Formation of bound states in the Lindblad approach

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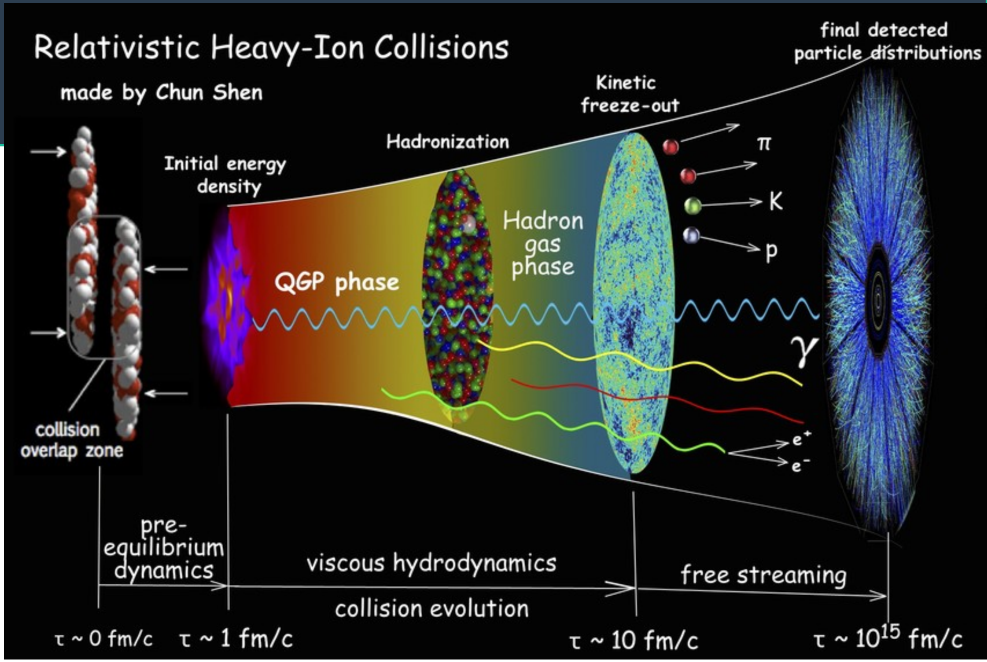
Bound states in HICs

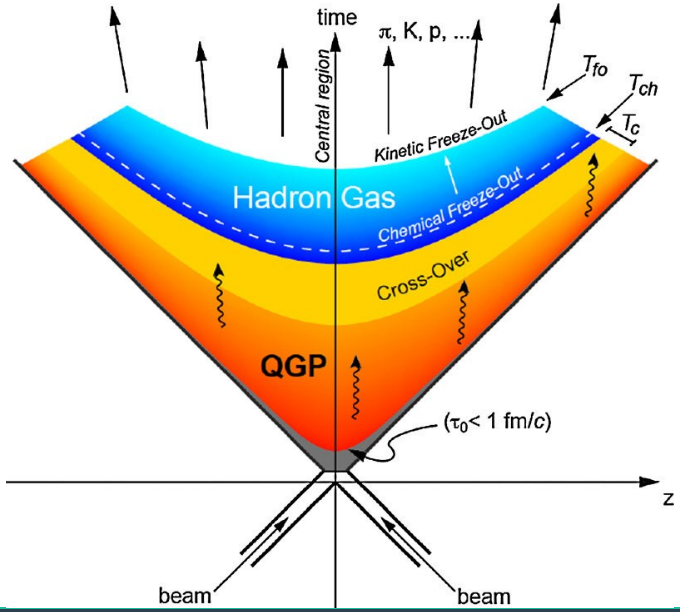
Probe for QGP
(Color Debye Sreening,
QGP Thermometer)

- Charmonia (J/ψ, Y)

- Light Nuclei (deuterons, tritons, and helium nuclei

- Exotic Bound States: (Hypernuclei)





Low binding energy (deuteron ~ 2.3 MeV): "Ice in a Fire"

- Probe of the freeze-out phase (chemical and kinetic) at ~ 150 MeV

- Measuring collective flow (non-static fireball → recombinations)

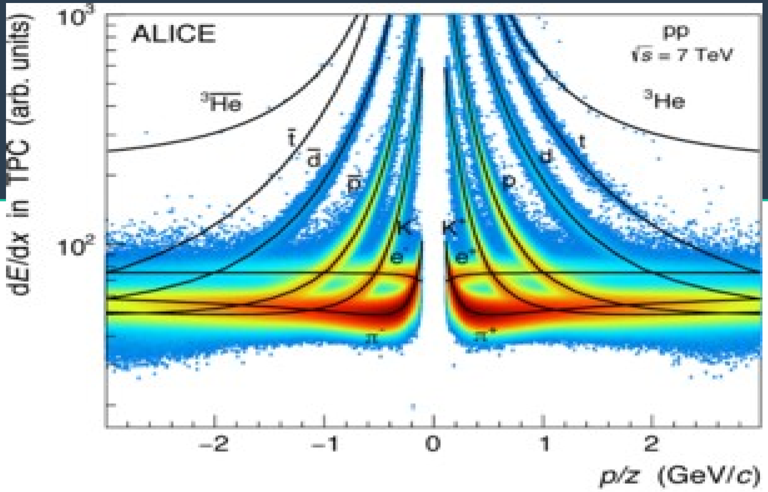
- Searching for the QCD Critical Point

Measuring deuteron with TPC

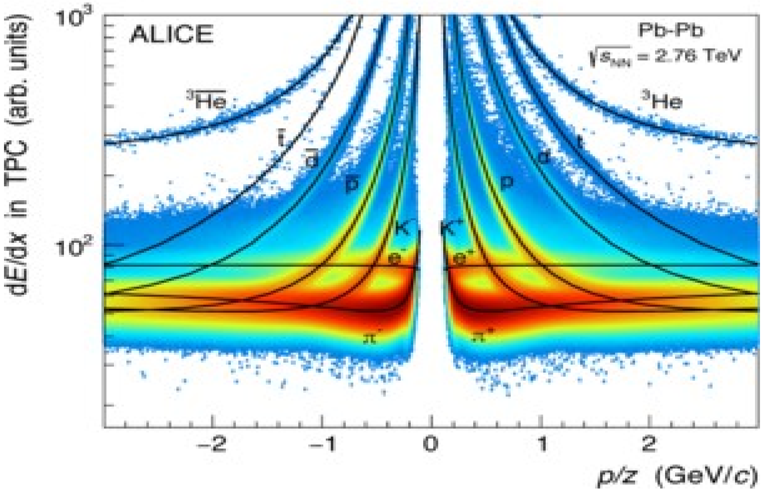
(Time Projection Chamber)

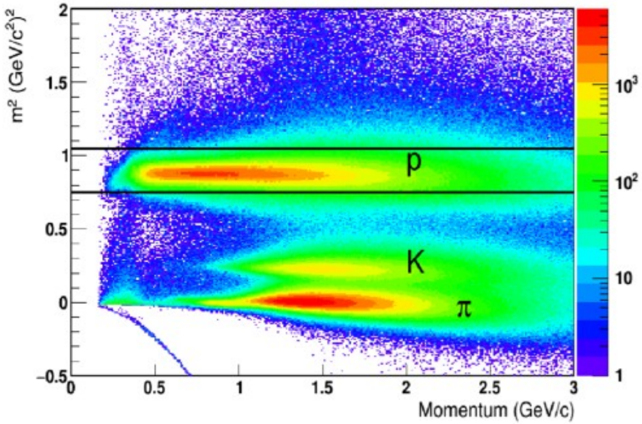
Calculation of dE/dx via Bethe-Bloch formula:

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = Kz^2\frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\left(\frac{2m_e c^2\beta^2\gamma^2 W_{\max}}{I^2}\right) - \beta^2 - \frac{\delta(\beta\gamma)}{2}\right]$$



Measuring of the specific ionization and TOF



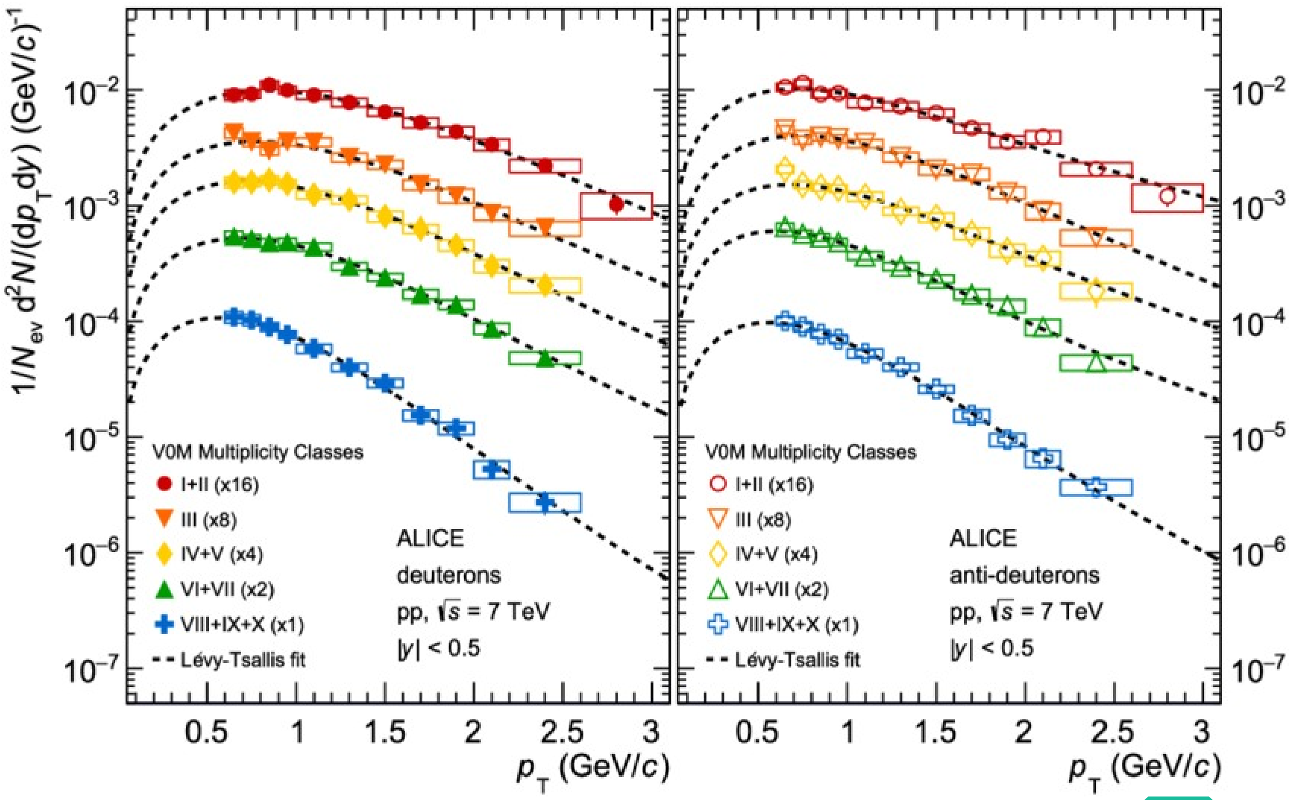


$$m = \frac{p}{\beta\gamma c} = \frac{p}{c}\sqrt{\frac{c^2t^2}{L^2} - 1}$$

p_T -spectra for (anti-)deuteron

- spectrum broad and exp-like for high p_T
- can be explained via coalescence:
- protons and neutrons in the bulk medium: thermal distribution at high p_T as the system cools.
- steepness is influenced by the kin. freeze-out temperature

