Instituto de Física
Universidade Federal Fluminense

Divergence of the gradient expansion and the applicability of fluid dynamics

Gabriel S. Denicol (IF-UFF)

arxiv:1608.07869, arXiv:1711.01657, arXiv:1709.06644

Frankfurt University 1.February.2018

Preview

- Introduction & motivation
- Hydro from kinetic theory: Method of moments
- Divergence of the Gradient expansion in KT
- Generalized Gradient expansion

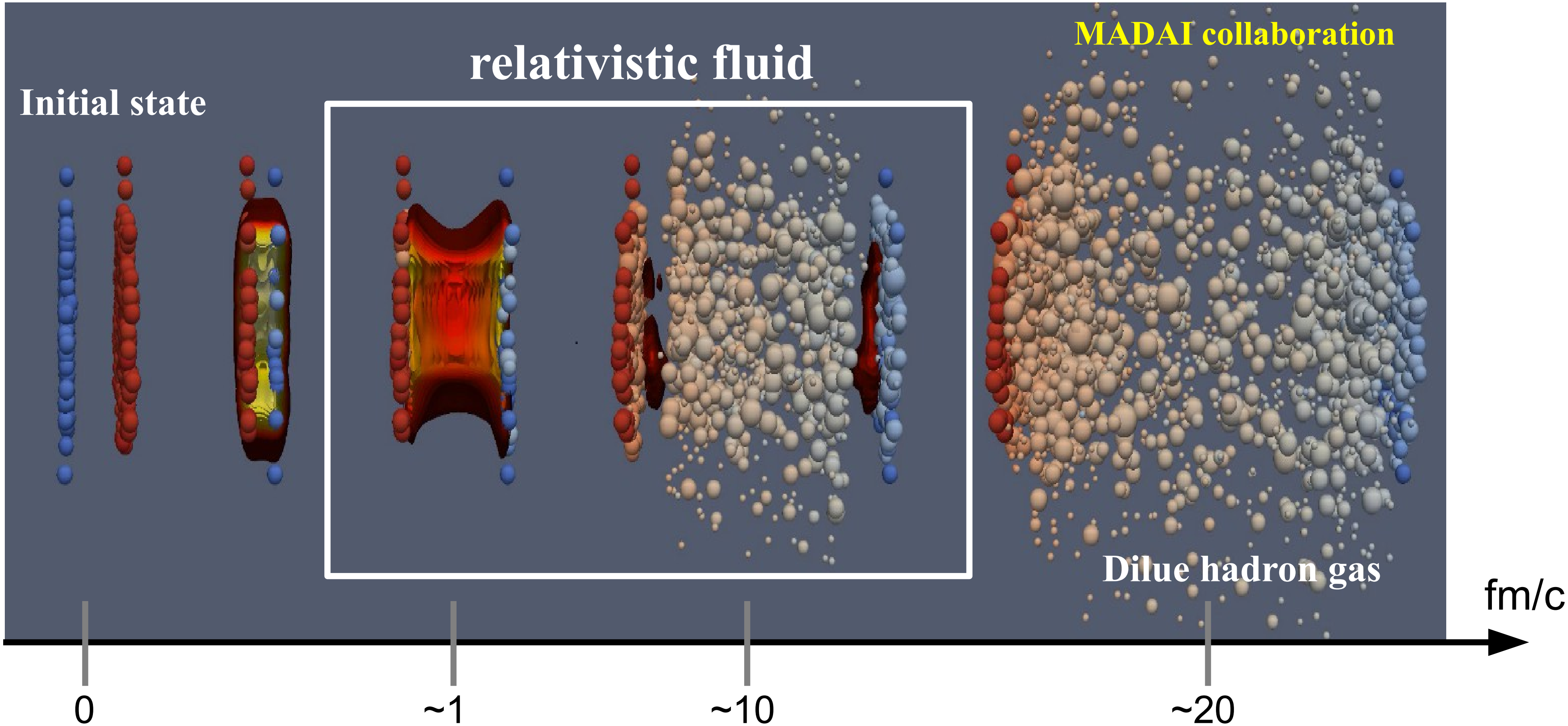
if(t < $T_{seminar}$){

- Israel Stewart theory and gradient expansion

}

Theoretical description of HIC

Empirical: Fluid-dynamical modeling of heavy ion collisions works well at RHIC and LHC energies



