# Testing the validity of fluid dynamics in (2+1)-dimensional boost-invariant expansion

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$$p^{\mu}\frac{\partial f}{x^{\mu}}+m\frac{\partial(K^{\mu}f)}{\partial p^{\mu}}=\frac{1}{2}\frac{g}{(2\pi\hbar)^{3}}\int_{\Re 3}\frac{\mathrm{d}^{3}\vec{p}_{2}}{E_{2}}\int_{\Re 3}\frac{\mathrm{d}^{3}\vec{p}_{1}'}{E_{1}'}\int_{\Re 3}\frac{\mathrm{d}^{3}\vec{p}_{2}'}{E_{2}'}W(p_{1}',p_{2}'\leftarrow p_{1},p_{2})\left(f_{1}'f_{2}'\,\overline{f}\,\overline{f}_{2}-ff_{1}\overline{f}_{1}'\overline{f}_{2}'\right)$$

with

**K. Gallmeister, C. Greiner, D. H. Rischke**

# Conservation laws & tensor decompositions

$$\partial_\mu N^\mu = 0$$
$$\partial_\mu T^{\mu\nu} = 0$$
$$N^\mu = n u^\mu + n^\mu$$
$$T^{\mu\nu} = e u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu}$$

| | |
|---|---|
| $n = u_\mu N^\mu$ | LRF particle density |
| $n^\mu = \Delta^\mu_\alpha N^\alpha$ | particle diffusion current |
| $e = u_\mu T^{\mu\nu} u_\nu$ | LRF energy density |
| $W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$ | energy diffusion current |
| $p(e,n) + \Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$ | isotropic pressure ($p_{eq}$ + *bulk*) |
| $\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}$ | shear stress tensor |

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$
$$T^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}$$

Fluid dynamics can be derived from the Boltzmann equation

- Close to thermal equilibrium: inverse Reynolds number $\frac{|\pi^{\mu\nu}|}{p} \lesssim 1$
- Separation of microscopic and macroscopic scales: Knudsen number $\lambda_{\mathrm{mfp}}\theta \lesssim 1$

Denicol, Niemi, Molnar, Rischke, Phys. Rev. D **85**, 114047 (2012)

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C\left[f\right]$$

$$\mathrm{Kn} \lesssim 1 \text{ and } R^{-1} \lesssim 1$$
$$\Downarrow$$

$$\dot{n}^{\langle\mu\rangle} + \frac{n^{\mu}}{\tau_n} = \frac{\kappa_n}{\tau_n}\nabla^{\mu}\alpha_0 + \mathcal{J}^{\mu} + \mathcal{R}^{\mu} + \mathcal{K}^{\mu} \ ,$$
$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} + \mathcal{K}^{\mu\nu}$$

- How small/large is the small Knudsen/Reynold number
- Conditions for validity of fluid dynamics
- A+A, p+A collisions: How good is the mapping from $v_2$ etc. to matter properties $\eta$, $\zeta$, etc.

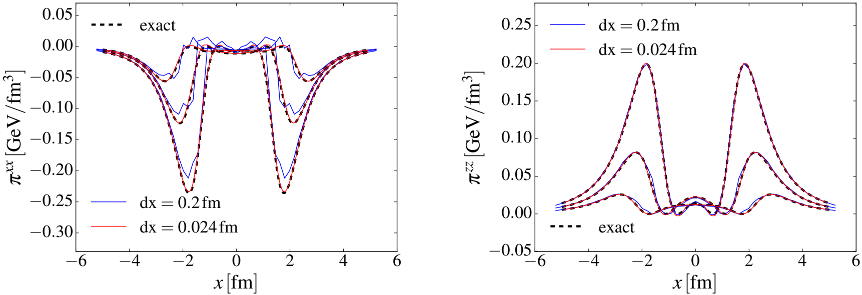
We test fluid dynamics by comparing to the direct solutions of the Boltzmann equation.