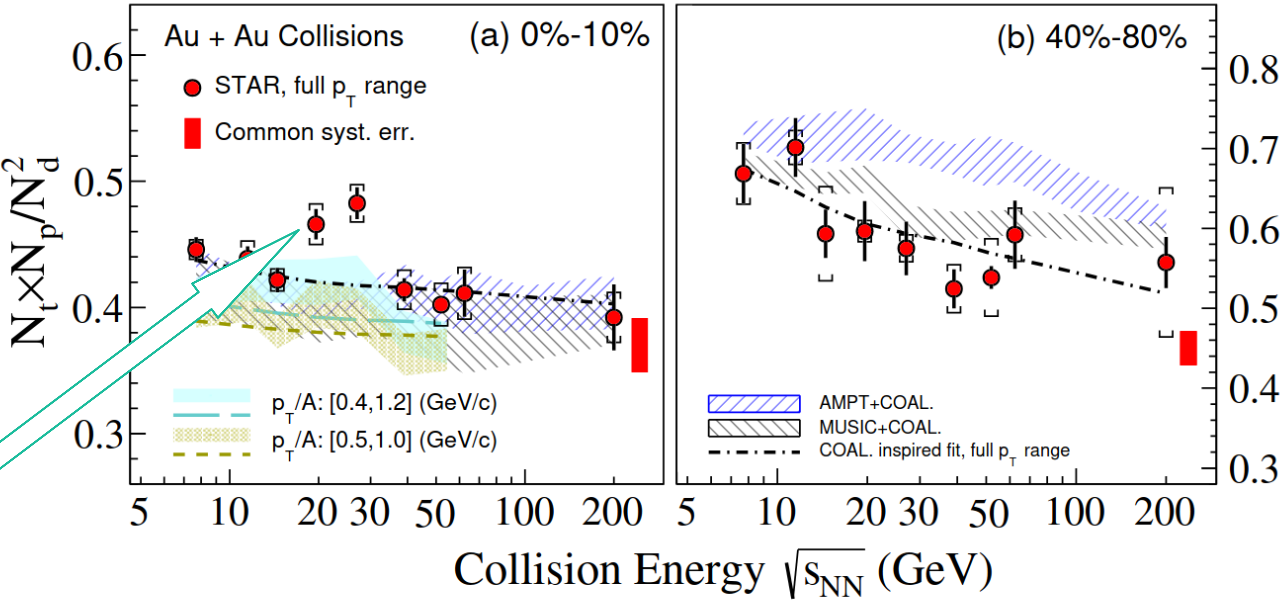
- spectrum flat for low p_T
  - radial flow
  - most (anti-)deuterons for low p_T



The deuteron also helps to understand phase diagram

Rate of tritons, protons and deuterons

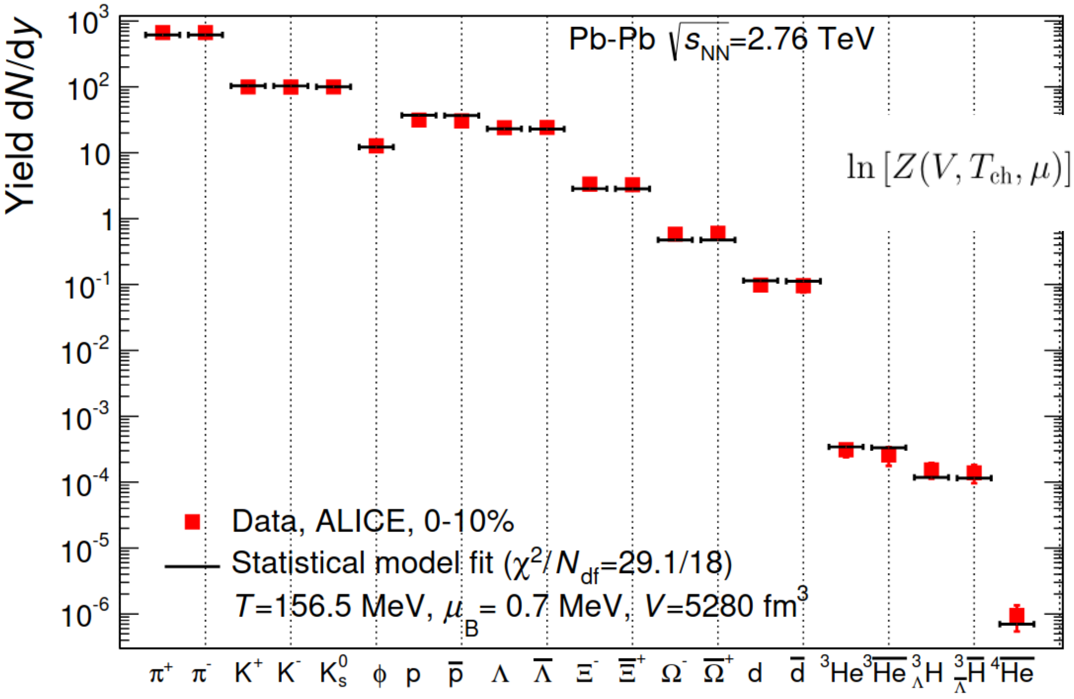
$$O_{tpd} = N_t N_p / N_d^2$$



Sensitivity to density fluctuations

Also helps to distinguish between thermal model and coalescence

Coalescence usually basis for theoretical description



Thermal model predicts particle yields (at chem. freeze out) very well

$$\ln\left[Z(V,T_{\mathrm{ch}},\mu)\right] = \sum_i \left[\ln\ Z_i(V,T_{\mathrm{ch}},\mu)\right] = \sum_i \frac{g_i V}{2\pi^2}\int_0^\infty \mathrm{d}p \left[\pm p^2 \ln\left(1 \pm \lambda_i e^{-\frac{E}{T_{\mathrm{ch}}}}\right)\right]$$

$$N_i = T_{\mathrm{ch}} \frac{\partial \ln\ Z_i(V,T_{\mathrm{ch}},\mu)}{\partial \mu_i} \approx \lambda_i \frac{q_{ii} g_i V T_{\mathrm{ch}}}{2\pi^2} m_i^2 K_2\left(\frac{m_i}{T_{\mathrm{ch}}}\right)$$

Can be used estimating the yields at a given temperature or T_{ch} for given yields.

→ no formation of the deuteron

The coalescence model

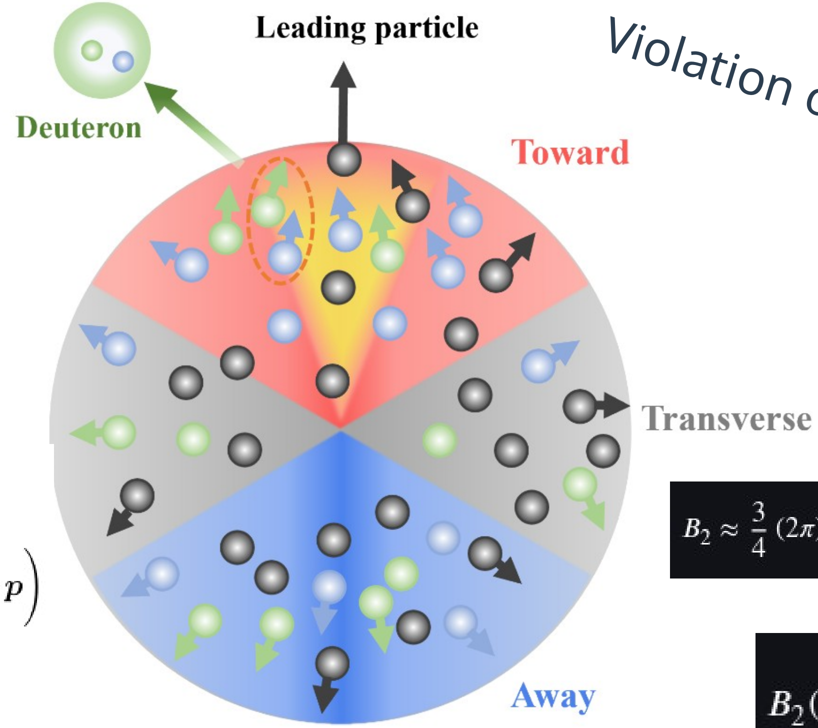
$$E_d \frac{d^3 N_d}{dp_d^3} = B_2 \left(E_p \frac{d^3 N_p}{dp_p^3} \right) \left(E_n \frac{d^3 N_n}{dp_n^3} \right)$$

In Wigner-representation:

$$\frac{\mathrm{d}^3 N_d}{\mathrm{d}\boldsymbol{P}_d^3} = \frac{S_d}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{r}_d \int \mathrm{d}^3\boldsymbol{r} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \mathcal{D}(\boldsymbol{r}, \boldsymbol{p}) \times f_p^W\left(\boldsymbol{r}_d + \frac{\boldsymbol{r}}{2}, \frac{\boldsymbol{P}_d}{2} + \boldsymbol{p}\right) f_n^W\left(\boldsymbol{r}_d - \frac{\boldsymbol{r}}{2}, \frac{\boldsymbol{P}_d}{2} - \boldsymbol{p}\right)$$

with

$$\mathcal{D}(\boldsymbol{r}, \boldsymbol{p}) = \int \mathrm{d}^3\xi e^{-\mathrm{i}q\xi} \varphi_d\left(\boldsymbol{r} + \frac{\boldsymbol{\xi}}{2}\right) \varphi_d^*\left(\boldsymbol{r} - \frac{\boldsymbol{\xi}}{2}\right)$$



