Covariant formulation of kinetic theory for systems without relativistic invariance

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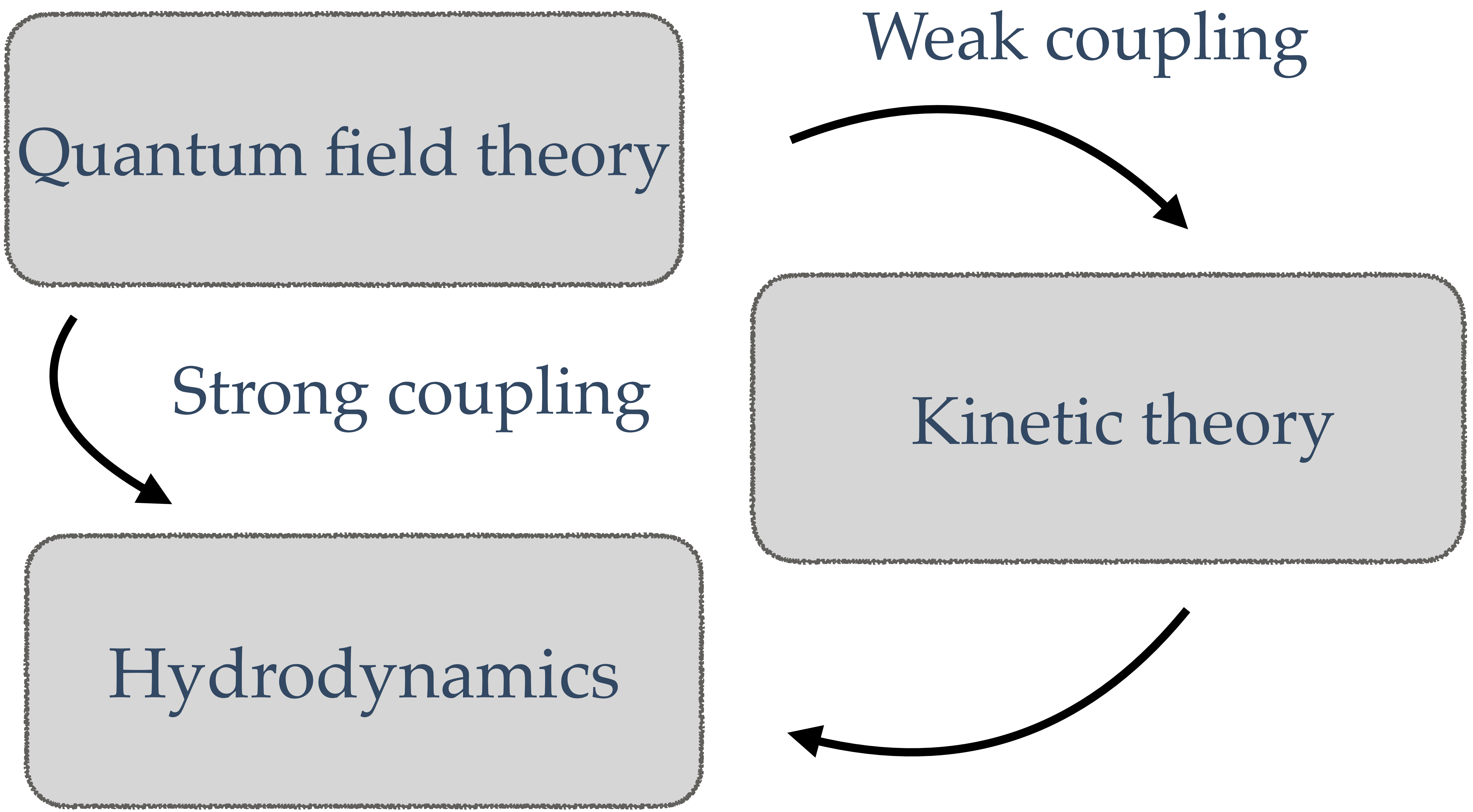
arXiv:2407.08805

