. (2018)

Result & phenomenological implications

- Condition for pion condensation to occur is $|l_e + l_\mu| \gtrsim 0.1$
- Increase in the amplitude of primordial gravitational waves
- Modified mass distribution of primordial black holes
## Summary

<table>
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<th>Location/cutoff of the heavy $Z_2$ boundary in the Columbia plot</th>
<th>$N_f$-dependence of the nature of the QCD chiral phase transition</th>
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<td>Insights onto the $(T, \mu_I)$ phase diagram from Dirac spectrum</td>
<td>Pion condensation in the early Universe for $</td>
</tr>
</tbody>
</table>
Summary

Insights onto the \((T, \mu_I)\) phase diagram from Dirac spectrum

Pion condensation in the early Universe for \(|l_e + l_\mu| \gtrsim 0.1\)

\[ N_f \text{-dependence of the nature of the QCD chiral phase transition} \]
Insights onto the \((T, \mu_I)\) phase diagram from Dirac spectrum

Pion condensation in the early Universe for \(|l_e + l_\mu| \gtrsim 0.1\)
Pion condensation in the early Universe for $|l_e + l_\mu| \gtrsim 0.1$
Thank you for your attention!
Backup slides
Sample the order parameter $\mathcal{O}$ and extract central moments of the distribution, which is shifted and deformed while varying $\beta$

- In our studies at $\mu = i\mu_i^c$, the measured observable $\mathcal{O}$ is also sensitive to the Roberge-Weiss phase transition on whose phase boundary we stand, while varying $\beta$

$$B_3(\beta) = 0 \quad \forall \beta$$

Crossing of $B_4(\beta_c) \forall N_{\sigma}$ pinpoints $\beta_c$ & reveals the order of the transition
Isospin chemical potential: positivity of the measure

Standard lattice fermion algorithms run into the well-known sign problem with a real chemical potential. In this paper we investigate the possibility of using an imaginary chemical potential and argue that it has advantages over other methods, particularly for probing the physics at finite temperature as well as density. As a feasibility study, we present numerical results for the partition function of the two-dimensional Hubbard model with an imaginary chemical potential. We also note that systems with a net imbalance of isospin may be simulated using a real chemical potential that couples to \( I_3 \) without suffering from the sign problem.

\[ \text{Systems with } n_I \neq 0 \text{ can be simulated with standard Monte Carlo importance sampling techniques using } \mu_I \in \mathbb{R} \text{ that couples to } I_3 = \frac{T_3}{2}. \]
$N_f$-QCD on the lattice - Setup

- QCD with $N_f = 2 + 1$ improved dynamical staggered quarks with **physical masses** at various $T$, $\mu_I$ and values of the I.R. regulator $\lambda$

\[ S_{ud} = \bar{\psi} \mathcal{M}_{ud} \psi, \quad \psi = (u, d)^\top, \]
\[ \mathcal{M}_{ud} = \gamma_\mu (\partial_\mu + iA_\mu) \mathbb{1} + m_{ud} \mathbb{1} + \mu_1 \gamma_4 \tau_3 + i \lambda \gamma_5 \tau_2, \]

- Explicit, unphysical **symmetry breaking term** in $\mathcal{M}_{ud}$ couples to the charged pion field $\pi^{\pm}$, the coupling $\lambda$ referred to as “pionic source”

\[ S_{ud} = S_{ud}(\lambda = 0) + \lambda \pi^{\pm}, \quad \pi^{\pm} \equiv \bar{\psi} i \gamma_5 \tau_2 \psi = \bar{u} \gamma_5 d - \bar{d} \gamma_5 u. \]

- $N_s^3 \times N_t$ lattices with spacing $a$, temperature $T = 1/(N_t a)$ and spatial volume $V = (N_s a)^3$, gauge coupling $\beta = 6/g^2$ and

\[ Z = \int \mathcal{D}U_\mu \ e^{-\beta S_{G}^{\text{Sym}}} \ (\det \mathcal{M}_{ud})^{1/4} (\det \mathcal{M}_s)^{1/4}, \quad U_\mu = \exp(i a A_\mu) \]