- Vary system size and cross section
- Gaussian number density profiles with width $w = 1$ and 3 fm.
- Glauber binary collision profile, with $b = 7.5$ fm.
- Constant $2 \leftrightarrow 2$ cross section $\sigma = 1 - 20$ mb
- massless particles (single component)
- Boost-invariant (2+1)-dimensional expansion

Fluid dynamics here: 14-moment approximation (Denicol, Koide, Rischke, PRL **105**, 162501 (2010))

- spacetime evolution of $T^{\mu\nu}$
- Freeze-out: freeze-out condition, $\delta f$-correction $\longrightarrow p_T$ spectrum

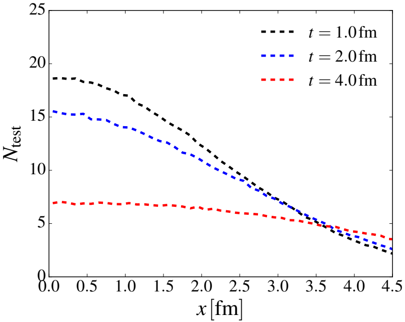
# Testing the numerics: Gubser flow

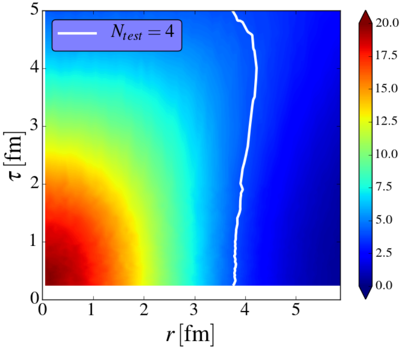



- fluid dynamics: SHASTA
- Test against exact solution (Gubser flow): Marrochio, Noronha, Denicol, Luzum, Jeon and Gale, Phys. Rev. C **91**, no. 1, 014903 (2015)
- Need to resolve 2 (short) timescales: longitudinal expansion $1/\tau$ and relaxation times $\tau_\pi$
- $\longrightarrow$ adaptive time-step.

# Solving the Boltzmann equation: BAMPS

- Boltzmann solver: BAMPS (Xu, Greiner, Phys. Rev. C **71** (2005) 064901)
- test particles represent the particle distribution function
- test particles per real particles = 1000-7000
- (test) particles can interact within the computational cell $\Delta^3 x$
- If the number of test particles in the cell $N_{\text{test}} < 4 \longrightarrow$ free gas



# Comparions of energy-momentum tensor

- BAMPS: cartesian $(t, x, y, z)$-coordinates
- $T^{\mu\nu}$ and $N^{\mu}$ components by averaging over space-time rapiditity interval $\Delta\eta_s$ at fixed cartesian time
- hydro: $(\tau, x, y)$
- Need to convert hydro results to the same coordinate system/averaging

In practice:

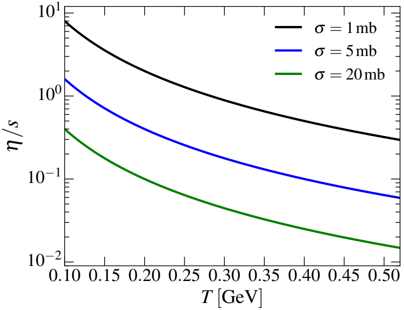
- BAMPS: every particle is boosted by $-\eta_s$, where $\eta_s$ is the space-time rapidity of the particle
- average over $\Delta\eta_s \longrightarrow T^{\mu\nu}$ and $N^{\mu}$
- decomposition of $T^{\mu\nu}$ and $N^{\mu}$

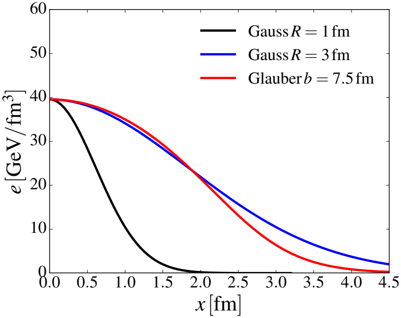
On the fluid dynamical side the same averaging corresponds

$$\langle T^{\mu\nu}\rangle_{\Delta\eta_s,t} = \frac{1}{2z_{\max}}\int_{-z_{\max}}^{z_{\max}} dz\, T^{\mu\nu}(\tau = \sqrt{t^2 - z^2}, x, y),$$

where $z_{\max} = t\tanh(\eta_{s,\max})$, $\eta_{s,\max} = 0.5$.

# Shear viscosity and Initial conditions




14-moment approximation:

