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Einführung in die Theoretische Festkörperphysik

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Exercise 13

(Due date: 10.02.2020)

Problem 1 (**Compressibility of a Fermi Gas**) (3 points)

In this exercise, we will review the thermodynamic properties of an ideal Fermi gas of $S = \frac{1}{2}$ particles in the limit of zero temperature ($T \to 0$). The compressibility κ is defined by the equation:

(1)
$$\kappa^{-1} = n^2 \left(\frac{\partial \mu}{\partial n}\right)_T$$

where the μ is the chemical potential and n is the particle density. For the ideal Fermi gas of spinfull particles, these are:

(2)
$$\mu|_{T\to 0} = \epsilon_F = \frac{p_F^2}{2m} \quad , \quad n = \frac{N_\uparrow + N_\downarrow}{V} = 2 \times \frac{1}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3$$

Show that the compressibility of an ideal Fermi gas is given by:

(3)
$$\kappa = \frac{2}{n^2} D(\epsilon_F)$$

where $D(\epsilon_F)$ is the (non-interacting) density of states at the Fermi energy per spin.

Problem 2 (**Compressibility of a Fermi Liquid**) (7 points)

In contrast to the ideal Fermi gas a part of the interactions between $S = \frac{1}{2}$ particles can be taken into account within the theory of Fermi liquids. There, one replaces the originally interacting particles by so-called quasi-particles that are non-interaction but have renormalized physical properties, e.g. an effective mass m^* and a density of states $D^*(E)$.

Consider a Fermi liquid of $S = \frac{1}{2}$ particles. In this case, the chemical potential can be considered as $\mu = \tilde{\epsilon}_{k_F}$, the quasiparticle energy at the Fermi wavevector. Then:

(4)
$$\left(\frac{\partial \mu}{\partial n}\right)_{T\to 0} = \frac{\partial \epsilon_{k_F}}{\partial n} + \sum_{\vec{k}'\sigma'} f_{k_F\sigma;\vec{k}'\sigma'} \frac{\partial n_{\vec{k}'\sigma'}}{\partial n}$$

where:

(5)
$$f_{\vec{k}\sigma;\vec{k}'\sigma'} = \frac{1}{2VD^*(\epsilon_F)} \sum_{l=0}^{\infty} (F_l^S + 4\sigma\sigma' F_l^A) P_l(\cos\vartheta)$$

and $D^*(\epsilon_F)$ is the interacting density of states at the Fermi energy.

a) Using the definition of the compressibility in exercise 14.1, show:

(6)
$$\kappa^* = \frac{1}{n^2}\frac{2D^*(\epsilon_F)}{1 + F_0^S}$$

where κ^* is the compressibility of the interacting Fermi liquid.

Hint #1: It is useful to consider the derivatives in terms of the Fermi wavevector by making the substitution: $\frac{\partial}{\partial n} \rightarrow \frac{\partial k_F}{\partial n}\frac{\partial}{\partial k_F}$. In this case, $\frac{\partial n_{\vec{k}\sigma}}{\partial k_F} = \delta(k_F - |\vec{k}|)$.

Hint #2: It is useful to perform the k-summation as an integral in spherical coordinates $\sum_{\vec{k}'} \rightarrow \frac{V}{(2\pi)^3}\int d\mathbf{k}'$, because $\int_0^{2\pi} d\phi \int_0^{\pi} d\vartheta \sin(\vartheta) P_l(\cos\vartheta) = 4\pi\delta_{l,0}$.

b) Using the definition of $D^*(\epsilon_F) := \frac{m^* p_F}{\pi\hbar^3}$ to show that:

$$\frac{\kappa^*}{\kappa} = \frac{1 + F_0^S/3}{1 + F_0^S} \tag{7}$$

where κ is the compressibility of the Fermi gas obtained from exercise 14.1.