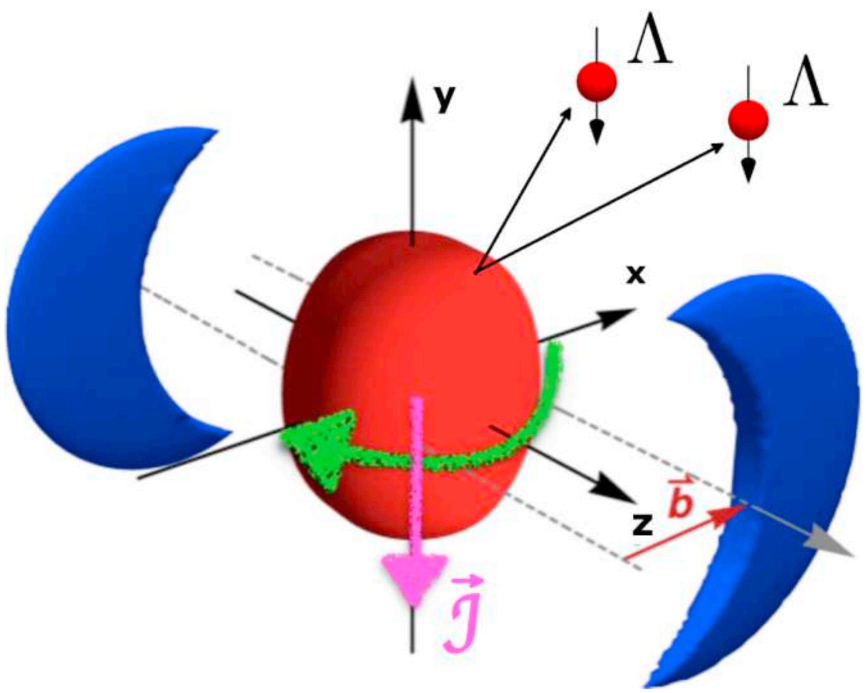
Frankfurt, 28.11.2024

Effective descriptions



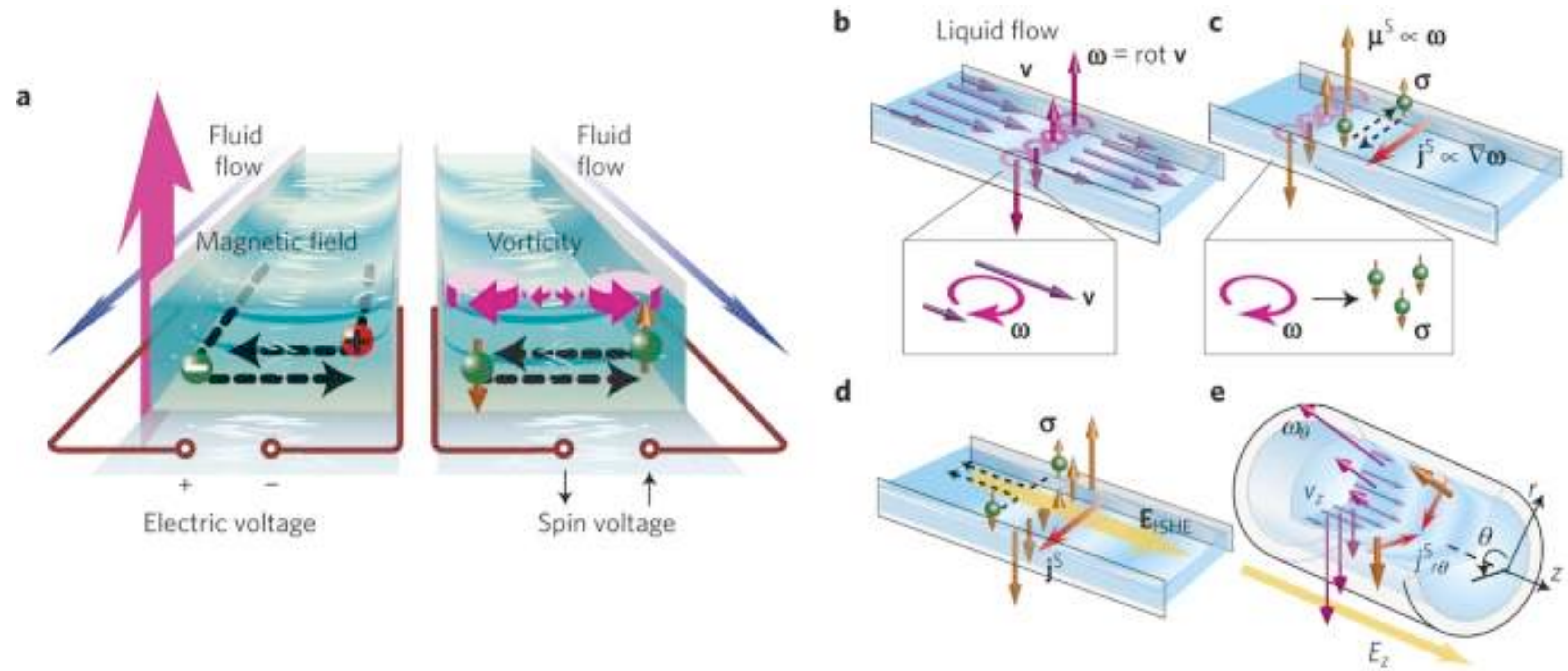
Hydrodynamics with spin and rotations

Heavy-ion collisions



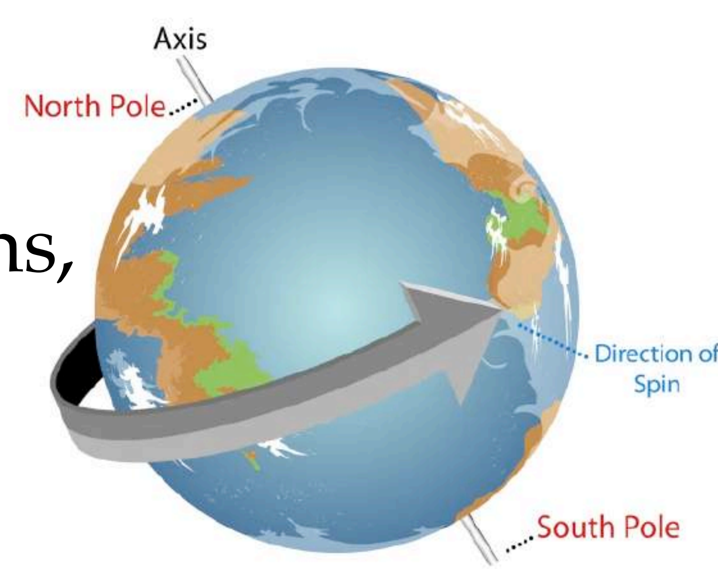
Florkowski et al.

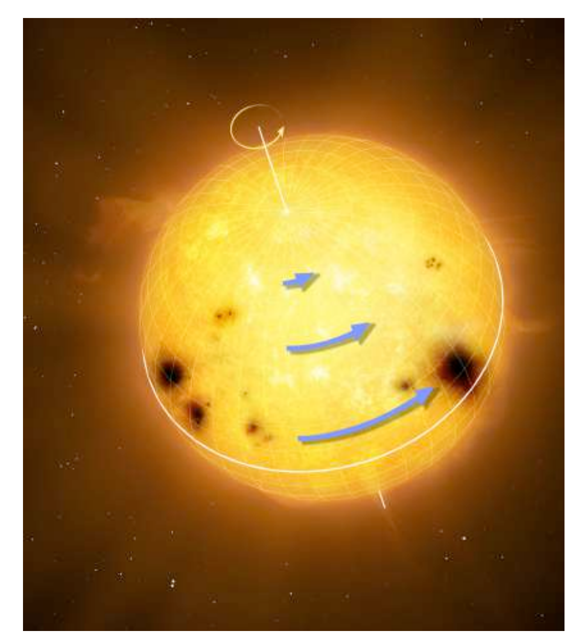
Electron flows



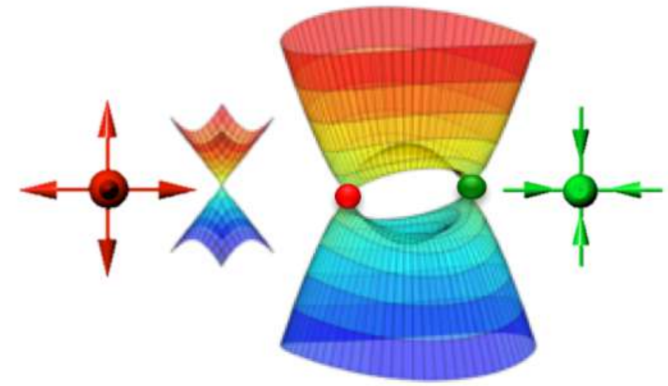
Takahashi et al.