Main assumption: fluid dynamics can be applied at the very early stages – *Why?*

Validity of fluid dynamics

- **proximity to (local) equilibrium**

- **"small" gradients**

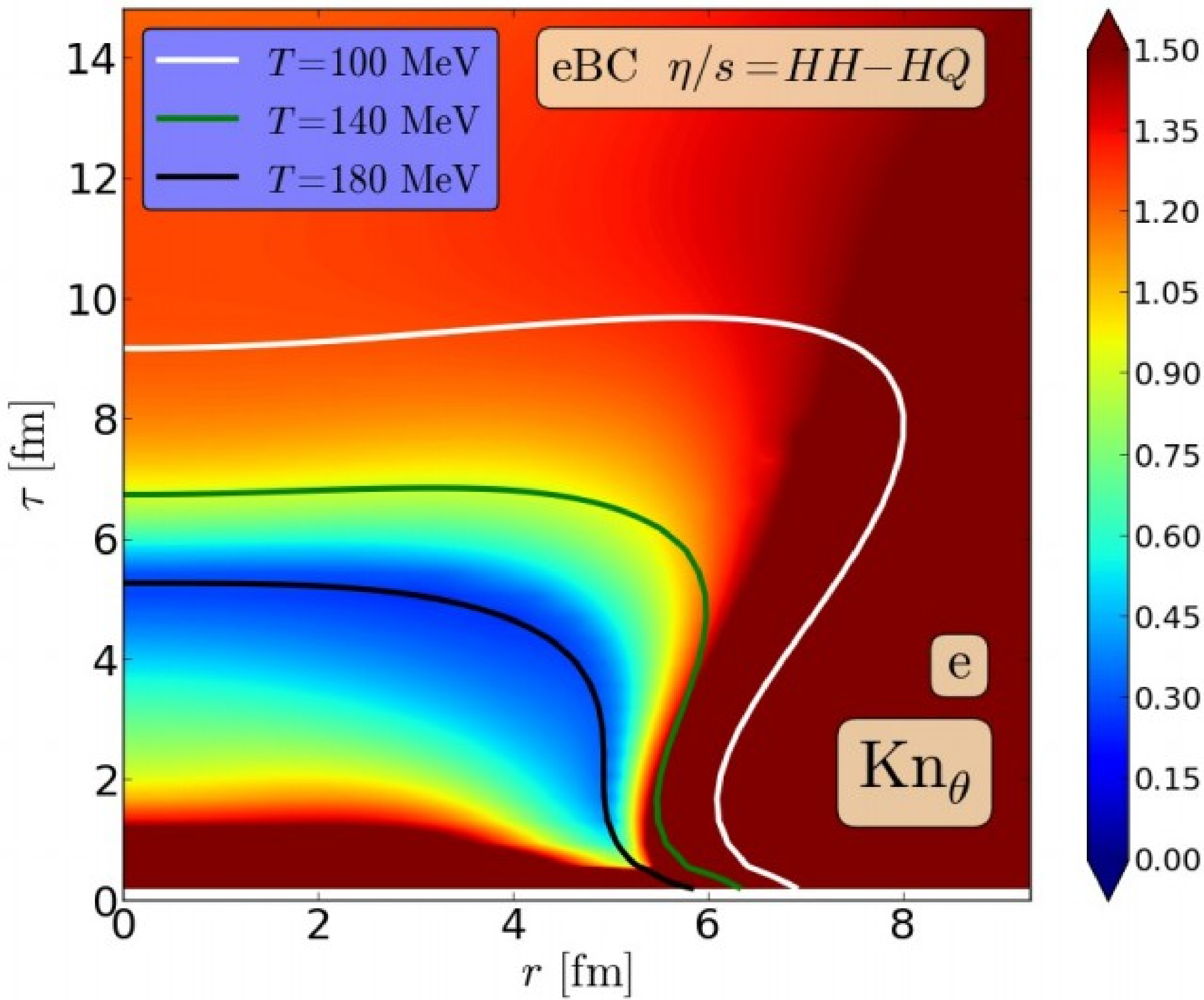
Separation of scales $\rightarrow$ **macroscopic:** L **microscopic:** ℓ

Knudsen number: $$K_N \sim \frac{\ell}{L} \ll 1$$

Do these things occur early in HIC?

Are the gradients small? no.