$$\eta = \frac{4}{3}\frac{T}{\sigma}$$

$$s = \frac{4g}{\pi^2} T^3$$

Symmetric Gaussian with $w = 1$ and 3 fm

$$n(\tau_0, \mathbf{x}) \propto \exp\left(\frac{-\mathbf{x}^2}{2w^2}\right)$$

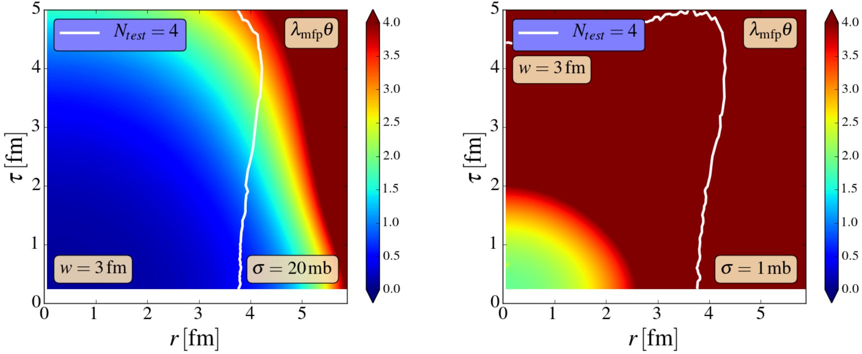
Binary profile (nBC) with $b = 7.5$ fm

$$n(\tau_0, \mathbf{x}) \propto T_A(\mathbf{x} - \mathbf{b}/2)\, T_A(\mathbf{x} + \mathbf{b}/2)$$

# Gaussian $n$ profile $w = 3$ fm

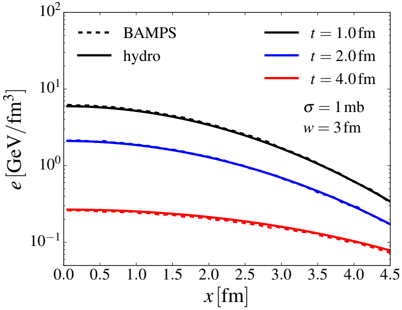
spacetime-evolution of Knudsen number $\lambda_{\rm mfp}\theta$

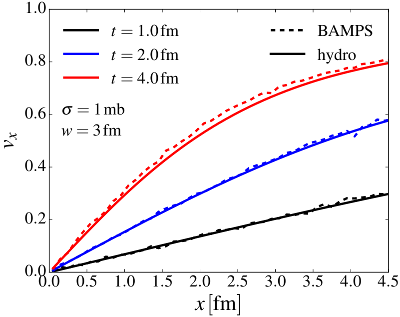
$\sigma = 20$ mb $\qquad\qquad \sigma = 1$ mb



# Gaussian profile $w = 3$ fm, $\sigma = 1$ mb

Energy density and velocity profiles, $\sigma = 1$ mb

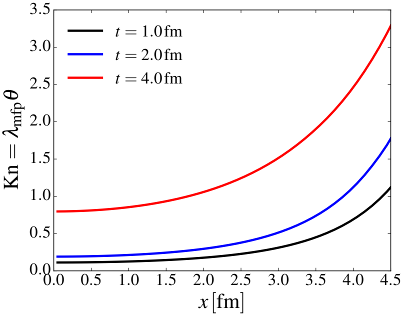


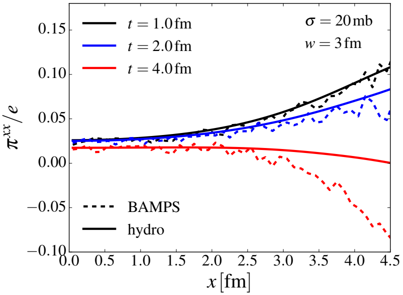




- Always well described, regardless of the cross section

# Gaussian profile $w = 3$ fm, $\sigma = 20$ mb

Knudsen number and $\pi^{xx}/e$, $\sigma = 20$ mb

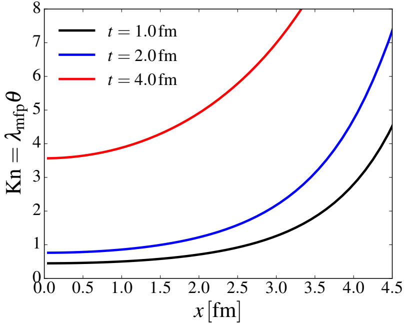


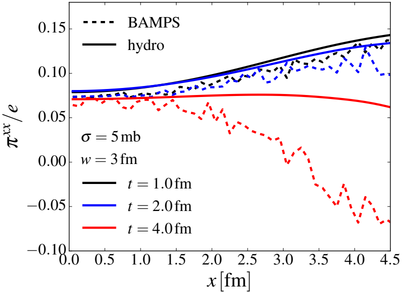




- Good agreement up to $t = 4$ fm and $r = 3$ fm. (where $\mathrm{Kn} \sim 1$)