Earth’s atmosphere and oceans,
Plasmas in stars

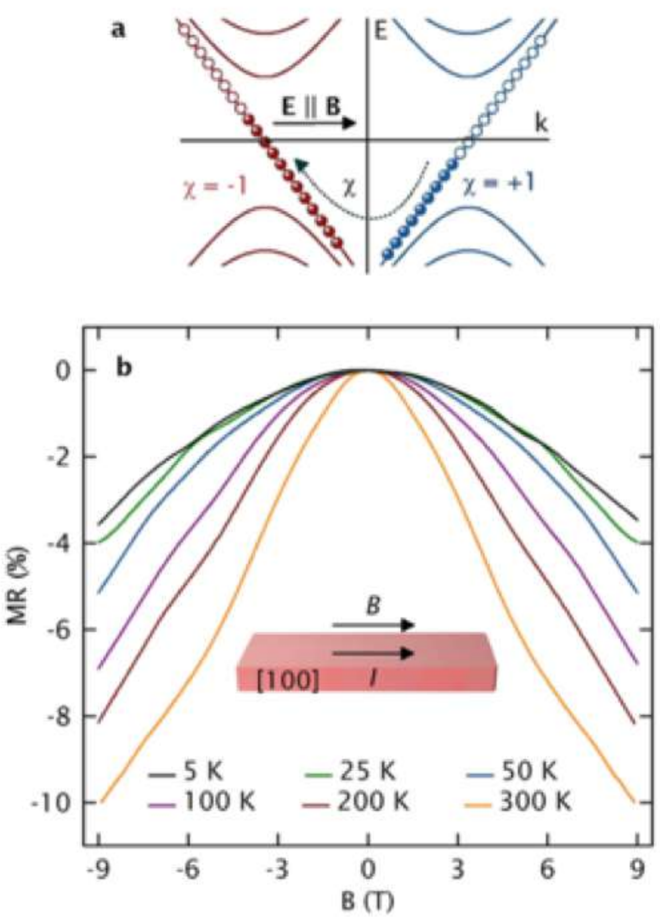




Weyl semimetal



$$H_{\text{Weyl}} = v_F \mathbf{q} \cdot \sigma$$



- The fermions have massless dispersion reaction with some effective velocity.
- There are multiple Weyl nodes with no net chirality (no static chiral magnetic effect)
- In weak fields metals with Weyl quasiparticles exhibit longitudinal negative magnetoresistance

$$\sigma_{zz} = \frac{e^4 v_F^3}{4\pi^2 \hbar c^2 \mu^2} B^2 \tau$$

Multi-Weyl semimetal

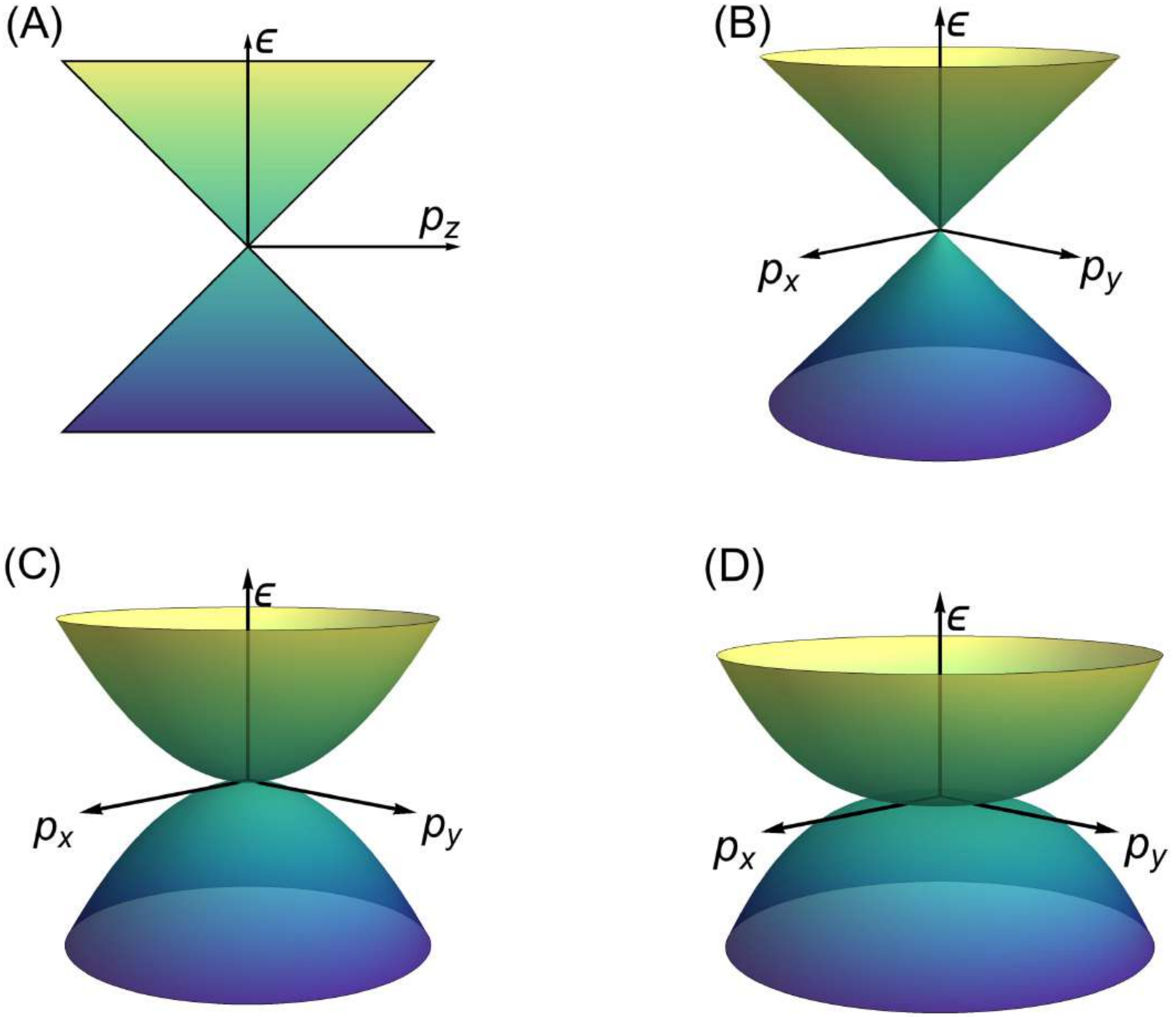
$$H_n(\mathbf{p}) = \alpha_n p_\perp^n \left[\cos(n\phi_p)\sigma_x + \sin(n\phi_p)\sigma_y\right] + v p_z \sigma_z$$

These generalized semimetals are characterized by pairs of (anti)-monopole of arbitrary integer charge n.