$\mathcal{M}_{ud}$ light quark matrix (in the $u$ and $d$ quarks basis), $\mathcal{M}_s$ s quark matrix.
n_f-QCD on the lattice - Breaking of $U_{\tau_3}(1)$ symmetry

- Spontaneous, by $\langle \pi^\pm \rangle$, and explicit, by $\lambda$, breaking of the $U_{\tau_3}(1)$ symmetry is completely analogous to the spontaneous, by $\langle \bar{\psi}\psi \rangle$, and explicit, by $m_{ud}$ breaking of the standard chiral symmetry at $\mu_I = 0$

Pion condensation $\rightarrow$ Chiral symmetry breaking

$U_{\tau_3}(1) \rightarrow \emptyset$ $\leftarrow$ breaking pattern $SU_L(2) \otimes SU_R(2) \rightarrow SU_V(2)$

1 $\leftarrow$ # Goldstones $3$

$\langle \bar{\psi}\gamma_5\tau_2\psi \rangle$ $\leftarrow$ condensates $\langle \bar{\psi}\psi \rangle$

$\lambda \rightarrow 0$ $\leftarrow$ explicit breaking $m \rightarrow 0$

$\rho^{|\phi(\mu_I)+m|^2}(0)$ $\leftarrow$ Banks-Casher $\rho^{(\phi)}(0)$

- While in nature $m_{ud} > 0$, $\lambda$ is unphysical: the limit $\lambda \rightarrow 0$ must be taken!
In our partition function $Z = \int \mathcal{D}U_\mu \, e^{-\beta S^\text{Sym}_G} \left( \det M_{ud} \right)^{1/4} \left( \det M_s \right)^{1/4}$

$$M_{ud} = \begin{pmatrix} \bar{\Phi}(\mu_1) + m_{ud} & \lambda \eta_5 \\ -\lambda \eta_5 & \bar{\Phi}(-\mu_1) + m_{ud} \end{pmatrix}, \quad M_s = \bar{\Phi}(0) + m_s$$

- $\det M_s \in \mathbb{R}^+$ due to the standard $\eta_5$-hermiticity relation $\eta_5 M_s \eta_5 = M_s^\dagger$ with $\eta_5 = \gamma^S_\gamma \otimes \gamma^F_\gamma = (-1)^{n_x+n_y+n_z+n_t}$ equivalent of $\gamma_5$ is the local staggered spin-flavor structure
- $\det M_{ud} \in \mathbb{R}^+$ due to

$$\begin{align*}
\Phi(\mu_1) \eta_5 + \eta_5 \Phi(\mu_1) &= 0 \\
\eta_5 \Phi(\mu_1) \eta_5 &= \Phi(-\mu_1)^\dagger
\end{align*} \implies \tau_1 \eta_5 M_{ud} \eta_5 \tau_1 = M_{ud}^\dagger$$

and

$$M'_{ud} = B M_{ud} B = \begin{pmatrix} \Phi(\mu_1) + m_{ud} & \lambda \\ -\lambda & [\Phi(\mu_1) + m_{ud}]^\dagger \end{pmatrix}, \quad B = \text{diag}(1, \eta_5)$$
The pion condensate and quark condensate obtainable from $Z$, via differentiation and measurable with noisy-estimator techniques:

$$\langle \pi^\pm \rangle = \frac{T}{V} \frac{\partial \log Z}{\partial \lambda} = \frac{T}{2V} \text{tr} \frac{\lambda}{|\Phi(\mu_I) + m_{ud}|^2 + \lambda^2}$$

$$\langle \bar{\psi} \psi \rangle = \frac{T}{V} \frac{\partial \log Z}{\partial m_{ud}} = \frac{T}{2V} \text{Re} \text{tr} \frac{\Phi(\mu_I) + m_{ud}}{|\Phi(\mu_I) + m_{ud}|^2 + \lambda^2}$$

then becoming, after appropriate multiplicative/additive renormalization,

$$\Sigma_{\bar{\psi} \psi} = \frac{m_{ud}}{m_{\pi}^2 f_{\pi}^2} \left[ \langle \bar{\psi} \psi \rangle_{T,\mu_I} - \langle \bar{\psi} \psi \rangle_{0,0} \right] + 1$$

$$\Sigma_{\pi} = \frac{m_{ud}}{m_{\pi}^2 f_{\pi}^2} \langle \pi^\pm \rangle_{T,\mu_I}$$