# Gaussian profile $w = 3$ fm, $\sigma = 5$ mb

Knudsen number and $\pi^{xx}/e$, $\sigma = 5$ mb

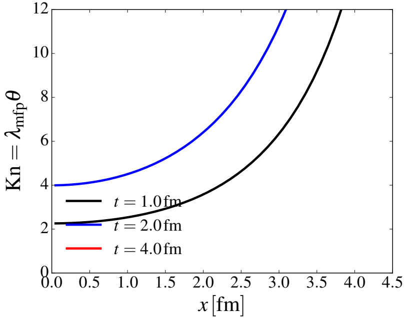


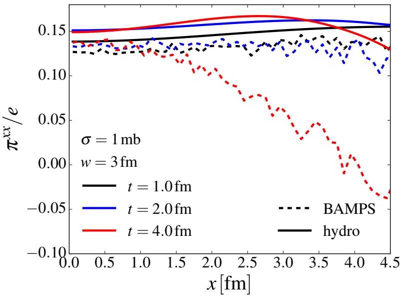




- Reasonable agreement, even if $\mathrm{Kn} \gtrsim 1$

# Gaussian profile $w = 3$ fm, $\sigma = 1$ mb

Knudsen number and $\pi^{xx}/e$, $\sigma = 1$ mb



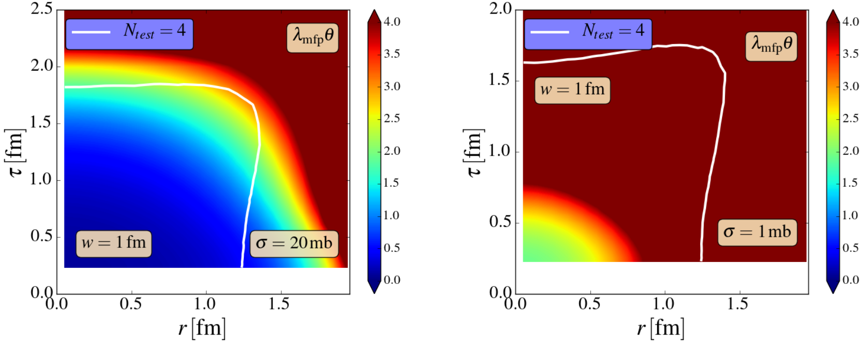




- The differences grow larger, but still ok up to $t \sim 2$ fm. Even when $\mathrm{Kn} \gg 1$

# Gaussian $n$ profile $w = 1$ fm

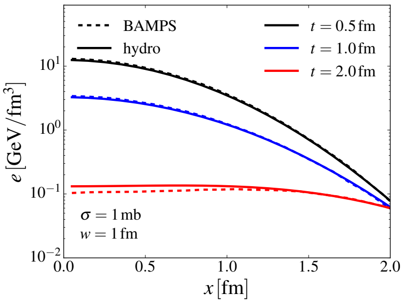
spacetime-evolution of Knudsen number $\lambda_{\mathrm{mfp}}\theta$

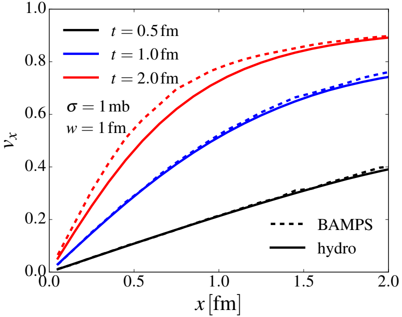
$\sigma = 20$ mb $\sigma = 1$ mb




- Smaller system, stronger transverse expansion

# Gaussian profile $w = 1$ fm, $\sigma = 1$ mb

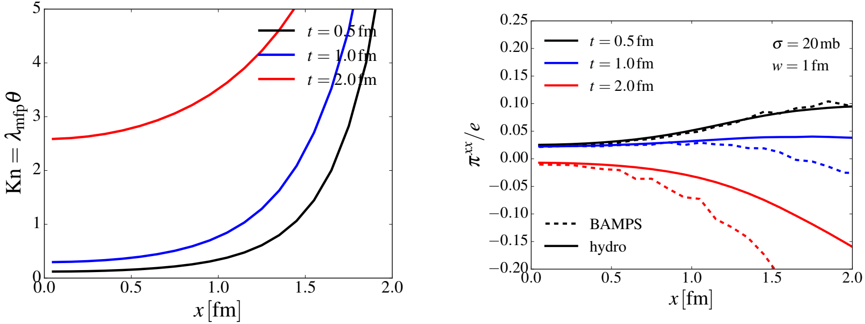



- As before good agreement holds for all cross sections
- Note: $t = 2$ fm already in the BAMPS free streaming region