The dispersion relation

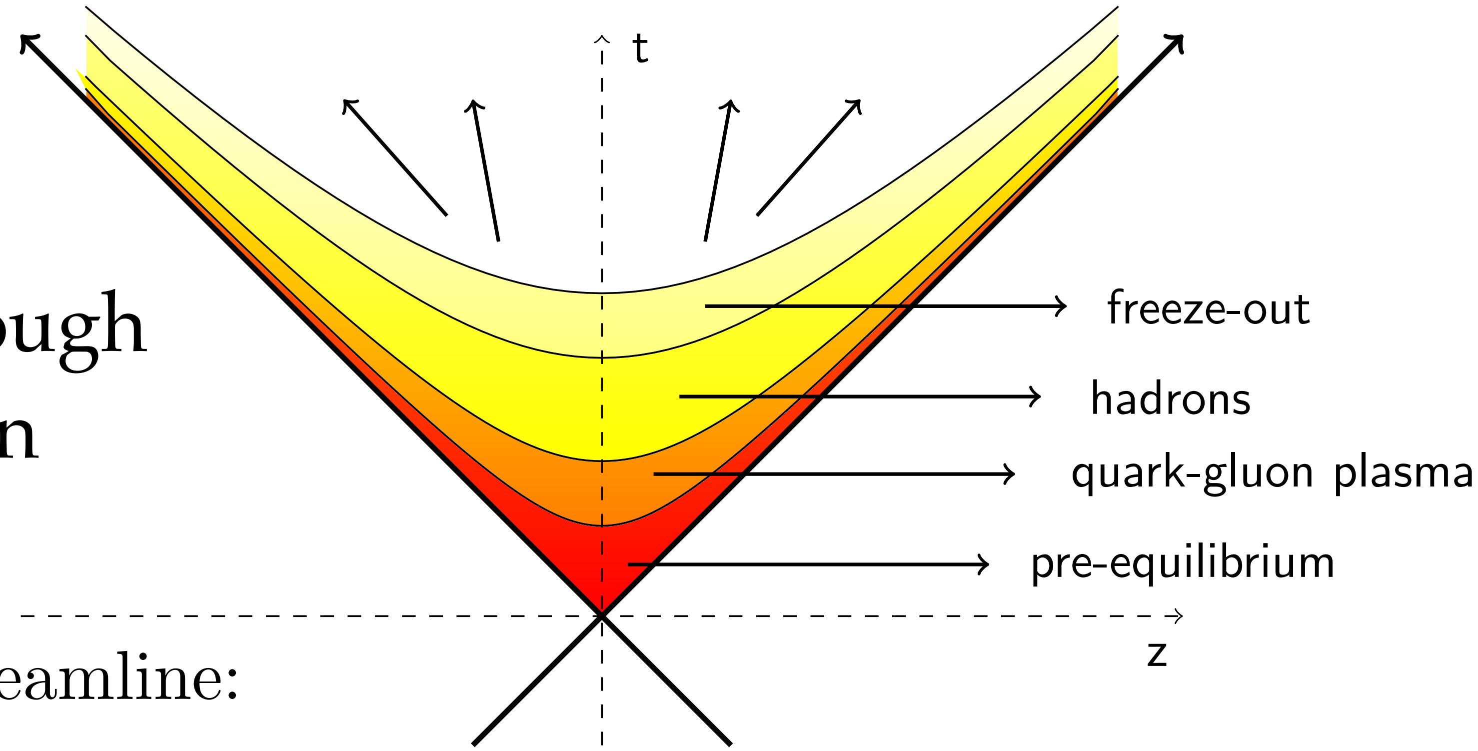
$$\epsilon_{\mathbf{p}} = \sqrt{\alpha_n^2 p_\perp^{2n} + v^2 p_z^2}$$

Symmetry: ISO(2)xSO(1,1)



Bjorken flow

Symmetry ISO(2)xSO(1,1), although
Conjectured also to be Carrollian



- Rotation invariance around the beamline:

$$\xi = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$$

- Translation invariance in the transverse plane:

$$\xi = \frac{\partial}{\partial x^1} \quad \text{and} \quad \xi = \frac{\partial}{\partial x^2}$$

- Boost invariance along the beamline:

$$\xi = x^3 \frac{\partial}{\partial x^0} + x^0 \frac{\partial}{\partial x^3}$$

So the Killing vectors for ISO(2) symmetry are $\xi^\mu = (0, 1, 0, 0)$, $\xi^\mu = (0, 0, 1, 0)$ and for SO(1,1) symmetry $\xi^\mu = (0, -x^2, x^1, 0)$ and $\xi^\mu = (x^3, 0, 0, x^0)$.

Anomalous transport

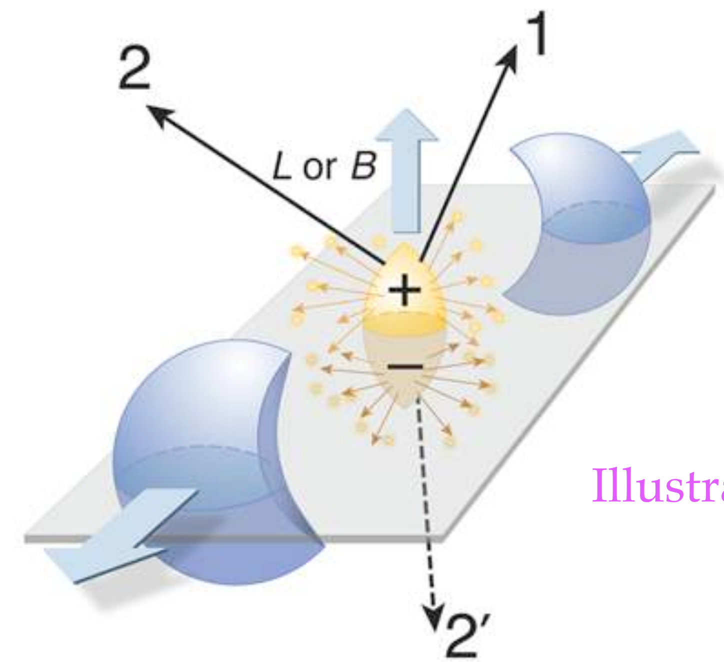


Illustration: Carin Cain

Relativistic fluid with one conserved charge described by conservation laws

no anomalies

$$\partial_\mu T^{\mu\nu} = 0,$$
$$\partial_\mu J^\mu = 0$$

with anomalies

$$\partial_\mu T^{\mu\nu} = F^\nu{}_\lambda J^\lambda_{L,R},$$
$$\partial_\mu J^\mu_{L,R} = C_{\mathrm{anom}} E^\mu B_\mu$$