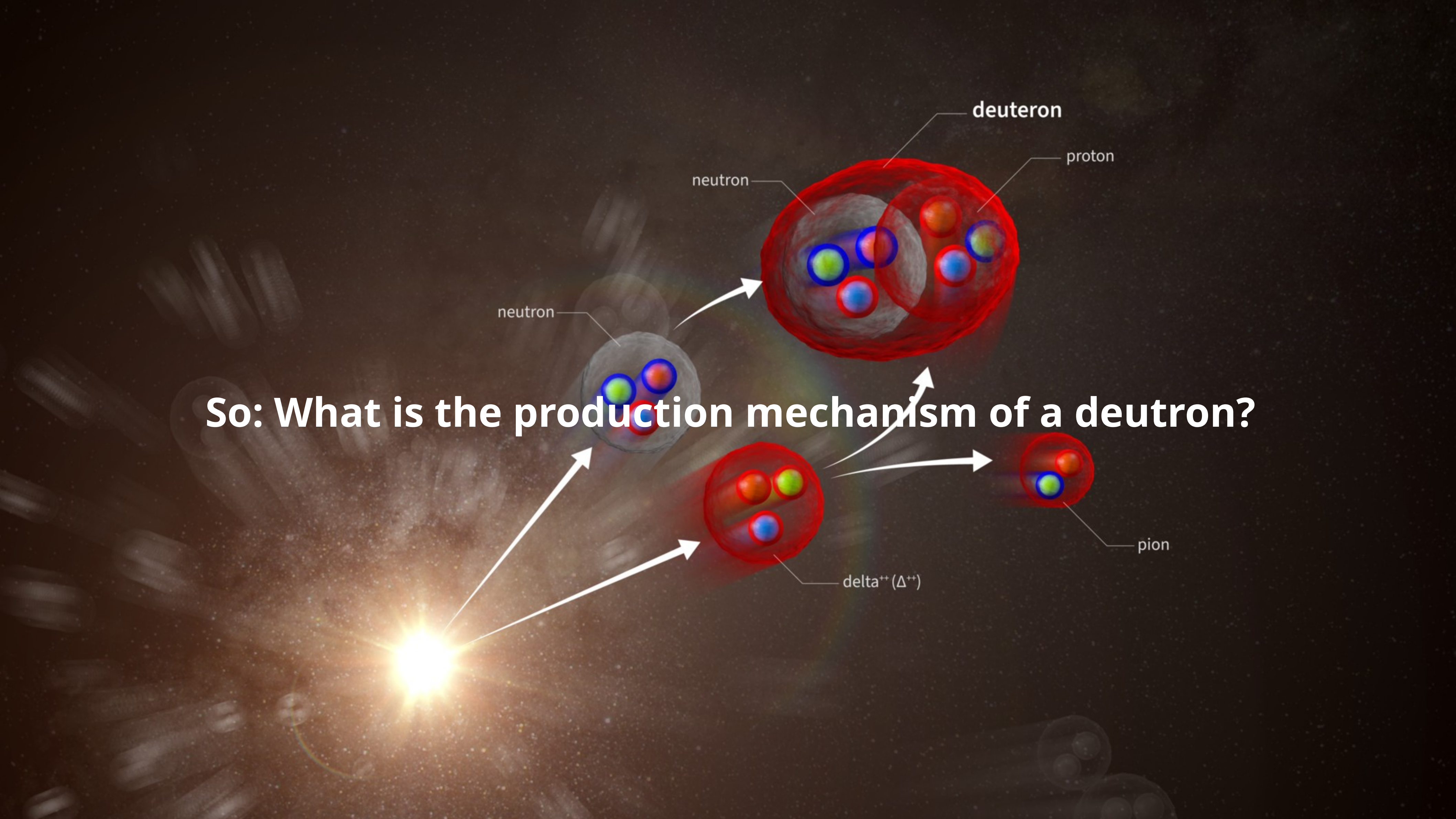
Violation of energy-conservation!

$$B_2 = \frac{3\pi^{3/2}\langle C_d \rangle}{2 m_T R^3}$$

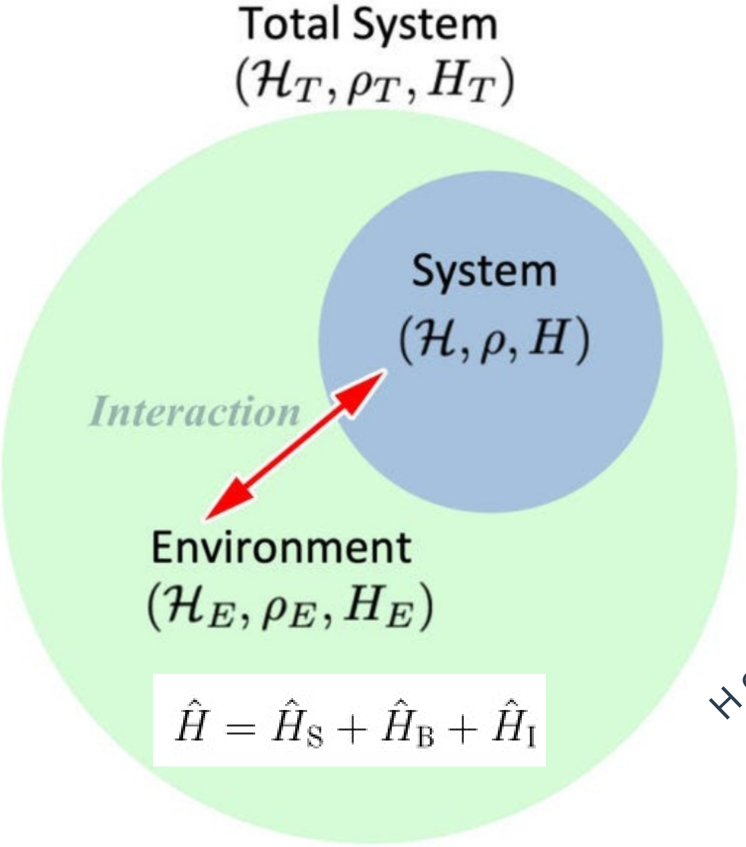
$$B_2 \approx \frac{3}{4}(2\pi)^3 \int d^3 r_1 d^3 r_2 \mathscr{D}(\mathbf{r}_1)\mathscr{D}(\mathbf{r}_2)|\psi_d(\mathbf{r}_1, \mathbf{r}_2)|^2$$

$$B_2(p_T) = \frac{\mathrm{Yield}_d(p_T)}{[\mathrm{Yield}_p(p_T/2)]^2}$$

So: What is the production mechanism of a deutron?



Open quantum system approach



Find dynamics of the reduced density matrix $\rho = \mathrm{Tr}_E\, \rho_T$

Use Caldeira-Leggett Modell for the full system:

$$H_T = \frac{1}{2m}P^2 + \frac{1}{2}M\Omega^2 X^2 + \sum_n \left(\frac{1}{2m_n}p_n^2 + \frac{1}{2}m_n\omega_n^2 x_n^2\right) - X\sum_n c_n x_n + X^2 \sum_n \frac{c_n^2}{2m_n\omega_n^2}$$

H of the Bath

H of the system

Linear coupling to the bath

Counter term including the bath spectrum

Caldeira-Leggett Master Equation

$$\begin{aligned}\rho(x,y,t) &= \int \mathrm{d}x_0 \int \mathrm{d}y_0\, K(x,y,t;x_0,y_0,0)\rho(x_0,y_0,0) \\ &= \int \mathrm{d}x_0 \int \mathrm{d}y_0 \int_{x_0}^{x} \mathcal{D}x_+ \int_{y_0}^{y} \mathcal{D}y_- \\ &\quad \exp\left(\mathrm{i}\int_0^t \mathrm{d}t' \left\{\frac{M}{2}\left[\dot{x}^2(t) - \dot{y}^2(t)\right] - \left[V_{\mathrm{ren}}(x(t)) - V_{\mathrm{ren}}(y(t))\right]\right\}\right) \\ &\quad \times \exp\left[\mathrm{i}\frac{M\gamma}{2}\int_0^t \mathrm{d}t'\,(x\dot{x} + x\dot{y} - y\dot{x} - y\dot{y})(t)\right] \\ &\quad \times \exp\left[-2M\gamma T \int_0^t \mathrm{d}t'\,(x-y)^2(t)\right]\rho(x_0,y_0,0)\,,\end{aligned}$$

- inserting Trotter formula
- motivate stationary state from statistical distribution

Approximations:

- Markovian System -- kills one time integral
- Ohmic heat bath:
- linear source term (coupling)
- high temperatures

Dissipative dynamics of system

But: not norm preserving
and
necessarily positive

$$\dot{\hat{\rho}} = -\mathrm{i}\left[\hat{H},\hat{\rho}\right] - \mathrm{i}\frac{\gamma}{2}\left[\{\hat{p},\hat{x}\},\hat{\rho}\right] - 2m\gamma T\left[\hat{x},\left[\hat{x},\hat{\rho}\right]\right] - \mathrm{i}\gamma\left(\left[\hat{x},\hat{\rho}\hat{p}\right] - \left[\hat{p},\hat{\rho}\hat{x}\right]\right)$$

Lindblad dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\mathrm{i}\left[\hat{H},\hat{\rho}\right] - \frac{1}{2}\left(\left\{\hat{L}^\dagger\hat{L},\hat{\rho}\right\} - 2\hat{L}\hat{\rho}\hat{L}^\dagger\right) = -\mathrm{i}\left[\hat{H},\hat{\rho}\right] + \mathcal{D}$$

Norm-conserving and positive!

$$\hat{L} = \mu\hat{x} + \mathrm{i}\nu\hat{p}$$

Linear combination of x and p?
What are the coefficients?

For the harmonic oscillator, coefficients can be evaluated analytically:

$$\mu^2 = 2\gamma m T\,,$$
$$2\mu\nu = \gamma\,,$$
$$\nu^2 = \frac{\gamma}{8mT}\,,$$

But: no general mechanism to derive Lindblad operators

Obtain coefficients from Wigner transform ⟺ „Diffusion"-coefficients connected to widths in xx, px, and pp of ρ

for example: $\frac{\langle p^2\rangle}{2m} = T$

Coefficients time dependent?