Niemi&GSD, arXiv:1404.7327



Knudsen number

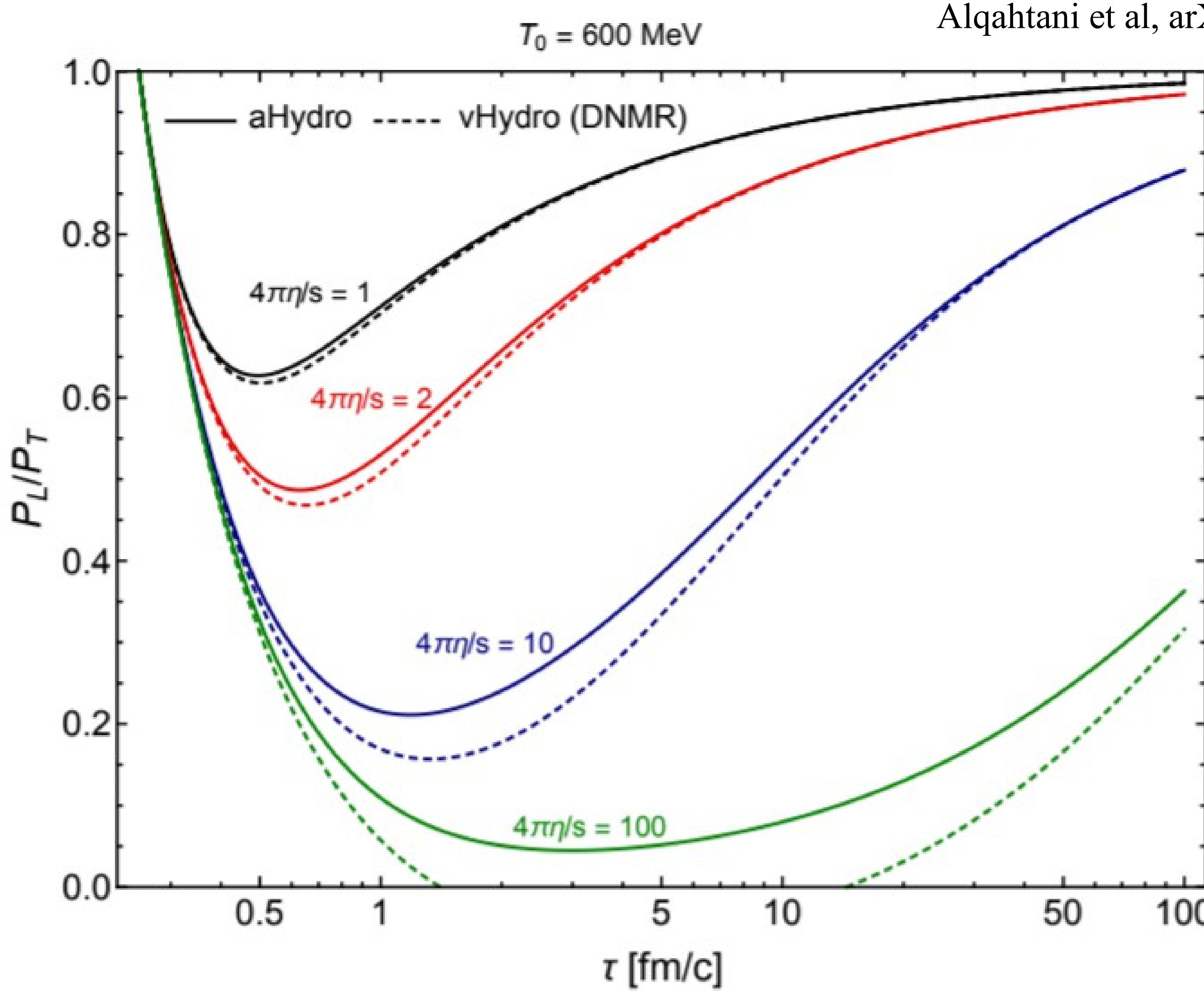
$$\mathrm{Kn} = \tau_\pi \nabla_\mu u^\mu$$

is *not* small at early times

Can this system really be close to equilibrium?