Non-vanishing values of the conductivities $\{\sigma_{\mathrm{VV}}, \sigma_{\mathrm{AV}}, \sigma_{\ \mathrm{V\Omega}}, \sigma_{\ \mathrm{A\Omega}}\}$ lead to the Chiral Magnetic Effect (CME), the Chiral Separation Effect (CSE) , the Chiral Vortical Effect (CVE) and the Chiral Vortical Separation Effect (CVSE).

$$J_V = \sigma_{\mathrm{VV}} B + \sigma_{\mathrm{VA}} B_5 + \sigma_{\mathrm{V\Omega}} \omega\,,$$
$$J_A = \sigma_{\mathrm{AV}} B + \sigma_{\mathrm{AA}} B_5 + \sigma_{\mathrm{V\Omega}} \omega\,.$$

Everything is tensorial.

Kinetic theory

Consider a gas of non-relativistic fermions with a Berry curvature on the fermi surface in the presence of electromagnetic field. The Lagrangian of such system is given by:

$$\mathcal{L} = \hbar \mathbf{k} \cdot \dot{\mathbf{r}} - \varepsilon_M(\mathbf{k}) + e\phi(\mathbf{r}) - e\dot{\mathbf{r}} \cdot A(\mathbf{r}, t) + \hbar \dot{\mathbf{k}} \cdot \mathcal{A}_n(\mathbf{k})$$

We can derive the EOMs

$$\dot{\mathbf{r}} = \frac{\partial \varepsilon_M(\mathbf{k})}{\hbar \partial \mathbf{k}} - \dot{\mathbf{k}} \times \mathbf{\Omega}(\mathbf{k}) \qquad\qquad \hbar \dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B}$$

where we have defined the Berry curvature

$$\mathbf{\Omega}^n_{\mu\nu}(R) = \frac{\partial}{\partial R^\mu} \mathcal{A}^n_\nu(R) - \frac{\partial}{\partial R_\nu} \mathcal{A}^n_\mu(R)$$

We see the so-called anomalous Karplus-Luttinger contribution to velocity

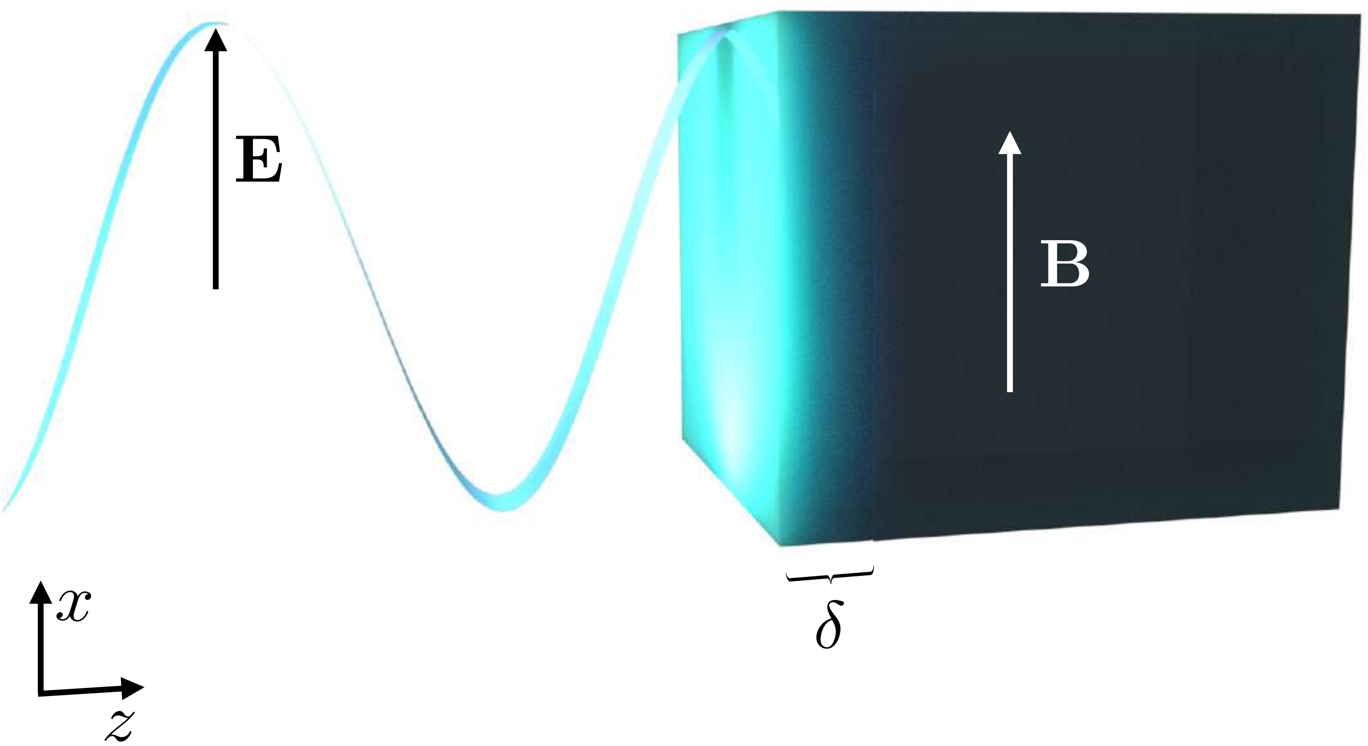
$$\partial_t f + \nabla_{\mathbf{x}} f \cdot \dot{\mathbf{x}} + \nabla_{\mathbf{p}} f \cdot \dot{\mathbf{p}} = C\left[f\right]$$

We solve Boltzmann equation. Anomaly is captured by a Berry monopole. Here everything is non-covariant.

Skin effects

Maxwell's equations give:

$$\nabla^2 \mathbf{E} - c^{-2}\partial_t^2 \mathbf{E} = \mu \partial_t \mathbf{J}$$



The relation between $\mathbf{J}$ and $\mathbf{E}$ is in general nonlocal. In the Fourier space

$$\mathbf{J}(q,\omega) = \sigma_{||}(q,\omega)\mathbf{E}(q,\omega)$$

Generically, electric field and current decay exponentially: $\mathbf{E}$, $\mathbf{J} \propto e^{-z/\delta(\omega)}$

Skin depth $\delta(\omega)$ plays the role of a frequency-dependent system size!