Lindblad equation as diffusion-advection equation

$$\partial_t \vec{u} + \partial_x \vec{f}^{\,x}[\vec{x},\vec{u}] + \partial_y \vec{f}^{\,y}[\vec{x},\vec{u}] = \partial_x \vec{Q}^{\,x}[\partial_x \vec{u}, \partial_y \vec{u}] + \partial_y \vec{Q}^{\,y}[\partial_x \vec{u}, \partial_y \vec{u}] + \vec{S}[t,\vec{x},\vec{u}]$$

$$\vec{f}^x[\vec{x},\vec{u}] = \begin{pmatrix} -2D_{px}\,(x-y)\,\rho_R + \gamma\,(x-y)\,\rho_I \\ +2D_{px}\,(x-y)\,\rho_I + \gamma\,(x-y)\,\rho_R \end{pmatrix},$$

$$\vec{f}^y[\vec{x},\vec{u}] = \begin{pmatrix} -2D_{px}\,(x-y)\,\rho_R - \gamma\,(x-y)\,\rho_I \\ +2D_{px}\,(x-y)\,\rho_I - \gamma\,(x-y)\,\rho_R \end{pmatrix},$$

$$\vec{Q}^x[\partial_x \vec{u}, \partial_y \vec{u}] = \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{1}{2m}\,\rho_R + D_{xx}\,\rho_I\right) + D_{xx}\,\frac{\partial}{\partial y}\,\rho_I \\ \frac{\partial}{\partial x}\left(-\frac{1}{2m}\,\rho_I + D_{xx}\,\rho_R\right) + D_{xx}\,\frac{\partial}{\partial y}\,\rho_R \end{pmatrix},$$

$$\vec{Q}^y[\partial_x \vec{u}, \partial_y \vec{u}] = \begin{pmatrix} \frac{\partial}{\partial y}\left(-\frac{1}{2m}\,\rho_R + D_{xx}\,\rho_I\right) + D_{xx}\,\frac{\partial}{\partial x}\,\rho_I \\ \frac{\partial}{\partial y}\left(\frac{1}{2m}\,\rho_I + D_{xx}\,\rho_R\right) + D_{xx}\,\frac{\partial}{\partial x}\,\rho_R \end{pmatrix},$$

$$\vec{S}[t,\vec{x},\vec{u}] = \begin{pmatrix} [V(y) - V(x)]\;\rho_R + \left[2\gamma - D_{pp}\,(x-y)^2\right]\rho_I \\ [V(x) - V(y)]\;\rho_I + \left[2\gamma - D_{pp}\,(x-y)^2\right]\rho_R\,, \end{pmatrix}$$

$$\vec{u} = \vec{u}(\vec{x},t) = (\rho_I(x,y,t), \rho_R(x,y,t))^T$$

$$\partial_t \tilde{\vec{u}} = \left[\begin{pmatrix} D_{xx} & 0 \\ 0 & D_{xx} \end{pmatrix} \partial_q^2 + \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \partial_r - 4\begin{pmatrix} D_{pp} & 0 \\ 0 & D_{pp} \end{pmatrix} r^2\right] \tilde{\vec{u}} + \left[\begin{pmatrix} 0 & \frac{1}{2m} \\ -\frac{1}{2m} & 0 \end{pmatrix} \partial_q \partial_r \right.$$
$$\left. +4\begin{pmatrix} 0 & D_{px} \\ -D_{px} & 0 \end{pmatrix} r\partial_q + \begin{pmatrix} 0 & V(r-q) - V(r+q) \\ V(r+q) - V(r-q) & 0 \end{pmatrix}\right] \tilde{\vec{u}}$$

Terms can be interpreted clearly:

- exponential suppression with D_{pp} – decoherence
- potential as source
- D_{xx} pure spatial diffusion
- D_{px} ~ velocity field towards diagonal

Decoherence

System gets less coherent during the process of thermalization
(deconstruction of quantum superposition)

Dominant term in Lindblad-equation: $$\partial_t \rho_S(x,x',t) = -\gamma \left\{ \frac{(x-x')^2}{\lambda_T} \right\}^2 \rho_S(x,x',t)$$

with

$$\lambda_T = \frac{(\hbar)}{\sqrt{2M(k_B)T}} \Longleftrightarrow \tau_D^{-1} = \gamma \left(\frac{x-x'}{\lambda_T} \right)^2$$

Wigner transform of CLME shows phase in phase space:

$$\dot{W}_{\text{int}} \approx -\left(\frac{D\Delta x^2}{\hbar^2} \right) W_{\text{int}} \Longleftrightarrow W_{\text{int}} \sim \cos\left(\frac{\Delta x}{\hbar} p \right)$$

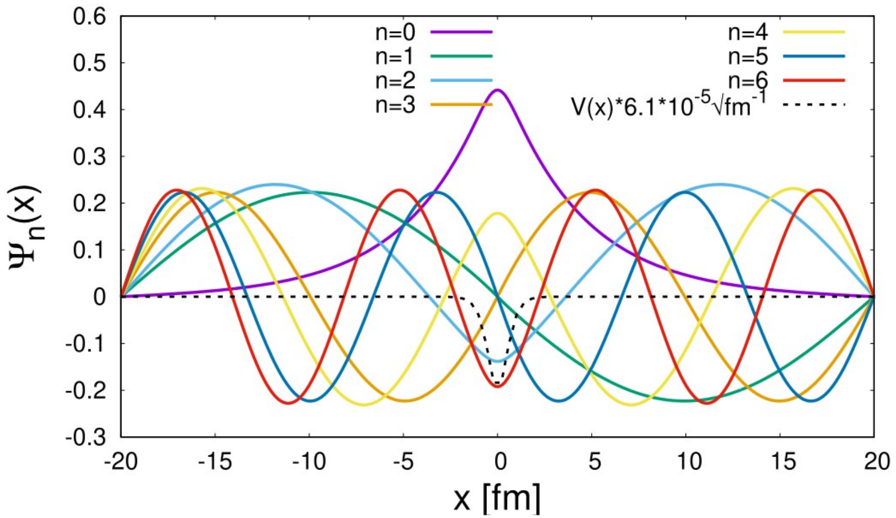
Study decoherence via purity of the system: $$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Tr}\rho^2 = -\frac{2M\gamma k_B T}{\hbar} \left(\langle x^2 \rangle - \langle x \rangle^2 \right)$$

Relaxation time is not directly connected with decoherence time!

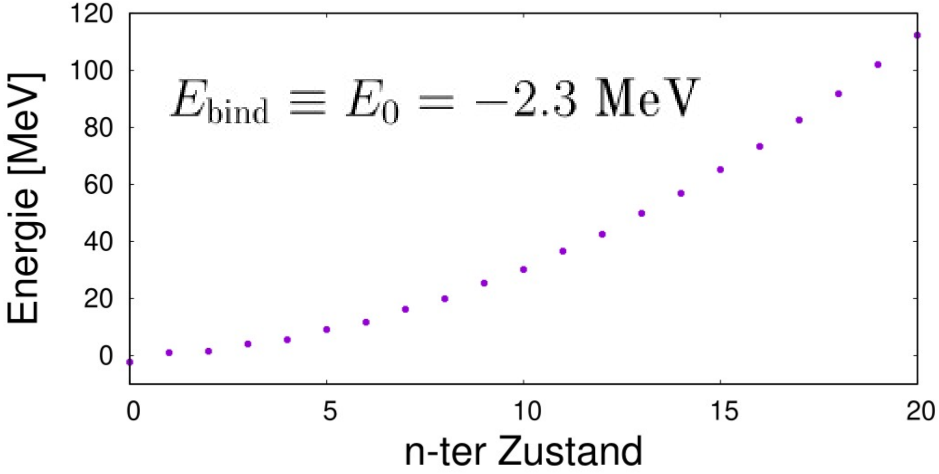
Bath in Lindblad approach is Markovian! -- No backflow from bath to system $$\hat{H}_{\text{int}} = x \sum_n c_n q_n$$

But!!: decoherence is bath dependent, and different for different baths (no momentum dependence)

A quantum bound state with deuteron parameters



$$V(x) = \begin{cases} -V_0 \frac{1}{\cosh^2(\alpha x)}, & \text{für } |x| \leq 20\text{fm} \\ \infty, & \text{für } |x| > 20\text{fm} \end{cases}$$