Simple example: Bjorken scalling

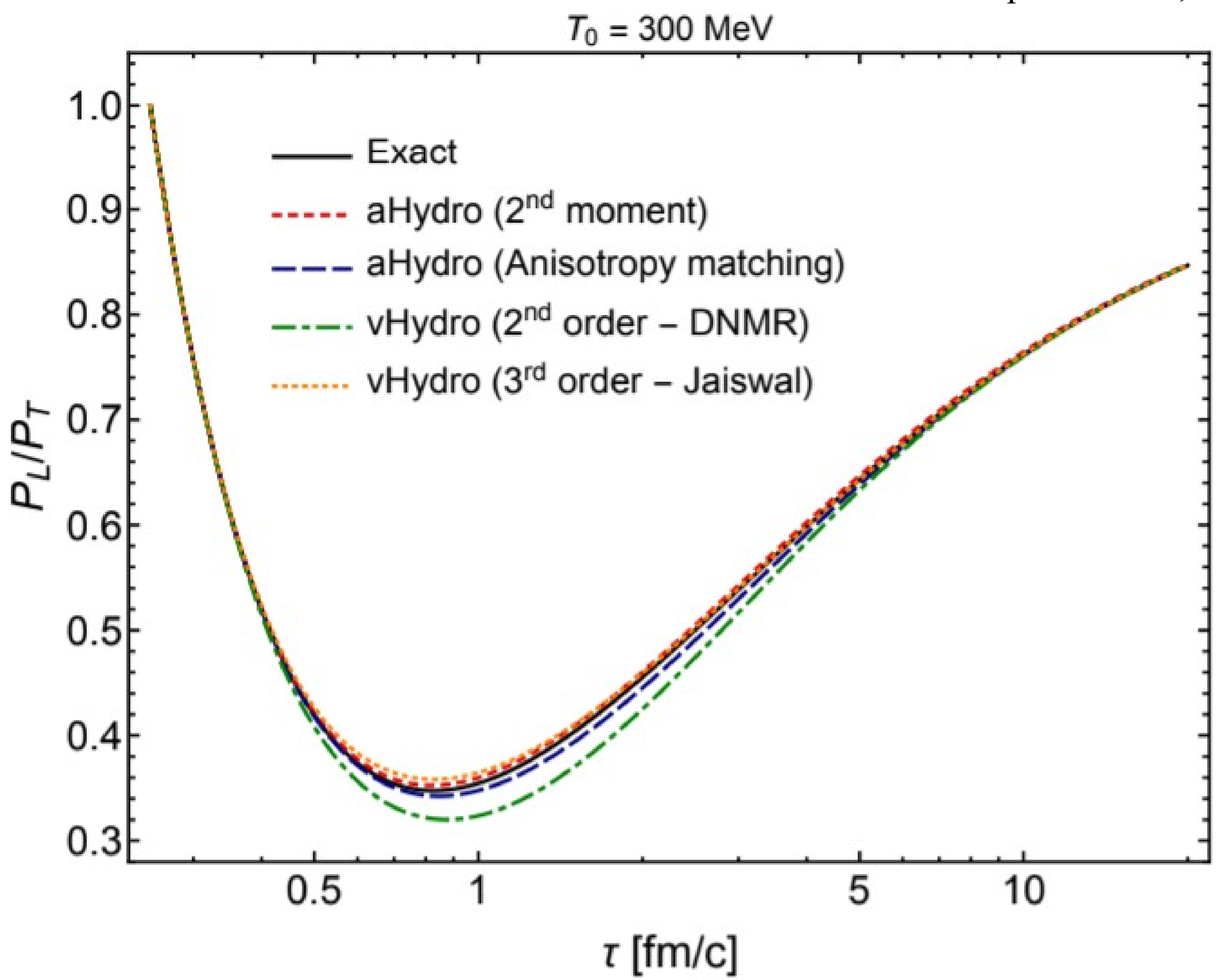
Alqahtani et al, arXiv:1712.03282v1



Approach to equilibrium is *inconsistent* with current "fluid-dynamical" theories