We can probe CME. Can we do the same for CVE?

A paradox

In 1972, I. Mueller presented calculations that seemed to prove that stress and heat flux in a gas computed from kinetic theory do not take consistent frame-covariant expressions. For example the heat flux reads

$$q_1^\mu = -\tau_{\rm coll}(2nT)h^{\mu\nu}\partial_\nu T\,.$$

Now let us change the frame of reference to one rotating with frequency Ω

$$q_2^\mu = -\frac{\tau_{\rm coll}(2nT)}{1+4\tau_{\rm coll}^2\Omega^2}h^{\mu\nu}\partial_\nu T - \frac{2\tau_{\rm coll}^2\Omega(2nT)}{1+4\tau_{\rm coll}^2\Omega^2}\tau_\rho\epsilon^{\rho\mu\nu}\partial_\nu T\,.$$

A more than 50 years old debate continues on the topic. Guesses exist what the answer should be.

Relativistic point particle

A particle in a curved spacetime moves along a timelike geodesic. If u is its velocity, the equation of motion is $\nabla_u u = 0$, or equivalently, $u^\alpha \nabla_\alpha u^\mu = 0$. This means that the velocity is parallelly propagated along the trajectory. This makes geodesics analogous to straight lines in flat spacetime. Another property of geodesics, namely that they are curves with the longest proper time between two given points, is also the straightforward generalization of the flat spacetime properties. The action for a particle of mass m reads

$$S[x^\mu(\tau)] = \int_{\tau_1}^{\tau_2} d\tau \, L, \quad L = -m\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}},$$

with fixed initial and final points is extremal for the geodesic. In order to derive the geodesic equation determined by this extremum, we impose the constraint

$$g_{\mu\nu}u^\mu u^\nu = -1,$$

which fixes the meaning of τ as the proper time. Then, the variation of the action leads to the equation of motion in the standard form

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0.$$

The parameter τ is the proper time along the geodesic.

The Hamiltonian $H = p_\mu \frac{dx^\mu}{d\tau} - L$, for the point particle vanishes identically. This is a direct consequence of the reparametrization invariance of the action. For a system with constraints one introduces them by using the Lagrange multipliers. Following Polyakov the action is

$$\tilde{S}[x^\mu(\tau), \eta(\tau)] = \int_{\tau_1}^{\tau_2} d\tau\, \tilde{L}, \qquad \tilde{L} = \frac{1}{2}\left(\eta^{-1} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - m^2\eta\right).$$

Following the formalism of Hamiltonian mechanics with gauge d.o.f. one has

$$p_\mu = \frac{\partial \tilde{L}}{\partial \dot{x}^\mu} = \eta^{-1} g_{\mu\nu} \frac{dx^\nu}{d\tau},$$

$$\tilde{H} = p_\mu \frac{dx^\mu}{d\tau} - \tilde{L} = \frac{1}{2}\eta\left[g^{\mu\nu} p_\mu p_\nu + m^2\right].$$

The Hamilton equations of motion are

$$\frac{dx^\mu}{d\tau} = \frac{\partial \tilde{H}}{\partial p_\mu}, \qquad \frac{dp_\mu}{d\tau} = -\frac{\partial \tilde{H}}{\partial x^\mu}.$$

The variation of the Hamiltonian action with respect to η gives the constraint equation

$$g^{\mu\nu} p_\mu p_\nu + m^2 = 0.$$

This is a mass shell constraint for a particle.

Consider a domain V_0 in the phase space P and determine its phase volume as follows:

$$V_0 = \int dq_1 \dots dq_n dp_1 \dots dp_n .$$

Let $z = z(t_0)$ be a point in the phase space representing the system at the initial moment of time t_0. As a result of motion its position at the moment t is $z(t)$. The Hamiltonian map $z \to z(t)$ determines a transformation of the phase space. Under this transformation, the initial domain V_0 transforms into a domain V_t. Let

$$V_t = \int dq_1 \dots dq_n dp_1 \dots dp_n$$

be its phase volume. According to the Liouville theorem, the phase volume remains the same under the Hamiltonian map, that is

$$V_t = V_0 .$$

For a particle motion in a spacetime the natural choice of the canonically conjugated quantities is $\{p_\mu, x^\mu\}$, where x^μ is a position of the particle and p_μ is its velocity/momentum. The time evolution of any phase space function $F(x,p)$ is computed as

$$\dot{F} = X_L[F] ,$$

where X_L is the Liouville operator.

Relativistic kinetic theory

e.g. Y. Choquet-Bruhat

It is assumed that the state of the matter in a spacetime (V, g) is represented by a 'one-particle distribution function'. This distribution function f is interpreted as the density of particles at a point $x \in V$ that have a momentum $p \in T_x V$, the tangent space to V at x. A distribution function f is a non-negative scalar function on the so-called phase space PV, a subbundle of the tangent bundle TV to the spacetime V:

$$f : PV \to \mathbb{R} \quad \text{by} \quad (x, p) \mapsto f(x, p), \quad \text{with} \quad x \in V, \quad p \in P_x \subset T_x V.$$

If (V, g) is a Lorentzian manifold, then the fibre P_x at x is such that $g_x(p, p) \leq 0$ and, in a time-oriented frame, $p^0 \geq 0$.

We denote by θ the 8-volume form on TV, i.e., with θ_x and θ_p respectively the volume forms on V and $T_x V$,

$$\theta := \theta_x \wedge \theta_p.$$

In local coordinates, θ_x and θ_p are given by

$$\theta_x = |\det g|^{\frac{1}{2}}\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n, \quad \theta_p = |\det g|^{\frac{1}{2}}\, dp^0 \wedge dp^1 \wedge \cdots \wedge dp^n.$$

The moments of f are functions or tensors on V obtained by integration on the fibres of the phase space PV of products of f by tensor products of p with itself.

Moment of order zero This is by definition the integral on the fibre P_x of the distribution function:

$$n(x) := \int_{P_x} f\theta_p.$$

It is the density of particles in spacetime.

First and second moments The first moment of f is a vector field on V defined by

$$P^\alpha(x) := \int_{P_x} p^\alpha f(x,p)\theta_p.$$

The second moment of the distribution function f is the symmetric 2-tensor on spacetime given by

$$T^{\alpha\beta}(x) := \int_{P_x} f(x,p)p^\alpha p^\beta \theta_p.$$

It is interpreted as the stress–energy tensor of the distribution f.