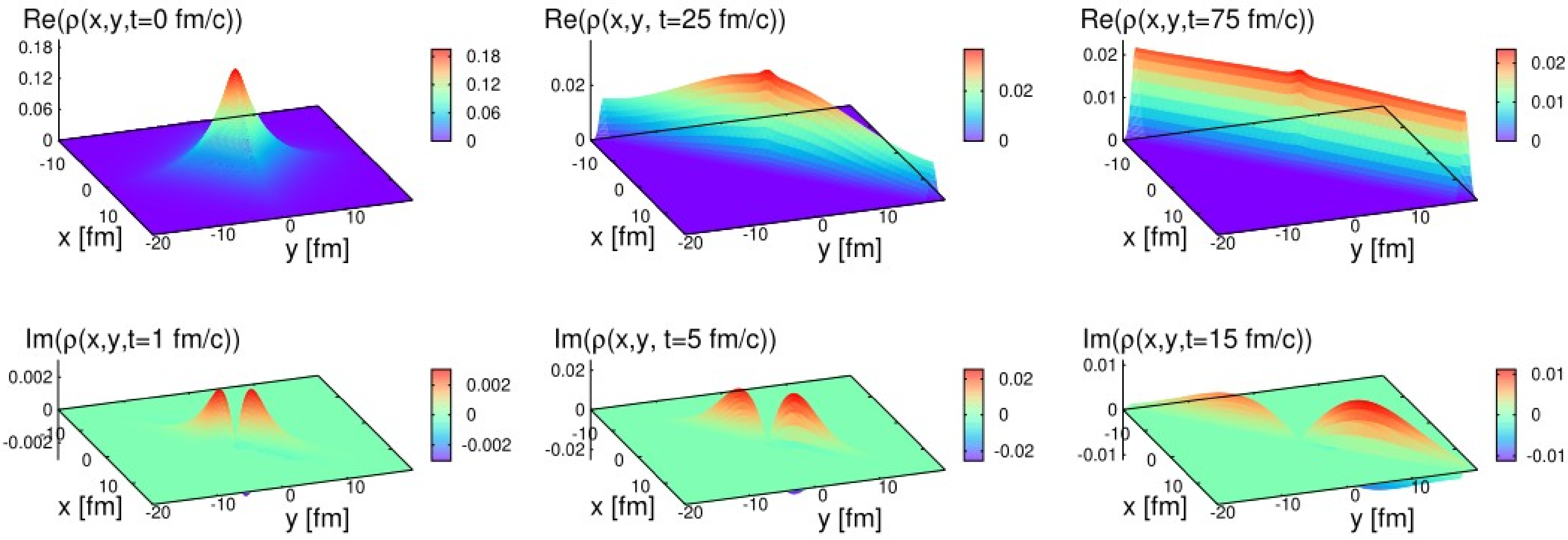


Compute ψ with shooting method

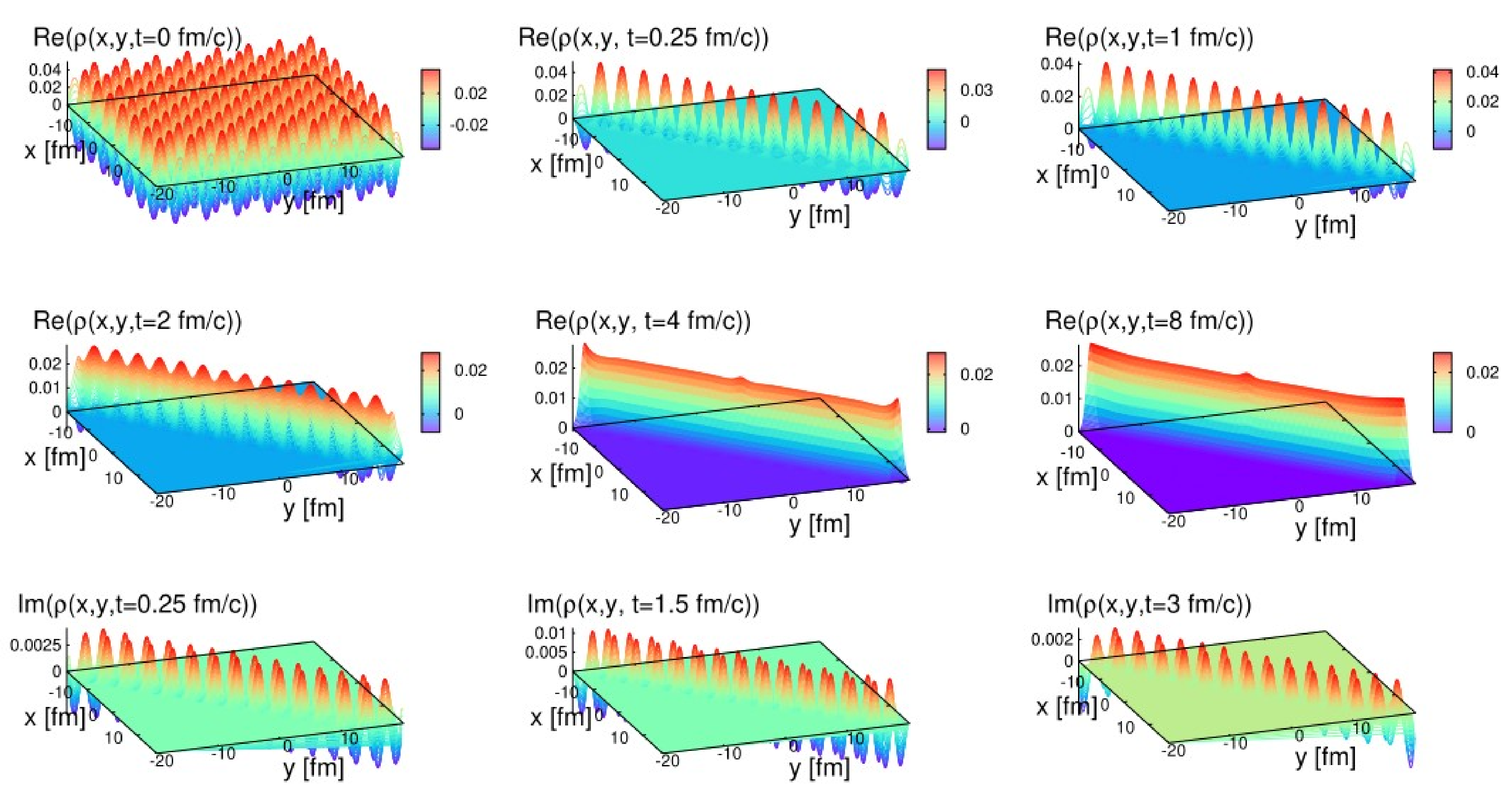
and use it for initial condition:

$$\rho(x, y, 0) = \sum_{m,n=0}^{N} c_{mn} \langle x|\psi_m\rangle \langle\psi_n|y\rangle$$

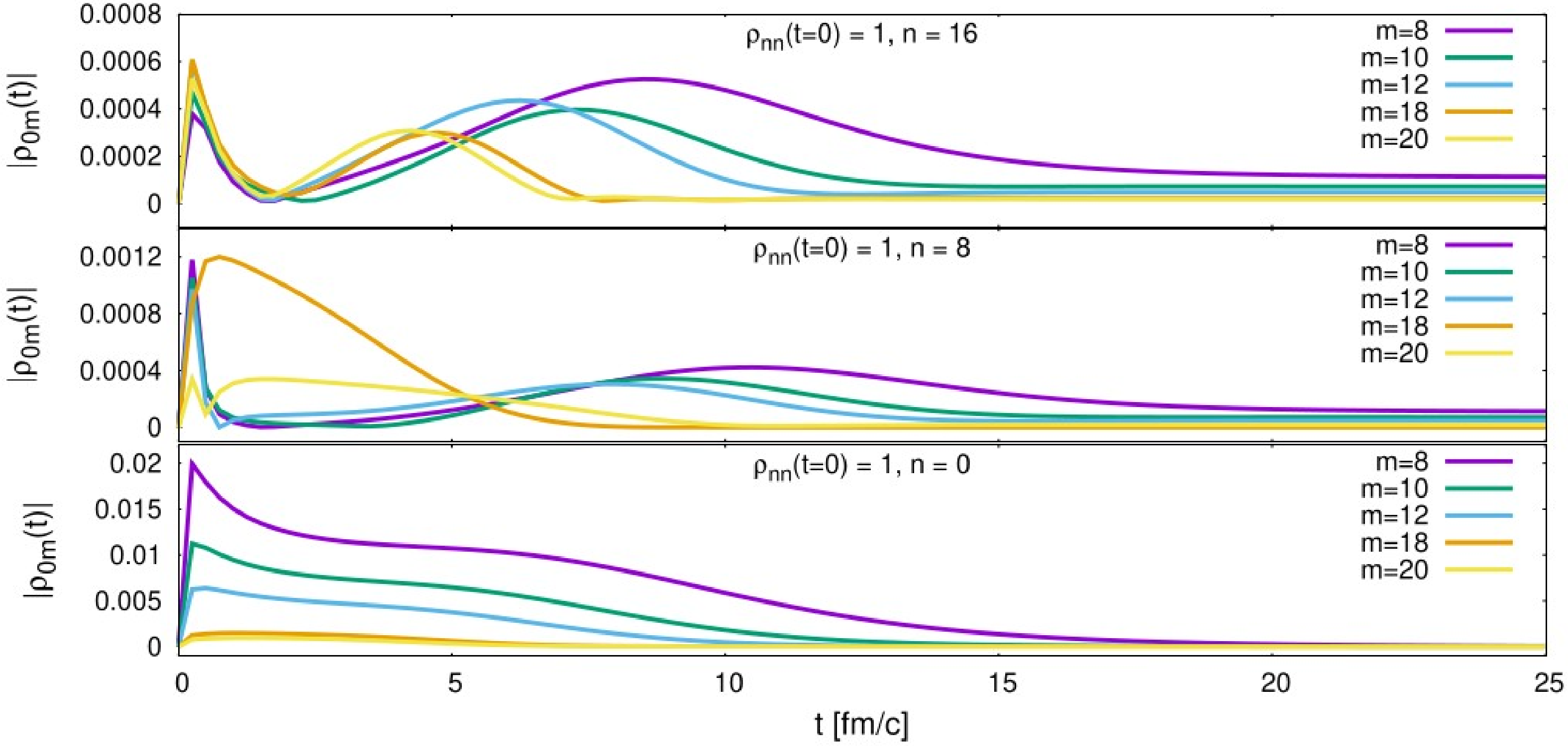
Initial conditions – final distribution – bound state



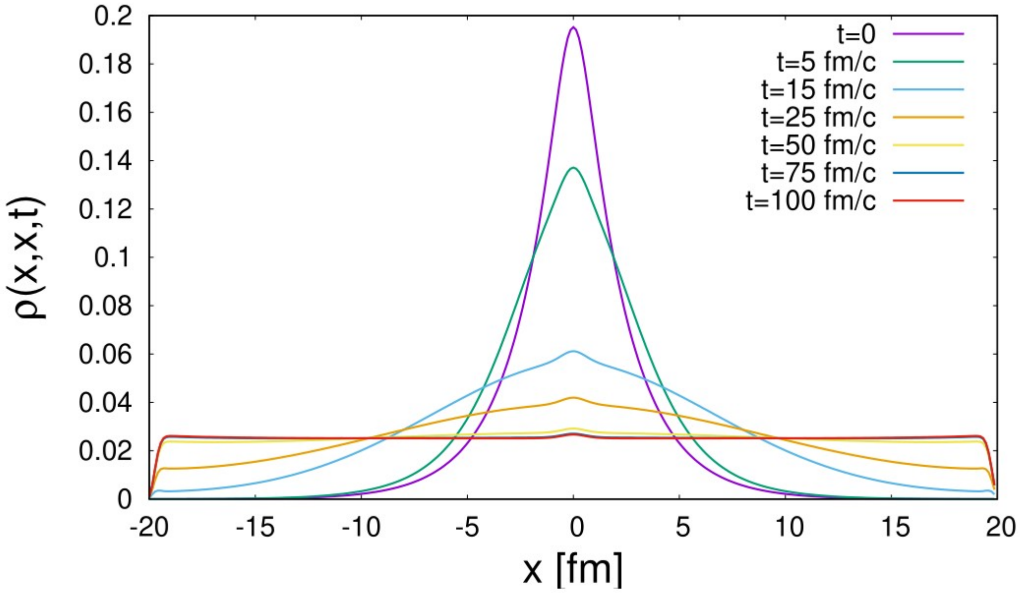
Initial conditions – final distribution – 16^{th} state