# Gaussian profile $w = 1$ fm, $\sigma = 20$ mb

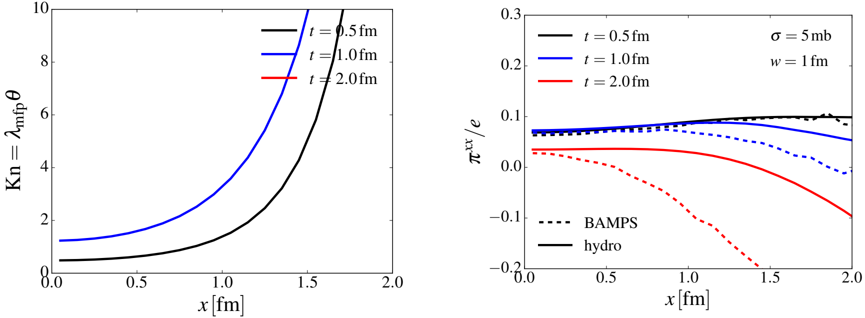
Knudsen number and $\pi^{xx}/e$, $\sigma = 20$ mb




- Similar to the $w = 3$ fm case: Good agreement up to $\mathrm{Kn} \sim 1$

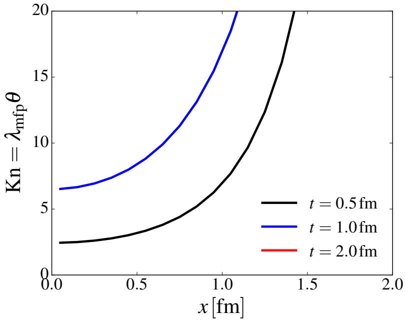
# Gaussian profile $w = 1$ fm, $\sigma = 5$ mb

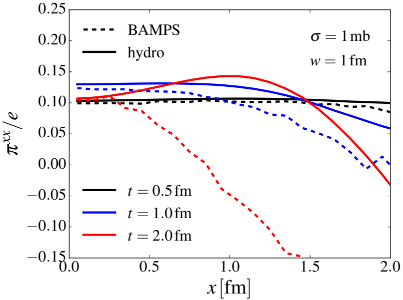
Knudsen number and $\pi^{xx}/e$, $\sigma = 5$ mb



# Gaussian profile $w = 1$ fm, $\sigma = 1$ mb

Knudsen number and $\pi^{xx}/e$, $\sigma = 1$ mb



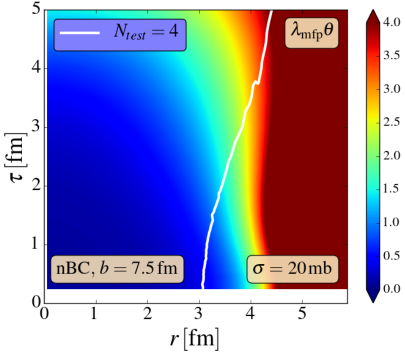




- up to $t = 1 - 2$ fm the agreement reasonable, even with very large Knudsen number.

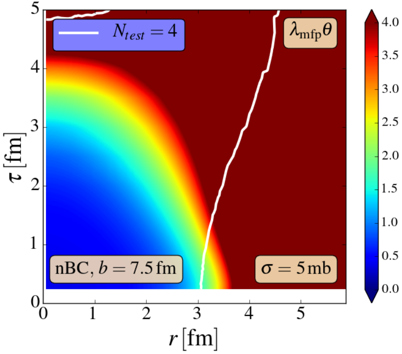
# Binary profile $b = 7.5$ fm

spacetime-evolution of Knudsen number $\lambda_{\mathrm{mfp}}\theta$

$\sigma = 20$ mb

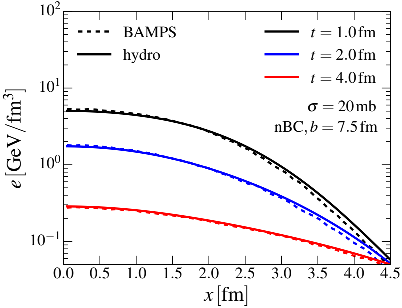


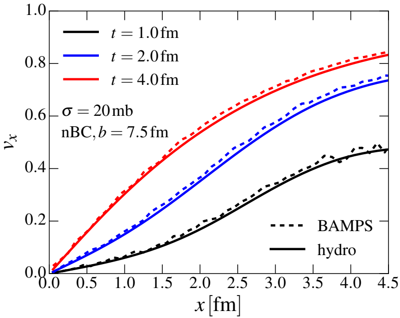

$\sigma = 5$ mb



# Binary profile $b = 7.5$ fm, $\sigma = 20$ mb

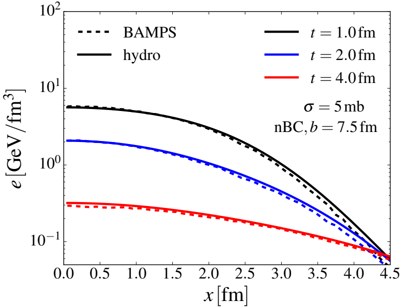
Energy density and velocity profiles, $\sigma = 20$ mb