Simple example: Bjorken scalling

Alqahtani et al, arXiv:1712.03282v1

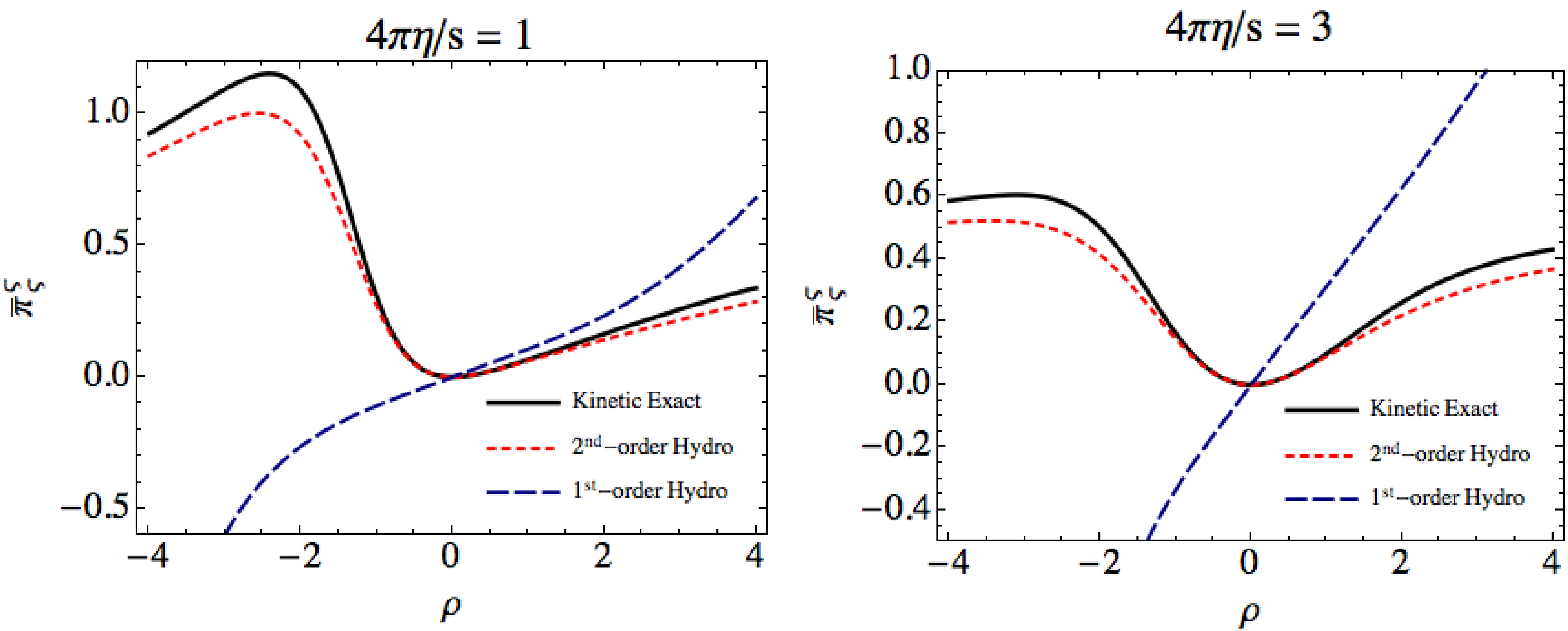


second-order fluid dynamics displays surprising accuracy

2nd order hydro works too well ...

Solution of the (conformal) Boltzmann equation under the relaxation time approximation

GSD et al, PRL 113 (2014) no.20, 202301
GSD et al, PRD 90 (2014) no.12, 125026



$$\rho(\tau, r) = -\text{arcsinh}\left(\frac{1 - q^2\tau^2 + q^2r^2}{2q\tau}\right)$$

$$\theta(\tau, r) = \arctan\left(\frac{2qr}{1 + q^2\tau^2 - q^2r^2}\right)$$

Validity of 2nd order fluid dynamics

↕ ???

Proximity to equilibrium, small gradients

I will argue that our intuition about the validity of hydro comes mostly from the gradient expansion

We can study this problem in Kinetic theory

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C\left[f\right]$$

Boltzmann eq.