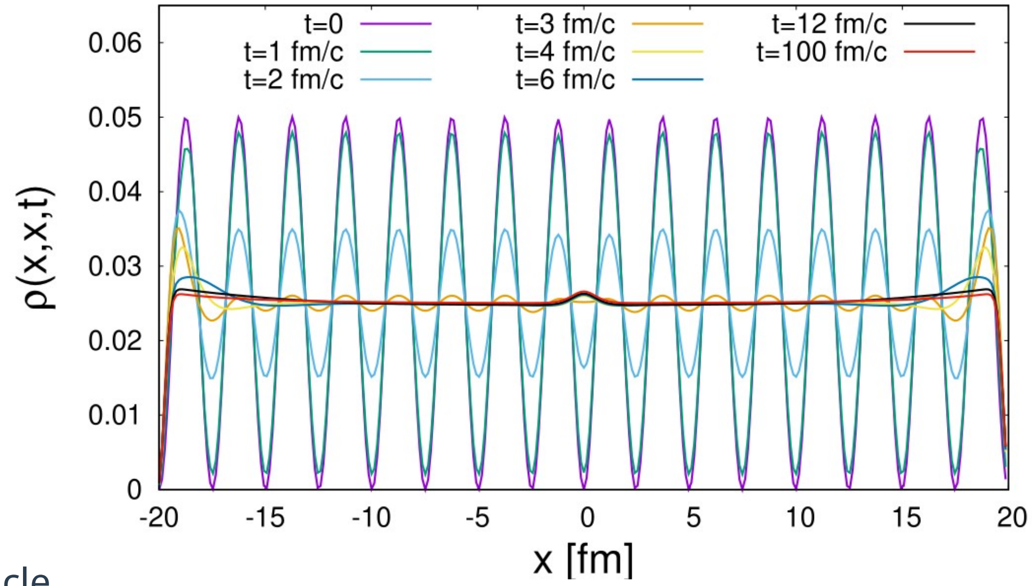


Decoherence



ρ(x,x,t) for initial n=0 and n=16

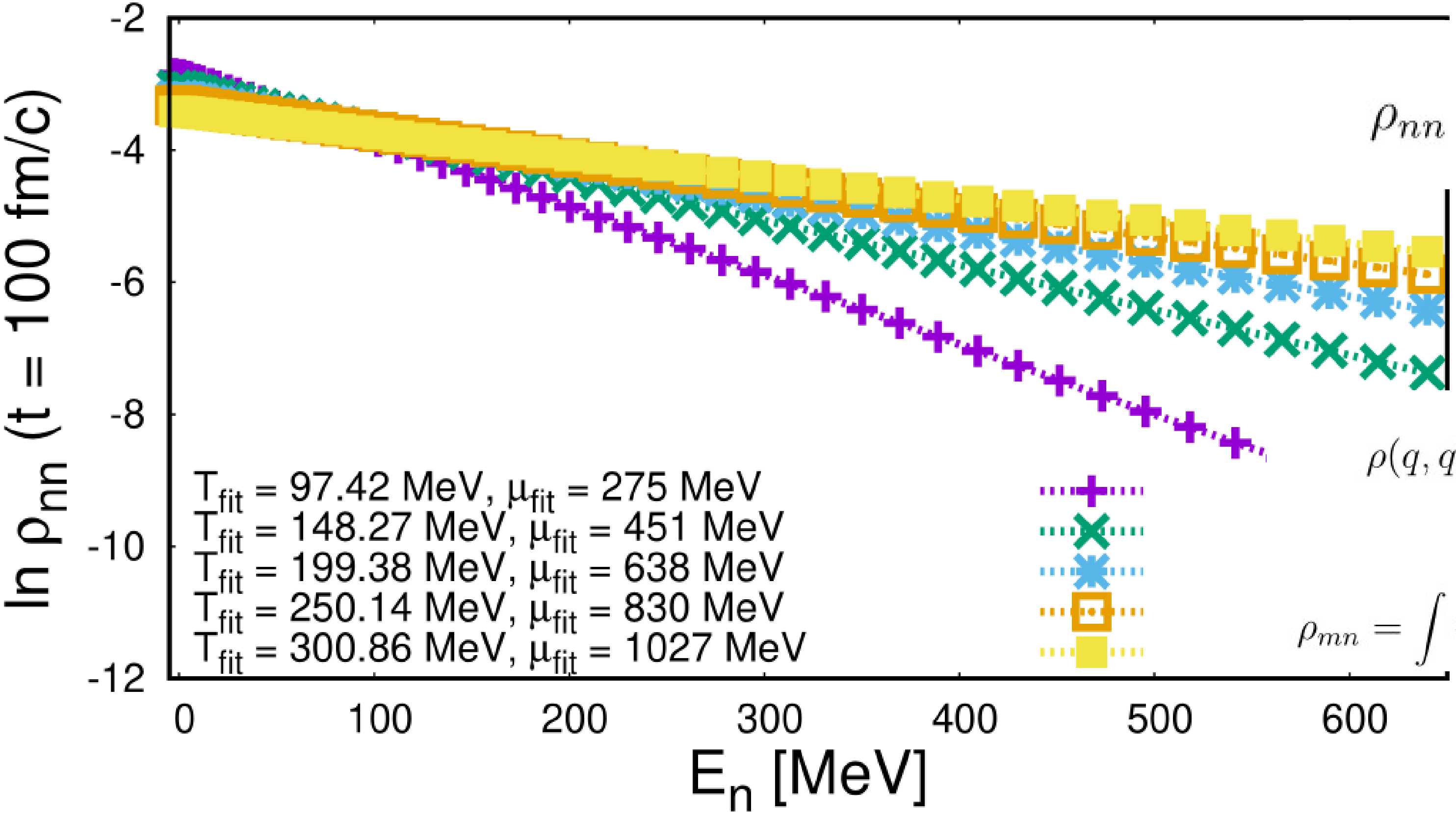




$$\rho_{\text{free}}(x, y, t \to \infty) = \frac{1}{L} e^{-\frac{mT}{2}(x-y)^2}$$

...valid to assume free particle

Distribution of ρ_{nn} for stationary system

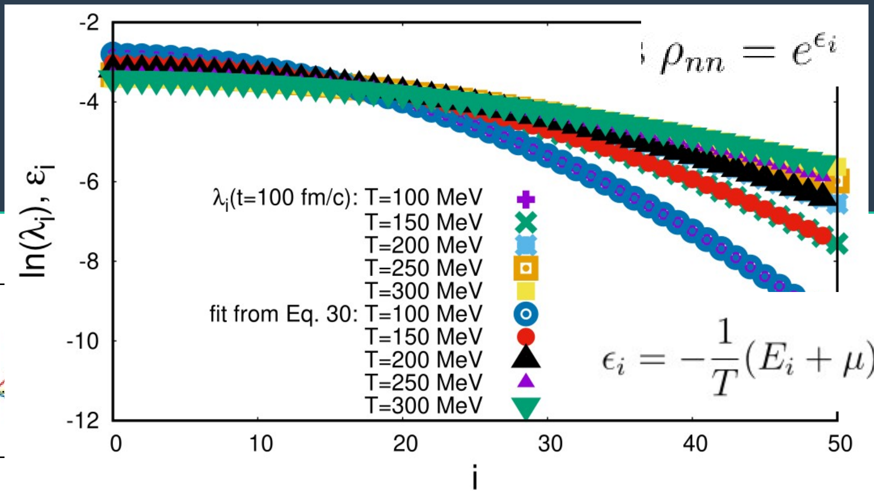


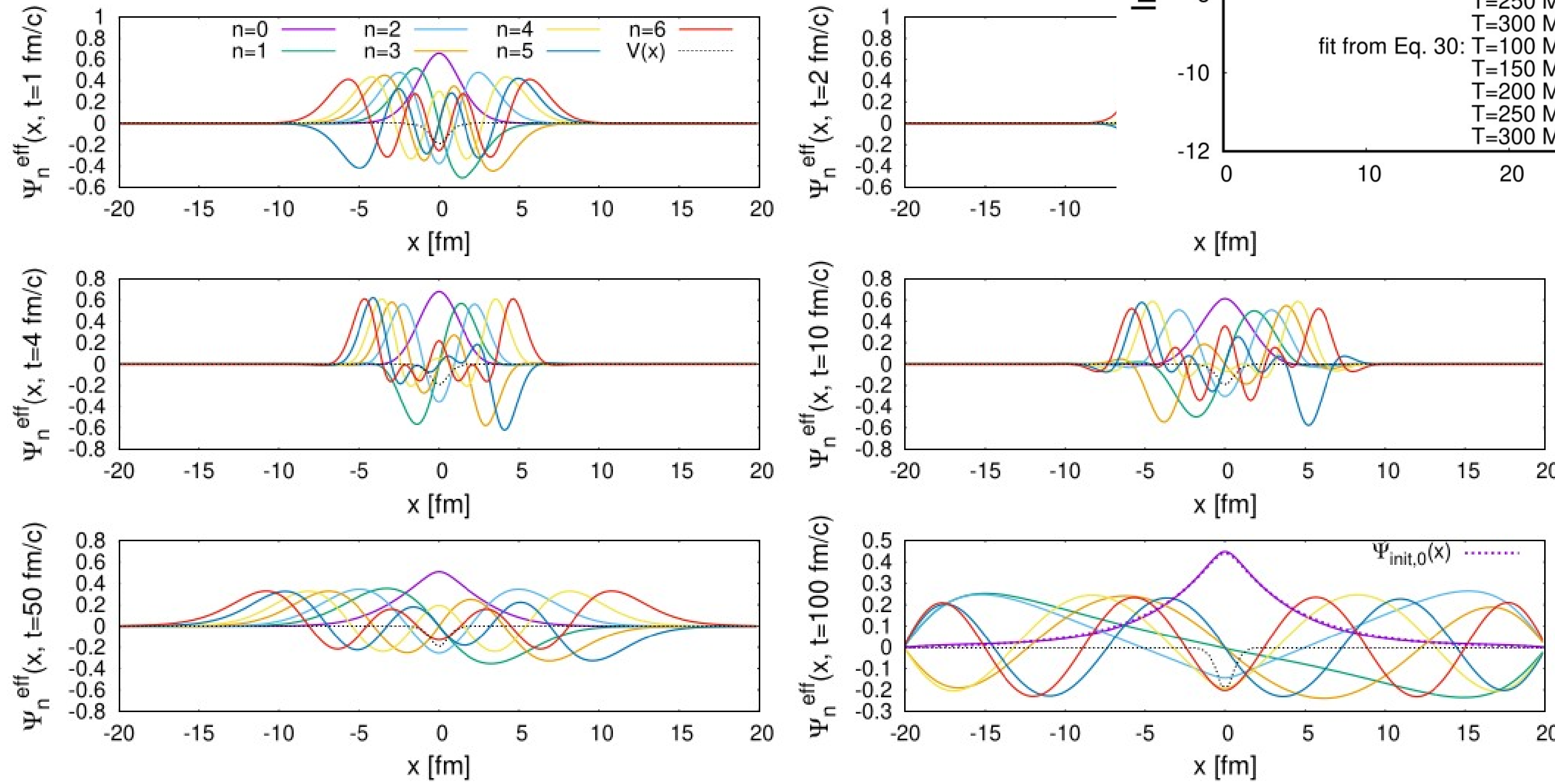
$$\rho_{nn} = \exp\left[-\frac{1}{T}(E_n - \mu)\right]$$

$$\rho(q, q') = \sum_{mn} \rho_{mn}\, \psi_n^*(q')\psi_m(q)$$

$$\rho_{mn} = \int \mathrm{d}q \int \mathrm{d}q'\, \psi_n^*(q')\rho(q, q')\psi_m(q)$$