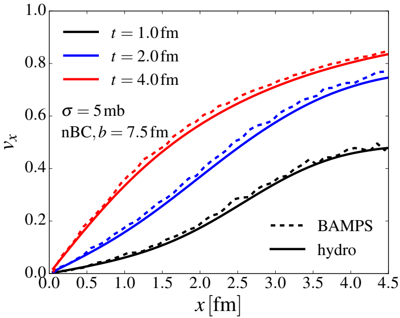


# Binary profile $b = 7.5$ fm, $\sigma = 5$ mb

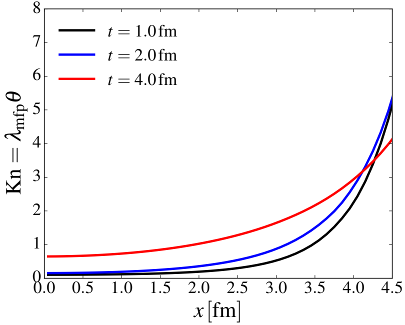
Energy density and velocity profiles, $\sigma = 5$ mb

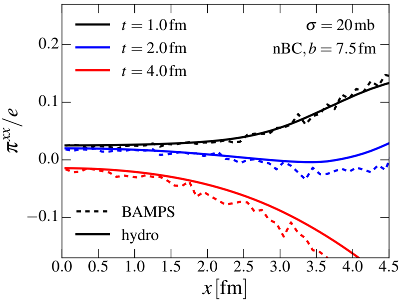


# Binary profile $b = 7.5$ fm, $\sigma = 20$ mb

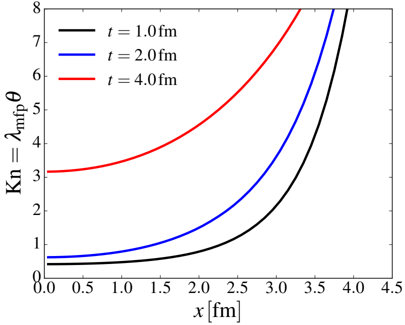
Knudsen number and $\pi^{xx}/e$, $\sigma = 20$ mb

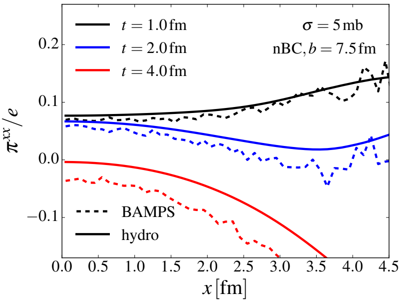


# Binary profile $b = 7.5$ fm, $\sigma = 5$ mb

Knudsen number and $\pi^{xx}/e$, $\sigma = 5$ mb



- Spacetime evolution of $T^{\mu\nu}$ well described when $\mathrm{Kn} \lesssim 1$
- Still resonable description when $\mathrm{Kn} = \mathrm{O}(1)$

Spacetime evolution cannot be directly observed $\longrightarrow$ $p_T$ spectrum, elliptic flow

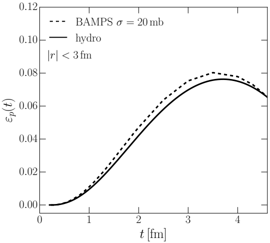
# Binary profile $b = 7.5$ fm

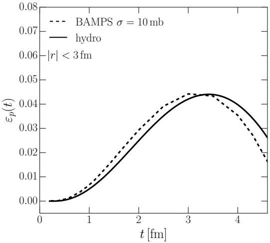
The momentum-space asymmetry of the solutions can be quantified by calculating momentum space eccentricity,