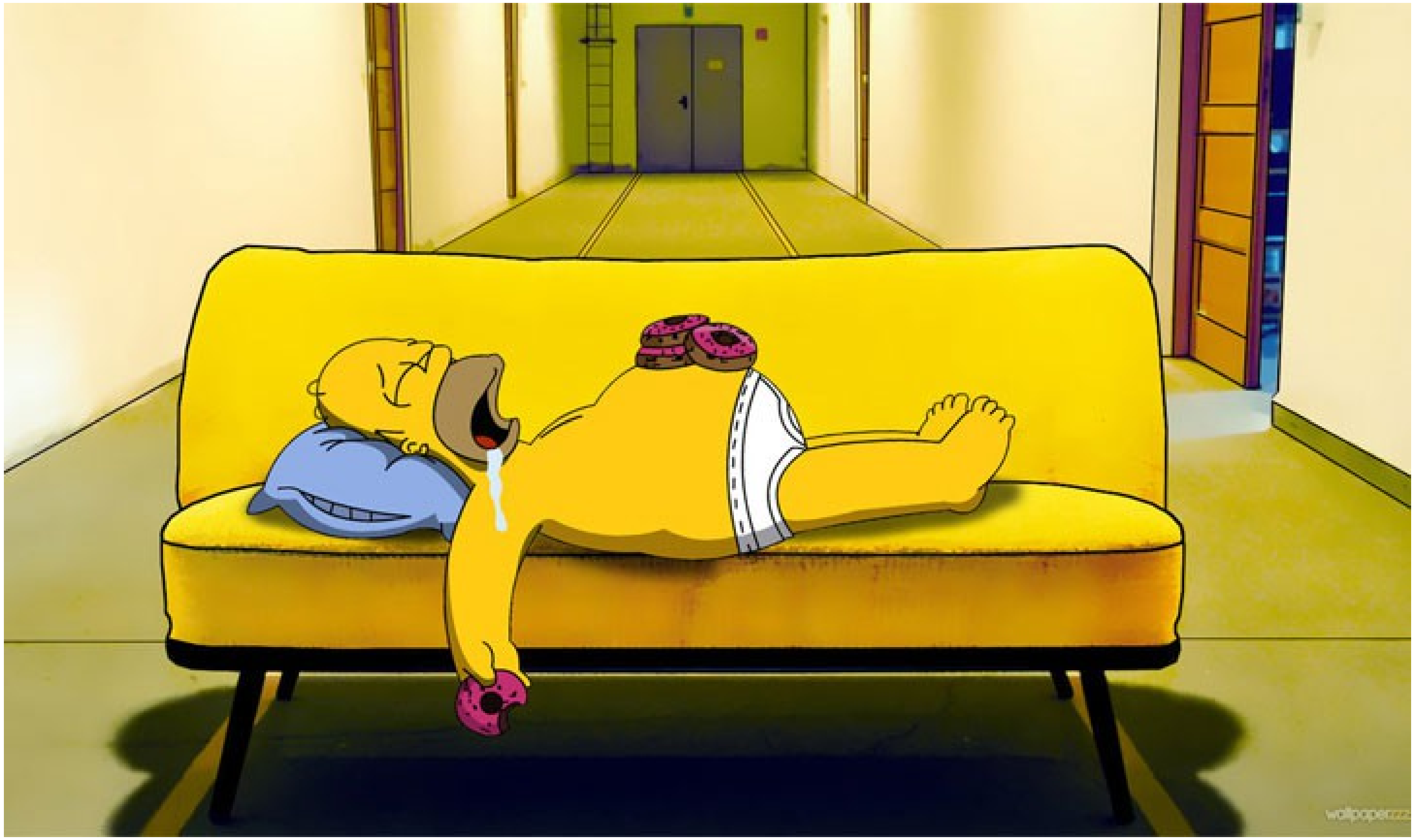
???

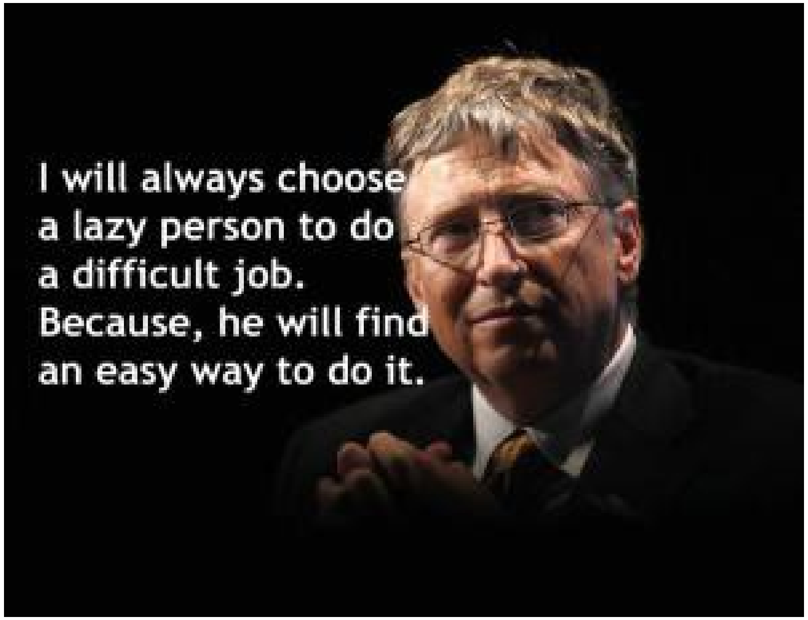
- Chapman-Enskog series
- Method of moments

$$\tau_{\Pi}\dot{\Pi} + \Pi = -\zeta\theta + \quad ...$$

$$\tau_{\pi}\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \quad ...$$

2nd-order hydro





Here, I will be lazy.