The mass shell

The phase space over (V, g), denoted by $P_{m,V}$, has for fibre $P_{m,x}$ the mass hyperboloid (also called the mass shell)

$$P_{m,x} \equiv P_x \cap \{g^{\mu\nu} p_\mu p_\nu = -m^2\}.$$

In the case of particles of a given mass m, the volume form $\theta_{m,p}$ on the mass shell $P_{m,x}$ is, taking the p_i as local coordinates on $P_{m,x}$ (then p^0 is a function of x and p_i),

$$\theta_{m,p} = \frac{|\det g|^{\frac{1}{2}}}{p^0} dp^1 \wedge \cdots \wedge dp^n.$$

Vlasov equation

In a curved spacetime, in the absence of nongravitational forces, each particle follows a geodesic of the spacetime metric g. The differential system satisfied by a geodesic in the tangent space TV of a pseudo- Riemannian manifold (V, g) reads in local coordinates, with λ called a canonical affine parameter,

$$p^\alpha := \frac{dx^\alpha}{d\lambda},$$

$$\frac{dp^\alpha}{d\lambda} = G^\alpha, \quad \text{with} \quad G^\alpha := -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu,$$

where $\Gamma^\alpha_{\lambda\mu}$ are the Christoffel symbols of the metric g. In other words, the trajectory of a particle in TV is an element of the geodesic flow generated by the vector field $X = (p, G)$. In a collisionless model, the physical law of conservation of particles, together with the invariance of the volume form in TV under the geodesic flow imposes that the distribution function f be constant under this flow, that is, that it satisfy the following first-order linear differential equation, which we call the Vlasov equation:

$$X_L[f] \equiv p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu \frac{\partial f}{\partial p^\alpha} = 0.$$

Newton-Cartan geometry

Newton-Cartan theory replaces the concept of gravitational force with the curvature of spacetime, similar to general relativity. As a starting point one constructs a spacetime interval, which is invariant under the Galilei group, giving the Newtonian spacetime a metrical structure. The Galilean transformations are rewritten as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + \zeta^\mu,$$

where $\Lambda^\mu{}_\nu$ is given by

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^0}{\partial x^i} \\ \frac{\partial x'^i}{\partial x^0} & \frac{\partial x'^i}{\partial x^j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}.$$

The restrictions of Galilean transformations on covariant and contravariant metrics give two degenerate solutions which are Galilei-invariant

$$\tau_{\mu\nu} = \tau_\mu \tau_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As such, one can only assign Galilei-invariant lengths to spatial separations using $h_{\mu\nu}$ or to temporal separations employing $\tau_{\mu\nu}$.

A NC geometry is a $d+1$ manifold with coordinates x^μ with $\mu = 0, 1, \ldots d$, a 1-form usually called the clock form or time metric, a degenerate symmetric tensor interpreted as an inverse spatial metric, and a covariant derivative $(\tau_\mu, h^{\mu\nu}, \nabla_\mu)$. The degeneracy condition manifests as $\tau_\mu h^{\mu\nu} = 0$. In addition, the derivative is required to obey $\nabla_\mu \tau_\nu = \nabla_\mu h^{\nu\rho} = 0$. We assume that the coordinates, the time metric, and the spatial metric have units $[x^\mu] = L$, $[\tau_\mu] = T/L$, and $[h^{\mu\nu}] = 1$ respectively.

Now let us introduce a set of observers with associated velocity fields $(v_\psi)^\mu$, where ψ labels the observer, normalized such that $\tau_\mu (v_\psi)^\mu = 1$. Any such velocity field can be expressed as

$$(v_\psi)^\mu = v^\mu + h^{\mu\nu} \psi_\nu \,,$$

with v^μ an arbitrary reference velocity (this is known as the Milne boost). The reference velocity v^μ makes it possible to define a metric $h_{\mu\nu}$ satisfying

$$h_{\mu\nu} v^\nu = 0 \,, \quad h^{\mu\nu} h_{\nu\rho} + v^\mu \tau_\rho = \delta^\mu_\rho \,.$$

Notice, however, that the ambiguity in the definition of v^μ implies that the metric $h_{\mu\nu}$ is observer-dependent.

A connection associated to the covariant derivative ∇_μ has the general form

$$\Gamma^{\mu}_{\alpha\beta} = v^{\mu}\partial_\alpha \tau_\beta + \frac{1}{2}h^{\mu\sigma}\left(\partial_\alpha h_{\beta\sigma} + \partial_\beta h_{\alpha\sigma} - \partial_\sigma h_{\alpha\beta}\right) + h^{\mu\sigma}\tau_{(\alpha}F_{\beta)\sigma} \ ,$$

with $F_{\mu\nu}$ an antisymmetric tensor with units $[F_{\mu\nu}] = T^{-1}$. The connection defined here is in general observer-dependent. However, if we introduce a frame-dependent gauge field m_μ such that $F_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu$ and postulate the transformation

$$(m_\psi)_\mu = m_\mu + P^{\nu}_{\mu}\psi_\nu - \frac{\psi^2}{2}\tau_\mu \ + \partial_\mu \Lambda \, ,$$

with Λ the gauge parameter, $P^{\mu}_{\nu} = \delta^{\mu}_{\nu} - v^{\mu}\tau_\nu$, the connection transforms such that it becomes observer-invariant if $d\tau = 0$, which is always the case when there exists a globally defined time function. This is equivalent to setting the anti-symmetric part of the connection or equivalently torsion to zero. In consequence Milne boost invariance can be interpreted as the independence of physical phenomena from the observer describing them. Note that such a connection is not invariant under $U(1)$ symmetry.

Point particle on NC spacetimes

In order to covariantly formulate kinetic theory on NC spacetimes it is necessary to be able to describe the dynamics of pointlike particles propagating on such spacetime. We define the reparametrization-, diffeomorphism-, and Milne-invariant action

$$S = \frac{m}{2}\int_\gamma \frac{d\lambda}{\tau_\rho \dot{x}^\rho} h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m\int d\lambda\, m_\mu \dot{x}^\mu\,,$$

where m is the mass of the particle. Under gauge transformations the Lagrangian transforms as a total derivative. The affinely parametrized ($\tau_\mu \dot{x}^\mu = 1$) trajectories satisfy the equations of motion

$$\dot{x}^\mu = \frac{p^\mu}{m} = 0\,,$$
$$\dot{p}^\mu + \Gamma^\mu_{\alpha\beta}\, p^\alpha \dot{x}^\beta = 0\,,$$

where $p^2 = h_{\mu\nu}p^\mu p^\nu$, and p^μ is the kinematic momentum satisfying the constraint $\tau_\mu p^\mu = m$.