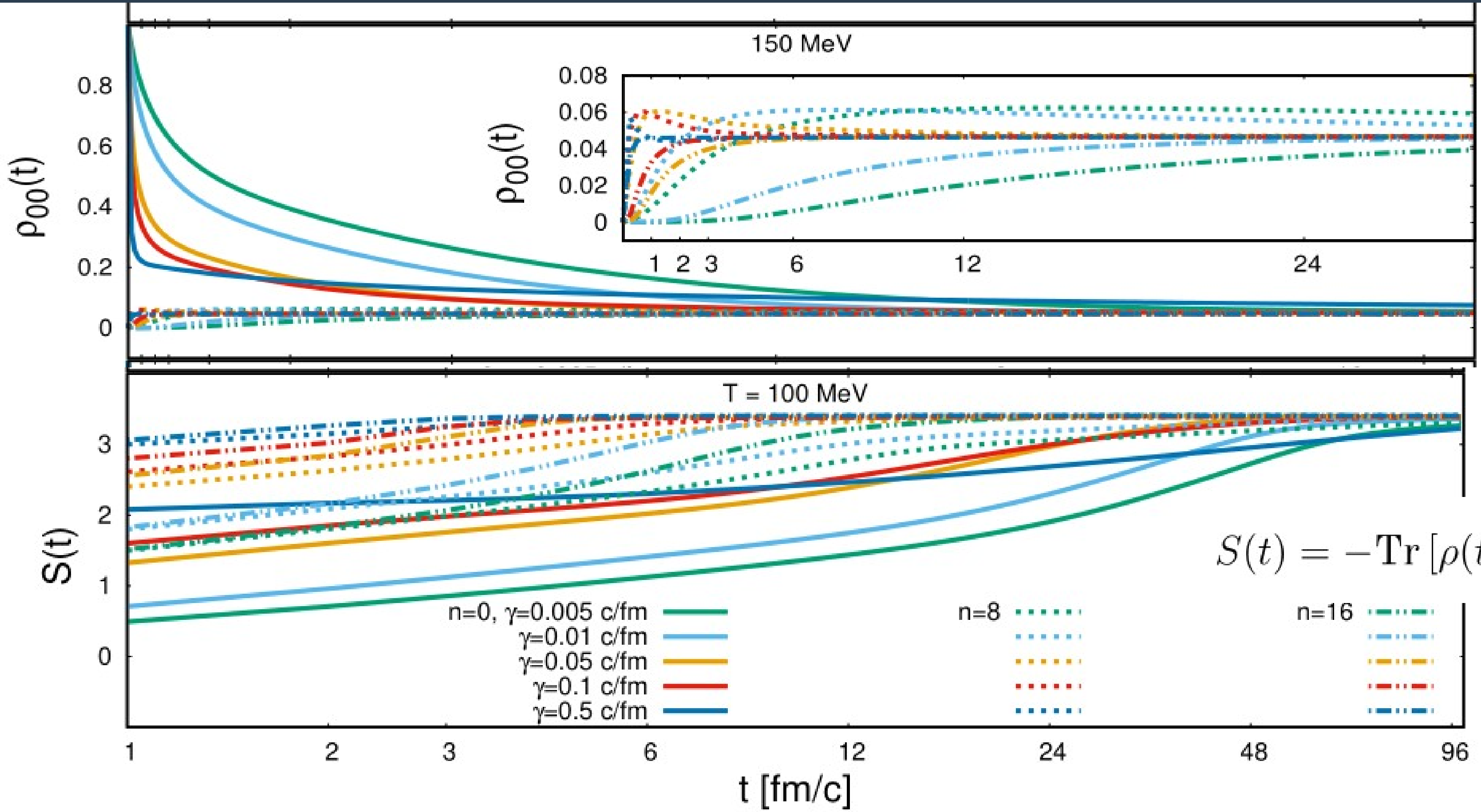
Does the system really thermalize?





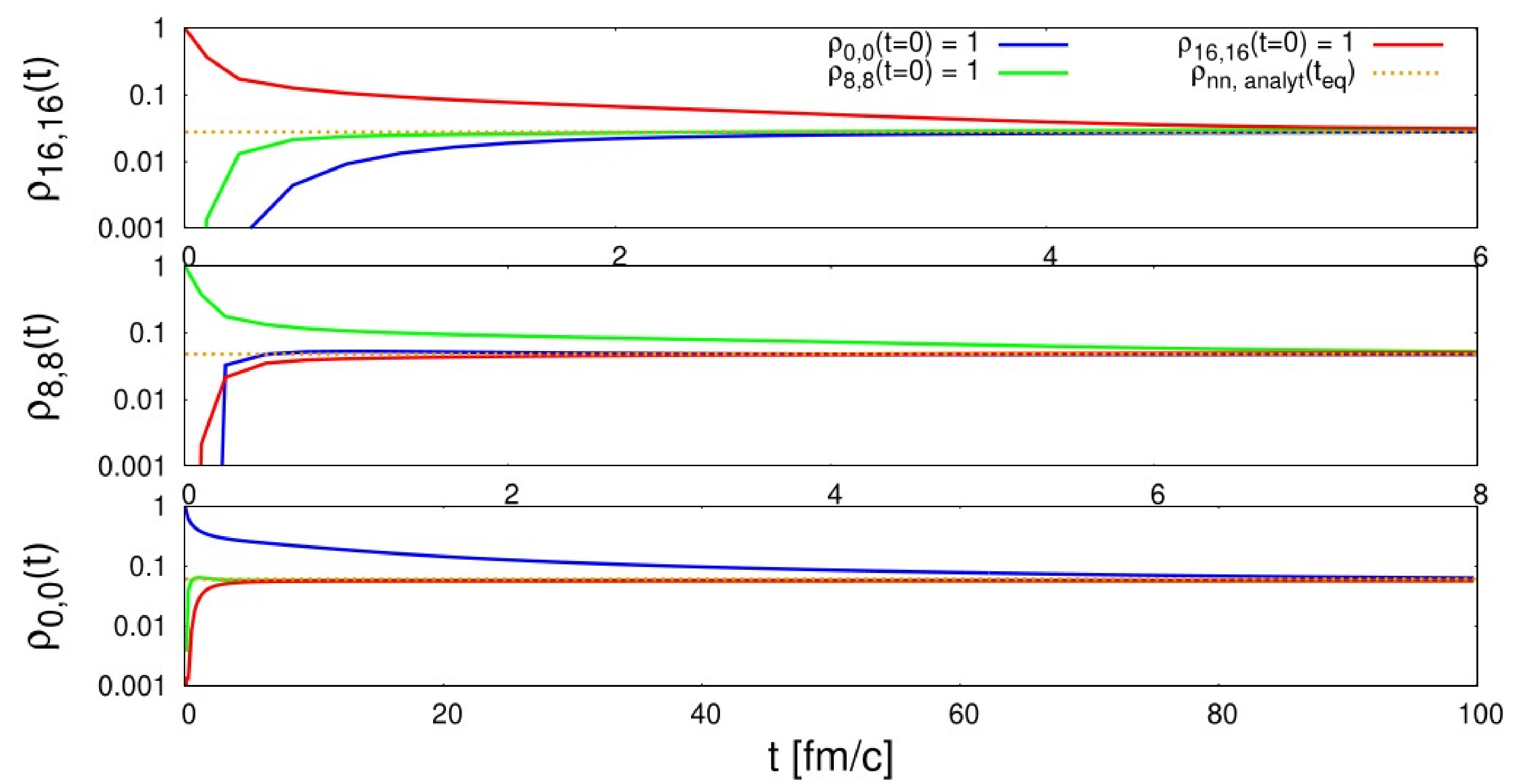
$$\ln\left(\lambda_i^{\rho}\right) = -\frac{\tilde{E}_i}{T}$$

Bound state and Entropy – different therm. times

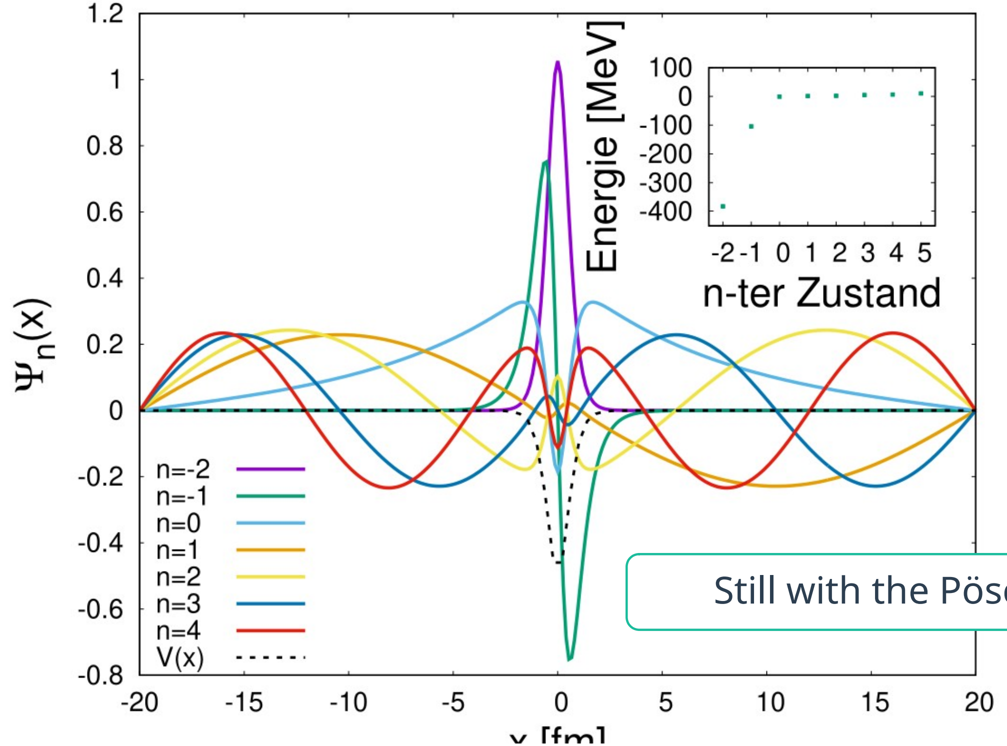


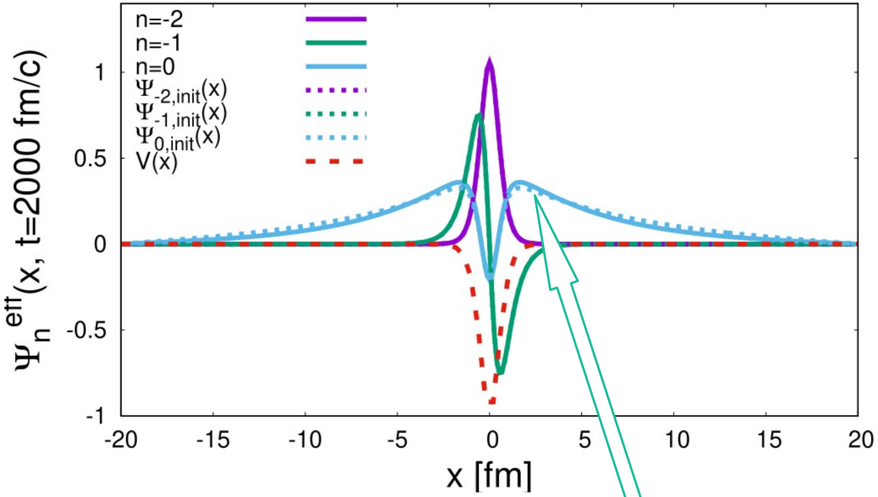
$$S(t) = -\mathrm{Tr}\left[\rho(t)\ln\rho(t)\right]$$

All (initially differerent) states reach the same result, but in different times!