We can study this problem in Kinetic theory

Bjorken scalling + RTA

$$k_0 \partial_\tau f_{\mathbf{k}} = \mathcal{C}[f]$$

$$f_{\mathbf{k}} \equiv f\left(\tau, k_\eta, k_0\right)$$

$$C\left[f\right] = -k_0 \frac{f_{\mathbf{k}} - f_{\mathrm{eq}}}{\tau_R}$$

Knudsen number: $K_N \sim \hat{\tau}^{-1} \equiv \tau_R/\tau$

Hydrodynamics from kinetic theory: method of moments

Basics of fluid dynamics

Effective theory describing the dynamics of a system over long-times and long-distances

Separation of scales → **macroscopic:** L **microscopic:** ℓ

Knudsen number: $K_N \sim \frac{\ell}{L} \ll 1$

Conservation laws

+

simple constitutive relations

Basics of fluid dynamics

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

tensor decomposition

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + \pi^{\mu\nu}$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

Bulk viscous pressure (Π)

Shear stress tensor ($\pi^{\mu\nu}$)

Closing the equations ...

Basics of fluid dynamics

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

tensor decomposition

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}\left(P_0 + \Pi\right) + \pi^{\mu\nu}$$

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \ \ldots$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \ \ldots$$

Method of moments

H. Grad, Comm. Pure Appl. Math. 2, 331 (1949)

H. Grad

f(x,p)

Expansion of f(**x**,**p**) using a complete basis

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}}^{(n\ell)} \rho_{(n)}^{\mu_1 \dots \mu_\ell} k_{\langle \mu_1} \dots k_{\mu_\ell \rangle} \,.$$

truncation leads to hydro – *no small parameter*

Expansion of distribution function

GSD et al, PRD 85, 114047 (2012)

Distribution function expressed in terms of its moments,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}}^{(n\ell)} \rho_{(n)}^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \; .$$

Orthogonality relations imply that,

$$\rho_{(r)}^{\mu_1 \cdots \mu_\ell} \equiv \left\langle (E_{\mathbf{k}})^r \, k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \right\rangle_\delta \; ,$$

$$\mathcal{H}_{\mathbf{k}}^{(n\ell)} \equiv \frac{W_\ell}{\ell!} \sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{\mathbf{k}}^{(m\ell)} \; .$$

$$\langle \cdots \rangle_\delta \equiv \int dK \, (\cdots) \, \delta f_{\mathbf{k}}$$

Equations of motion for moments

Obtain the **exact** equations of motion for the moments,

$$\dot{\rho}_{(r)}^{\langle\mu_1\ldots\mu_\ell\rangle} = \Delta^{\mu_1\ldots\mu_\ell}_{\nu_1\ldots\nu_\ell}\frac{d}{d\tau}\int dK\,(E_{\mathbf{k}})^r\,k^{\langle\nu_1}\ldots k^{\nu_\ell\rangle}\delta f_{\mathbf{k}},$$

They have the following form,

$$\dot{\rho}_{(r)} + \sum_{n=0}^{\infty}\mathcal{A}_0^{(rn)}\rho_{(n)} = \beta_\zeta^{(r)}\theta + \ldots$$

$$\dot{\rho}_{(r)}^{\langle\mu\rangle} + \sum_{n=0}^{\infty}\mathcal{A}_1^{(rn)}\rho_{(n)}^{\mu} = \beta_\kappa^{(r)}\nabla^\mu\alpha_0 + \ldots$$

$$\dot{\rho}_{(r)}^{\langle\mu\nu\rangle} + \sum_{n=0}^{\infty}\mathcal{A}_2^{(rn)}\rho_{(n)}^{\mu\nu} = 2\beta_\eta^{(r)}\sigma^{\mu\nu} + \ldots$$

$$\vdots$$

$\mathcal{A}_\ell$ → Collision terms

Contains the information of the microscopic theory

Bjorken scalling + RTA

$$k_0 \partial_\tau f_{\mathbf{k}} = C[f]$$

$$f_{\mathbf{k}} \equiv f(\tau, k_\eta, k_0)$$

$$C[f] = -k_0 \frac{f_{\mathbf{k}} - f_{\text{eq}}}{\tau_R}$$

Moments of the distribution function:

$$\rho_{n,\ell} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 \tau} \left(k^0\right)^n \left(\frac{k_\eta}{k^0 \tau}\right)^{2\ell} f_{\mathbf{k}}$$

Fluid-dynamical variables: $\varepsilon = \rho_{1,0}$ $\quad \rho_{1,1} = \frac{1}{3}\varepsilon - \pi^\eta_\eta$