Generalised Noether theorem

For generic geometries the system will not have any Noether charge; however, our ultimate goal is to construct a kinetic theory where a gas of interacting NC particles can equilibrate. In that case it is mandatory to restrict the problem to geometries with at least one time-like Killing transformation. Therefore, we assume the existence of a set of parameters $\chi_K = \left(\xi_K^\mu, \psi_\mu^K, \Lambda_K\right)$ such that the NC data is invariant under the action of δ_{χ_K}, i.e., $\delta_{\chi_K}\tau_\mu = \delta_{\chi_K}h^{\mu\nu} = \delta_{\chi_K}v^\mu = \delta_{\chi_K}m_\mu = 0$. The on-shell variation of the point particle Lagrangian is

$$\delta_{\text{on}}\mathcal{L} = \xi^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} + \dot{\xi}^\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{d}{d\lambda}\left(\xi^\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) \ .$$

On the other hand, under a Killing transformation χ_K, the symmetry variation of the Lagrangian up to a boundary term is given by

$$\delta_{\text{sy}}\mathcal{L} = \frac{m}{2N}\dot{x}^\alpha \dot{x}^\beta \pounds_{\xi_K} h_{\alpha\beta} - \frac{m}{2N^2} h_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \dot{x}^\mu \pounds_{\xi_K}\tau_\mu + m\dot{x}^\alpha \pounds_{\xi_K} m_\alpha = -m\frac{d}{d\lambda}\Lambda_K \ .$$

Comparing $\delta_{\text{sy}}\mathcal{L}$ with $\delta_{\text{on}}\mathcal{L}$ for $\xi^\mu = \xi_K^\mu$ we get the conserved charge

$$Q_K = \xi_K^\mu \pi_\mu + m\Lambda_K \ .$$

Thermal NC spacetimes

NC spacetimes were generalized by adding to the manifold and extra timelike vector field u^μ, which without loss of generality can be normalized as $u^\mu \tau_\mu = 1$. As we will see later, such a field can always be defined in the presence of a gas. For such spaces we could argue that the "Milne symmetry" is spontaneously broken since u^μ introduces a distinguished observer having velocity $v_U^\mu = u^\mu$. Notice that this frame is connected to one with arbitrary v^μ via a boost with the Milne parameter $\psi_\mu = h_{\mu\nu} u^\nu$. The distinguished spatial metric $g_{\mu\nu}$ and $U(1)$ gauge field A_μ obtained by Milne boosting $h_{\mu\nu}$ and m_μ are

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu \tau_\nu - u_\nu \tau_\mu + u^2 \tau_\mu \tau_\nu \,,$$

$$A_\mu = u_\mu + m_\mu - \frac{u^2}{2} \tau_\mu \,.$$

Using the extended NC data we can construct covariant derivatives and subsequently tensors invariant under all symmetries. NC gases equilibrate on spacetimes with such a structure; therefore, we will dub them Thermal Newton-Cartan (TNC) manifolds.

Equilibrium distribution function

Firstly, we define the Milne- and gauge-invariant temperature, velocity, and chemical potential

$$T = \frac{1}{\tau_\mu \xi^\mu}\,, \qquad u^\mu = T\xi^\mu \qquad \frac{\mu}{mT} = \xi^\mu A_\mu + \Lambda\,.$$

We can now write the charges as

$$Q_{\text{ideal}} = -\frac{1}{T}\left[\frac{1}{2m}h_{\alpha\beta}(p^\alpha - mu^\alpha)(p^\beta - mu^\beta) - \mu\right]\,.$$

The equilibrium distribution function f_{ideal} can be written as

$$f_{\text{ideal}}(x,p) \equiv \mathcal{N}\exp[Q_{\text{ideal}}],$$

where

$$Q_{\text{ideal}} = \xi^\mu \pi_\mu + m\Lambda.$$

Example: 2D rotating gas

Let us first consider the case of a two-dimensional gas. In 2D there is only one spatial component of the field strength. We define the *vorticity scalar* as

$$B = \frac{1}{2} u^\rho \epsilon_{\rho\mu\nu} f^{\mu\nu} \,,$$

where $f = dA$ is Milne-invariant. The viscosities can then be determined systematically in any coordinate systems; for example, choosing polar coordinates (ρ, ϕ) we have

$$\lambda^{(e)} = \frac{\tau_{\rm coll}(2nT)}{1 + 4\tau_{\rm coll}^2 B^2}, \qquad \lambda^{(o)} = \frac{2\tau_{\rm coll}^2 B(2nT)}{1 + 4\tau_{\rm coll}^2 B^2}.$$

The even and odd thermal conductivities equal to

$$\kappa^{(e)} = \frac{\tau_{\rm coll}(2nT)}{1 + \tau_{\rm coll}^2 B^2}, \qquad \kappa^{(o)} = \frac{\tau_{\rm coll}^2 B(2nT)}{1 + \tau_{\rm coll}^2 B^2}.$$

Resolution to the paradox

The traditional approach used e.g. by Mueller is to perform the gradient expansion in the frame comoving with the *observer*, rather than in the frame comoving with the *fluid*. In other words, the gradient expansion is performed using v^μ of the observer and the corresponding $h_{\mu\nu}$ and $F_{\mu\nu}$ instead of the Milne-invariant u^μ, $g_{\mu\nu}$ and $f_{\mu\nu}$. At the first-order, this is equivalent to using the Milne-boost-dependent vorticity scalar $B = \frac{1}{2} v^\rho \epsilon_{\rho\mu\nu} F^{\mu\nu}$ instead of the Milne-invariant one. The source of the problem, then, is the arbitrariness in the choice of v^μ. The paradox can be resolved by fixing v^μ in terms of some distinguished timelike (in the sense $\tau_\mu v^\mu = 1$) vector field. For generic gases, there exists only one such field, namely the fluid velocity u^μ. Therefore, one can obtain a truly covariant kinetic theory by setting $v^\mu = u^\mu$. In some papers the answer was guessed but the explanation remained obscure.

Conclusions

- Kinetic theory can be developed in a frame-indifferent way
- A systematic tool to include rotations
- Coupling to geometry serves as a guiding principle
- Now many rotating semi-classical systems can be understood