The renormalized Polyakov loop $P_r(T, \mu_I) = Z \cdot \left\langle \frac{1}{V} \sum_{n_x,n_y,n_z} \text{Tr} \prod_{n_t=0}^{N_t-1} U_t(n) \right\rangle$

with $Z = \left( \frac{P_*(T_{\mu_I=0})}{P(T_*,\mu_I=0)} \right)^{T_*/T}$, and $T_* = 162$ MeV, hence $P_* = 1$
Order & universality class of the BEC transition

Volume scaling & collapse hinting for a second order phase transition (consistent with NLO $\chi$PT \cite{Carignano2017}), in the $O(2)$ universality class

\[ \Sigma = h^{1/\delta} \cdot f_G \left( \frac{t}{h^{1/(\beta \delta)}} \right) + a_1 th + b_1 h + b_3 h^3 \]

\[ h = \frac{\lambda}{\lambda_0}, \quad t = \frac{\mu_{I,c} - \mu_I}{t_0} \]

with critical exponents $\beta$, $\delta$ & the universal scaling function $f_G$ \cite{Ejiri2009}
**Pion condensate**

- BEC phase boundary, $\mu_{l,c}(T)$, by onset of $\Sigma_\pi$
- $\mu_{l,c}(T, a)$, 4th order polynomial in $(T - T_0)$ with $a$–dependent coefficients and $T_0 = 140$ MeV

**Quark condensate**

- Chiral crossover $T_c(\mu_l)$, by the inflection points of $\Sigma_{\bar{\psi}\psi}(T)$
- $T_c(\mu_l, a)$, even-in-$\mu_l$ polynomial, including data up to $\mu_{l,c}(0) = m_\pi/2$
The idea is...

- $\lambda > 0$ triggers pion condensation, but $\lambda$ is unphysical so a $\lambda \to 0$ extrapolation is needed

The problems are...

1. Observables exhibit pronounced $\lambda$-dependence
2. The condition number of the fermion matrix $\kappa(M_{ud})$ is strongly affected by $\lambda$, because $\lambda$ acts as a I.R. regulator
3. Fluctuations in the fermion force are regulated/influenced by $\lambda$

Taking the $\lambda \to 0$ limit requires an **improvement strategy** to be devised

1. To inhibit the observable’s $\lambda$ dependence
2. To reduce simulation costs

The needed improvement is a twofold one concerning both...

- the valence sector $\to$ operators modified on the basis of the *singular value representation* of $\mathcal{D}(\mu_I) + m_{ud}$ to remove explicit dependence on $\lambda$
- the sea sector $\to$ configurations reweighted to $\lambda = 0$ (reweighting factor to leading order in $\lambda$)
Signatures of the BCS phase from complex Dirac spectrum

- Banks-Casher relation extensible to the case of complex Dirac eigenvalues for QCD at zero-temperature, nonzero isospin chemical potential
  - The necessary condition for the derivation is the positivity of the fermionic measure (\( \rightarrow \) QCD inequalities \( \rightarrow \) exclusion of symmetry breaking patterns)
  - For \(|\mu_I| \gg \Lambda_{QCD}\) attractive channel between quarks near the Fermi surface lead to diquark pairing of the BCS type
- The density of the complex Dirac eigenvalues at the origin is proportional to the BCS gap squared
  \[
  \Delta^2 = \frac{2\pi^3}{3N_C}\rho(0)
  \]
  - \(\Delta\) is the BCS gap
  - \(\rho(\nu)\) is a 2d spectral density
  - BC relations derived considering \(Z(M)\) as function of the quark mass matrix \(M\)
    - in the fundamental \(n_l\)-QCD theory. Suitable derivatives/limits yield \(\rho(0)\)
    - in the corresponding effective theory. Suitable derivatives/limits yield \(\Delta^2\)