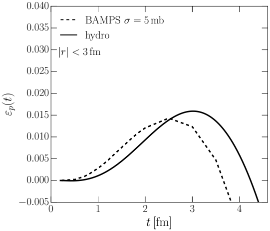
$$\varepsilon_p = \frac{\langle T^{xx} - T^{yy} \rangle}{\langle T^{xx} + T^{yy} \rangle},$$

where the angular brackets denote the integral over the transverse plane at fixed time,

$$\langle \cdots \rangle = \frac{1}{\Delta z} \int dxdydz (\cdots),$$



# $p_T$ spectrum and $v_2$

Transverse momentum spectrum: Cooper-Frye integral over decoupling surface

$$E\frac{dN}{d^3k} = \frac{dN}{dyd^2\mathbf{p}_T} = \int_\Sigma d\Sigma_\mu k^\mu f(x,k),$$

Momentum distribution function: 14-moment approximation

$$f_\mathbf{k} = f_{0\mathbf{k}} + \delta f_\mathbf{k} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(\frac{1}{8p_0T^2}k_{\langle\mu}k_{\nu\rangle}\pi^{\mu\nu} - \frac{5}{p_0}k_\mu n^\mu + \frac{1}{p_0T}E_\mathbf{k}k_\mu n^\mu\right),$$

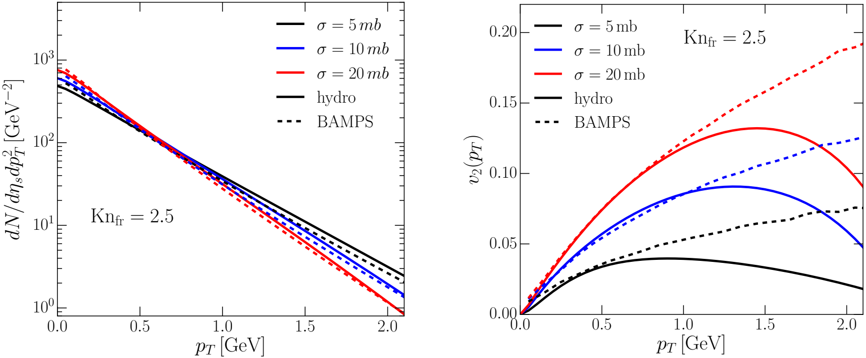
Decoupling conditions

- $\mathrm{Kn} = \lambda_{\mathrm{mfp}}\theta =$ constant
- $\lambda_{\mathrm{mfp}} =$ constant
- $T =$ constant

Note: $N_{\mathrm{test}} = 4$ must always be part of the decoupling surface as BAMPS is free streaming after this (In practice we use $N_{\mathrm{test}} = 5$)
Note2: Also need to include particles that decouple immediately

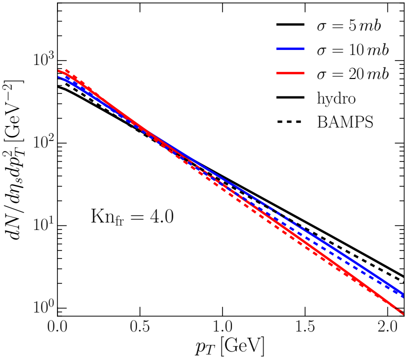
# Binary profile $b = 7.5$ fm, $\mathrm{Kn}_{fr} = 2.5$

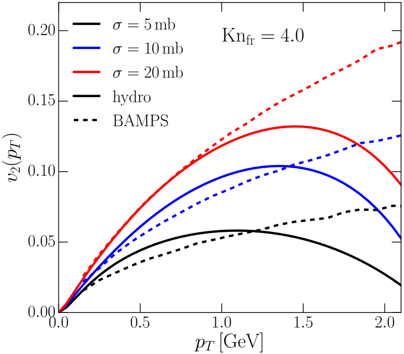
- Decoupling condition: $\mathrm{Kn} = 2.5$ and $N_{\mathrm{test}} = 5$



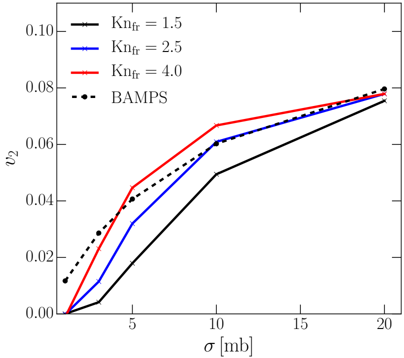
# Binary profile $b = 7.5$ fm, $\mathrm{Kn}_{fr} = 4$

- Decoupling condition: $\mathrm{Kn} = 4$ and $N_{\mathrm{test}} = 5$
- 5mb: Not possible to get low $p_T$ $v_2(p_T)$ and $p_T$-integrated $v_2$ simultaneously