Conception of a system with three bound states

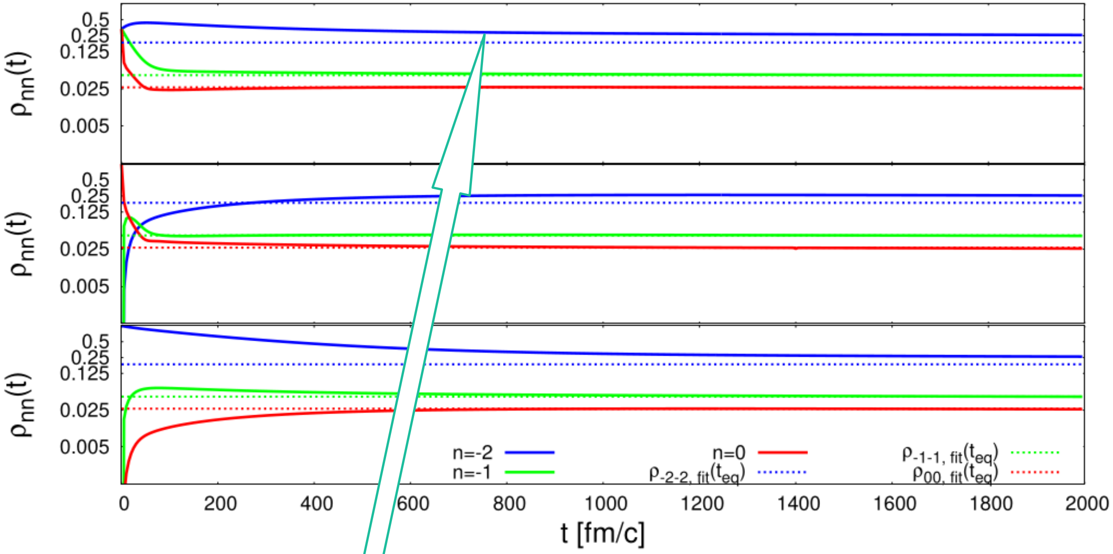


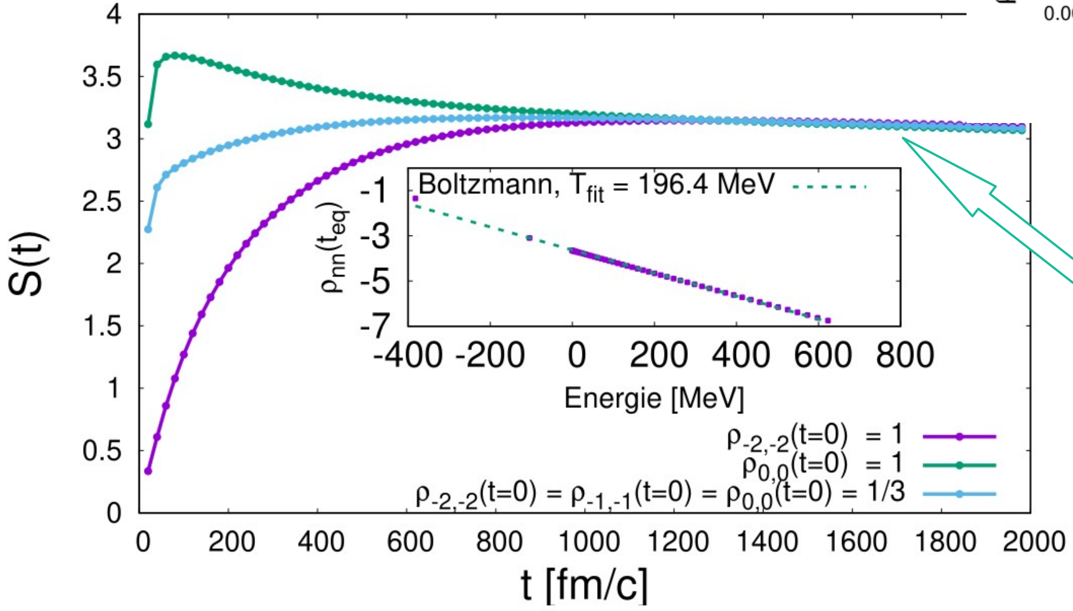


Still with the Pöschl-Teller Potential

Small difference between the diagonalized „thermal" eigenstates and the initial wave functions

Do Lindblad dynamics thermalize for all systems?

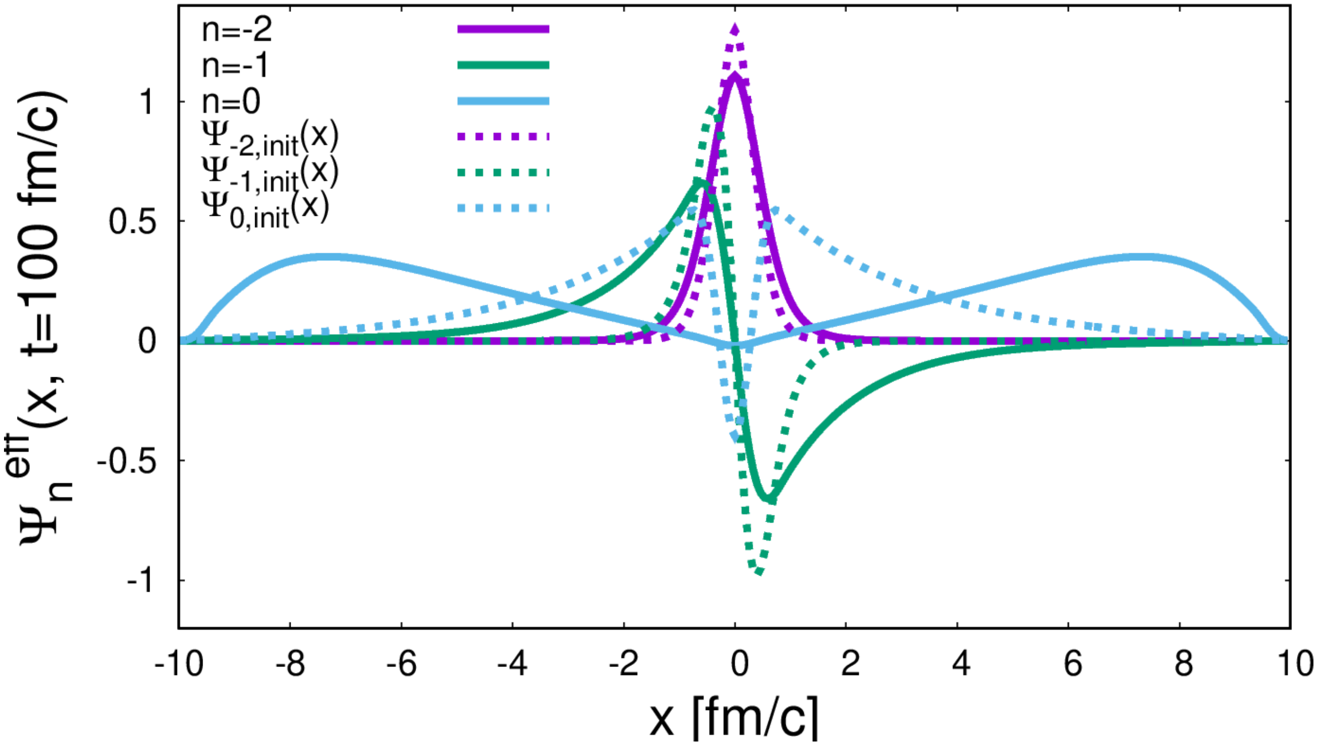




Lowest bound state stays above the analytic result

Entropy shows, that system is thermalized

So what about other potentials? J/ψ?



$$V = \begin{cases} \frac{1}{2}k|x| - V_{SB} < 0 \\ 0 \end{cases}$$

$k = 1.724$ GeV

$V_{SB} = 0.7652$ GeV

To conclude…

-Thermalization is reached for systems, where the impact of the potential is weak

- … this results form the choice of the parameters (motivated from h.o.)

- diffusion coefficients are motivated from „right“ equilibrium

- goal to find a more general method to motivate diffusion coefficients

- … diffusion coefficients are connected to the Lindblad operators

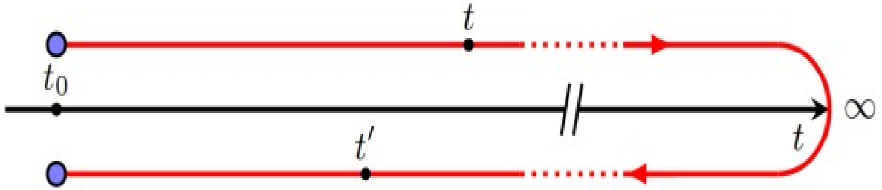
- … to describe the formation of more strongly bound states, as of the J/ψ

Derivation of a master equation of the CLM

Starting point is the path intergral:

$$(x_b\, t|\, x_a t_0) = \prod_i \oint \mathcal{D}q_i \int_{x_a}^{x_b} \mathcal{D}x \, \frac{1}{\prod_i Z_i} \exp\left\{-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t \left[\frac{M}{2}\dot{x}^2 - V(x)\right]\right\}$$
$$\times \exp\left\{-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t \sum_i \left[\frac{m_i}{2}(\dot{q_i}^2 + \Omega_i^2 q_i^2) - \sum_i c_i q_i(t) x(t)\right]\right\}$$
$$= \int_{x_a}^{x_b} \mathcal{D}x \, \exp\left\{-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t \left[\frac{M}{2}\dot{x}^2 - V(x)\right]\right\}$$
$$\times \prod_i \oint \mathcal{D}q_i \frac{1}{\prod_i Z_i} \exp\left\{-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t \sum_i \left[\frac{m_i}{2}(\dot{q_i}^2 + \Omega_i^2 q_i^2) - \sum_i c_i q_i(t) x(t)\right]\right\}$$

Solve at keldysh-contour



Dissipation

$$C(t,t') = -\mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \rho_{\mathrm{Bad}}(\omega') \sin(\omega'(t-t'))$$

$$\rho_{\mathrm{Bad}}(\omega') = 2\pi \sum_i \frac{c_i^2}{2M_i\Omega_i} \delta(\omega' - \Omega_i)$$

Fluctations

$$A(t,t') = \sum_i c_i^2 \frac{1}{M_i\omega_i} \coth\left(\frac{\omega_i}{2k_B T}\right) \cos(\omega_i(t-t'))$$
$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \rho_{\mathrm{Bad}}(\omega') \coth\left(\frac{\omega'}{2k_B T}\right) \cos(\omega'(t-t'))$$

$$\frac{1}{2}[A(t,t') + C(t,t')] = G(t,t')$$