Kanazawa, Wettig, Yamamoto (2013)
Complex spectrum of the Dirac operator

\[
[\mathcal{D}(\mu I)] \psi_n = (\nu_n) \psi_n
\]

\[\begin{array}{c}
\text{up sector, } \mu I \\
\eta_5 \text{-- hermiticity}
\end{array}
\]

\[
\begin{array}{c}
\text{chiral symmetry}
\end{array}
\]

\[
\tilde{\psi}_n^\dagger [\mathcal{D}(-\mu I)] = \tilde{\psi}_n^\dagger (\nu_n^*)
\]

\[\begin{array}{c}
\text{down sector, } -\mu I, \tilde{\psi}_n = \gamma_5 \psi_n
\end{array}
\]

- Complex eigenvalues \( \nu_n \in \mathbb{C} \)
- \([\mathcal{D}(\mu I), \mathcal{D}^\dagger(\mu I)] \neq 0\), so left and right eigenvectors of \( \mathcal{D}(\mu I) \) do not coincide
- \( \forall \) eigenvalue \( \nu_n \) in the up sector, complex conjugate \( \nu_n^* \) in the down sector
- Simulations at nonzero quark mass: instead of \( \rho(0) \), we look at \( \rho(m + i \ast 0) \) neglecting corrections at first.
Complex spectrum of $\mathcal{D}(\mu_I)$ - Measurement & analysis

**Measurement**  
Scalable Library for Eigenvalue Problem Computations (SLEPc)

- Spectrum measured with SLEPc (Scalable Library for Eigenvalue Problem Computations), set up to obtain, via the Krylov-Schur method, $\sim 150$ complex eigenvalues of $\mathcal{D}(\mu_I)$ (the closest, in modulo to the origin).

**Analysis**

- Spectral density $\rho(\nu)$ extrapolated to $m_{ud}$, by
  - Using kernel density estimation (KDE) as a non-parametric way to estimate the multivariate probability density function from the measured spectrum.
Complex spectrum of \( \Phi(\mu_I) \) - Results, quantitatively

- Match \( \mu_I \)- and \( T \)- dependence of \( \rho(m_{ud}) \) with the boundary of the BEC phase and with the deconfinement crossover.
Complex spectrum of $\Phi(\mu_I)$ - Results, quantitatively

- Match $\mu_I$- and $T$- dependence of $\rho(m_{ud})$ with the boundary of the BEC phase and with the deconfinement crossover.
Modified HRG model \textit{vs} Lattice QCD

\[ \Delta p(T, \mu_I) = \int_0^{\mu_I} d\mu' I_n(T, \mu'_I), \]

\[ \Delta I(T, \mu_I) = \mu_I n_I(T, \mu_I) + \int_0^{\mu_I} d\mu'_I \left(T \frac{\partial}{\partial T} - 4\right) n_I(T, \mu'_I) \]

\[ \frac{p}{T^4} = \frac{c_{2}^{I,Q}}{2} \left( \frac{\mu_I, Q}{T} \right)^2 + \frac{c_{4}^{I,Q}}{24} \left( \frac{\mu_I, Q}{T} \right)^4 + \ldots \]

\[ \frac{r_2(\chi_I, Q)}{T} = \sqrt{2 \frac{c_{2}^{I,Q}}{c_{4}^{I,Q}}}, \quad \chi_I, Q = \frac{\partial^2 p}{\partial \mu_{I,Q}^2}, \quad \mu_{Q}^{\text{crit}} \approx \mu_{I}^{\text{crit}} \cdot \frac{r_2(\chi_Q)}{r_2(\chi_I)} \]
Primordial gravitational waves and black holes

PGW relic density (amplitude of scalar perturbation $A_s = 2.1 \times 10^{-9}$, scale invariant $n_T = 0$ (solid lines), and scale dependent $n_T = 0.25$ (dashed lines) tensor power spectrum from the upper bound on the tensor to scalar perturbation ratio $r = 0.07$ of PLANCK.

fPBH fraction of PBHs with respect to total cold dark matter (CDM) abundance for different lepton asymmetry cases.