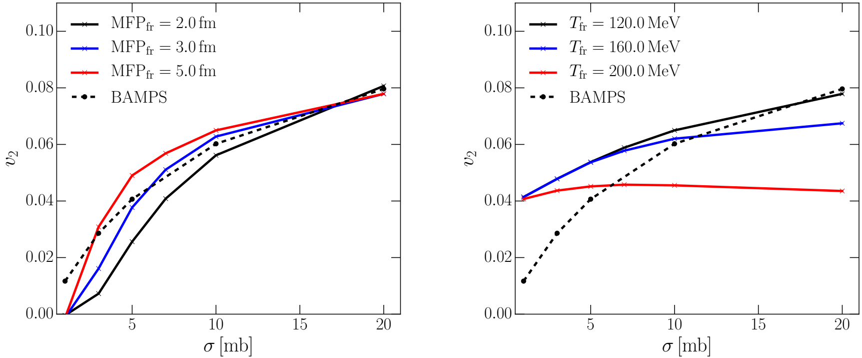


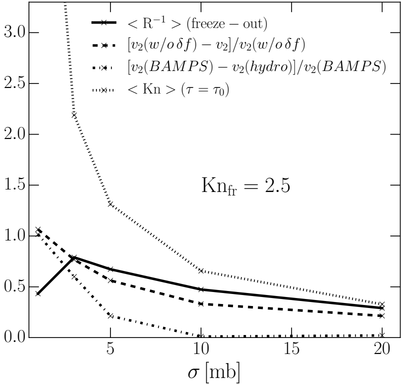
## $p_T$-integated $v_2$ as a function of cross section




- Decoupling condition can be tuned to describe large cross section results
- Gradual failure with smaller cross section
- $\mathrm{Kn_{fr}} = 2.5$
- Significant amount of $v_2$ generated during $\mathrm{Kn} > 1$ phase




- Constant $\lambda_{\mathrm{mfp}}$ freeze-out: similar to constant $\mathrm{Kn}$ freeze-out
- Constant temperature freeze-out fails
- Note: The usual constant $T$ freeze-out in heavy-ion collisions correspond rather $\lambda_{\mathrm{mfp}} =$ constant decoupling (system size changes, not cross sections)

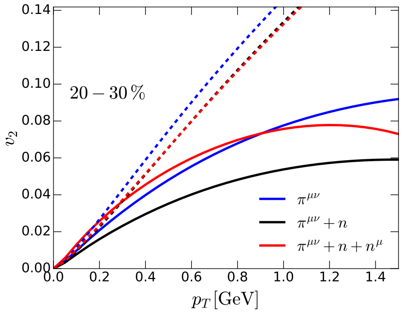


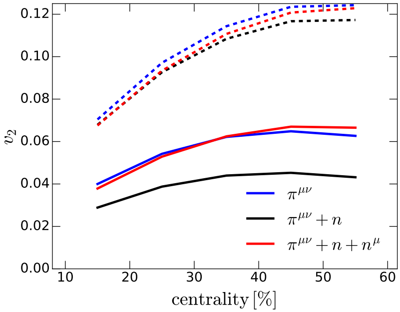

- Initial average $\mathrm{Kn}$ (entropy density weighted)
- Relative difference in $v_2$ (BAMPS vs. hydro)
- Average inv. Reynolds number on the decoupling surface
- $\delta f$ correction on $v_2$

- Difference between BAMPS and hydro starts to grow when $\sigma = 5 - 10$ mb, or when $\mathrm{Kn} \gtrsim 1$.
- Previous slides: significant amount of $v_2$ still generated when $\mathrm{Kn} \sim 1 - 2.5$.
- $\delta f$ corrections rather large (but still good agreement)
- Here we cannot really separate $R^{-1}$ from $\mathrm{Kn}$.

# Effects of diffusion

- Same setup as before, $n \propto T_A T_A$, $\sigma = 5$ mb
- 3 models:
  - I. Shear only, no conserved particle number (always in chemical equilibrium)
  - II. Shear and conserved particle number, no diffusion
  - III. Shear, conservation and diffusion
- The main effect of diffusion is coming from the $\delta f$ correction.



# Effects of $\delta f$ from shear and diffusion

$$\delta f_{\mathbf{k}} = \frac{f_{0\mathbf{k}}}{p_0}\left[\frac{1}{8T^2}k_{\langle\mu}k_{\nu\rangle}\pi^{\mu\nu} - \left(5 - \frac{E_{\mathbf{k}}}{T}\right)k_\mu n^\mu\right],$$