Bjorken scalling + RTA

$$k_0 \partial_\tau f_{\mathbf{k}} = \mathcal{C}[f] \quad \Longleftrightarrow \quad C[f] = -k_0 \frac{f_{\mathbf{k}} - f_{\text{eq}}}{\tau_R}$$

$$f_{\mathbf{k}} \equiv f(\tau, k_\eta, k_0)$$

Moment equations:

$$\partial_\tau \rho_{n,\ell} + \frac{1+2\ell}{\tau}\rho_{n,\ell} + \frac{n-2\ell}{\tau}\rho_{n,\ell+1} = -\frac{1}{\tau_R}\rho_{n,\ell} + \frac{1}{\tau_R}\rho^{\text{eq}}_{n,\ell}.$$

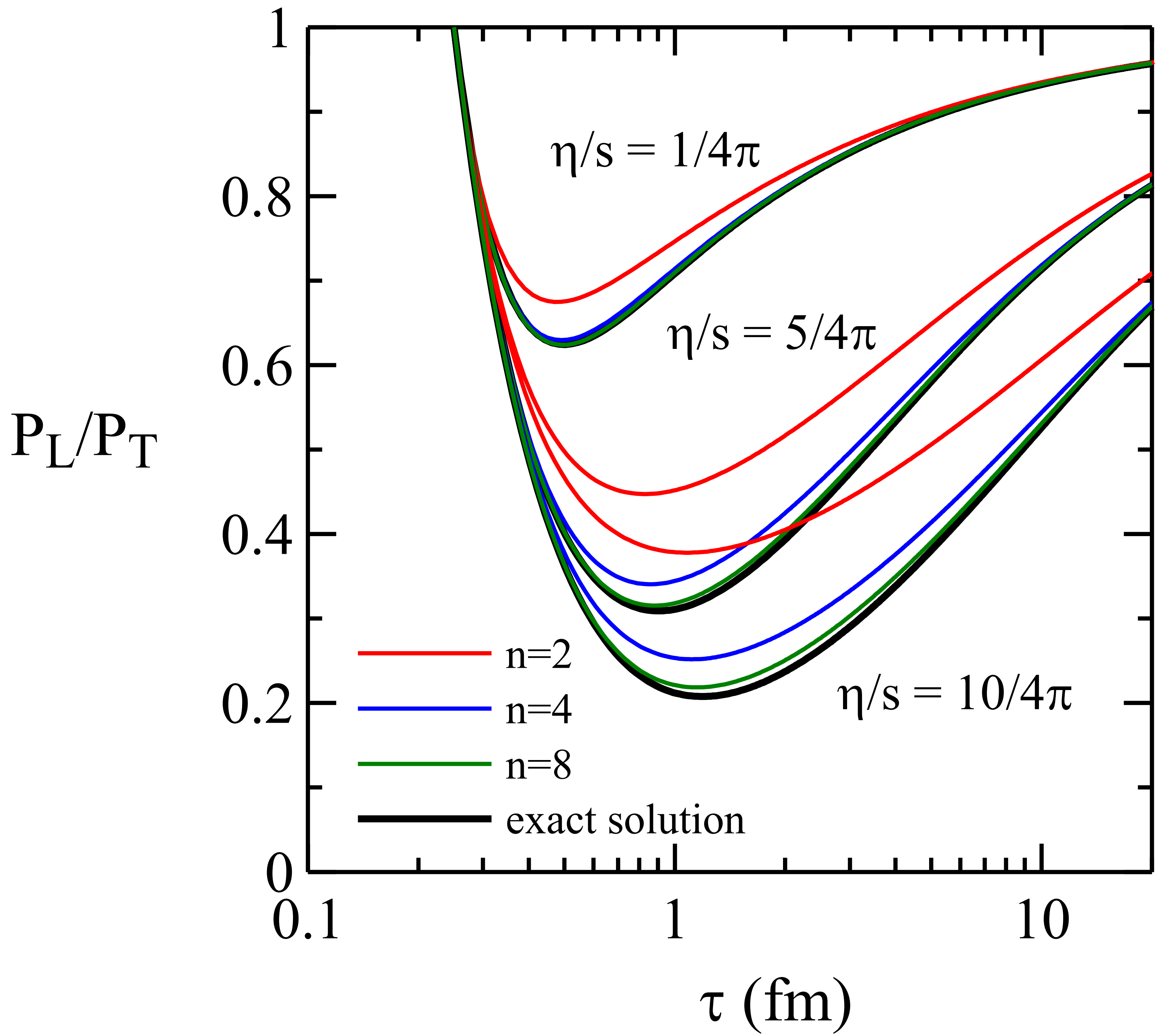
Lower rank moments always couple to higher rank ones

$$\rho^{\text{eq}}_{n,\ell} = \frac{(n+2)!}{2\ell+1}\frac{T^{n+3}}{2\pi^2},$$

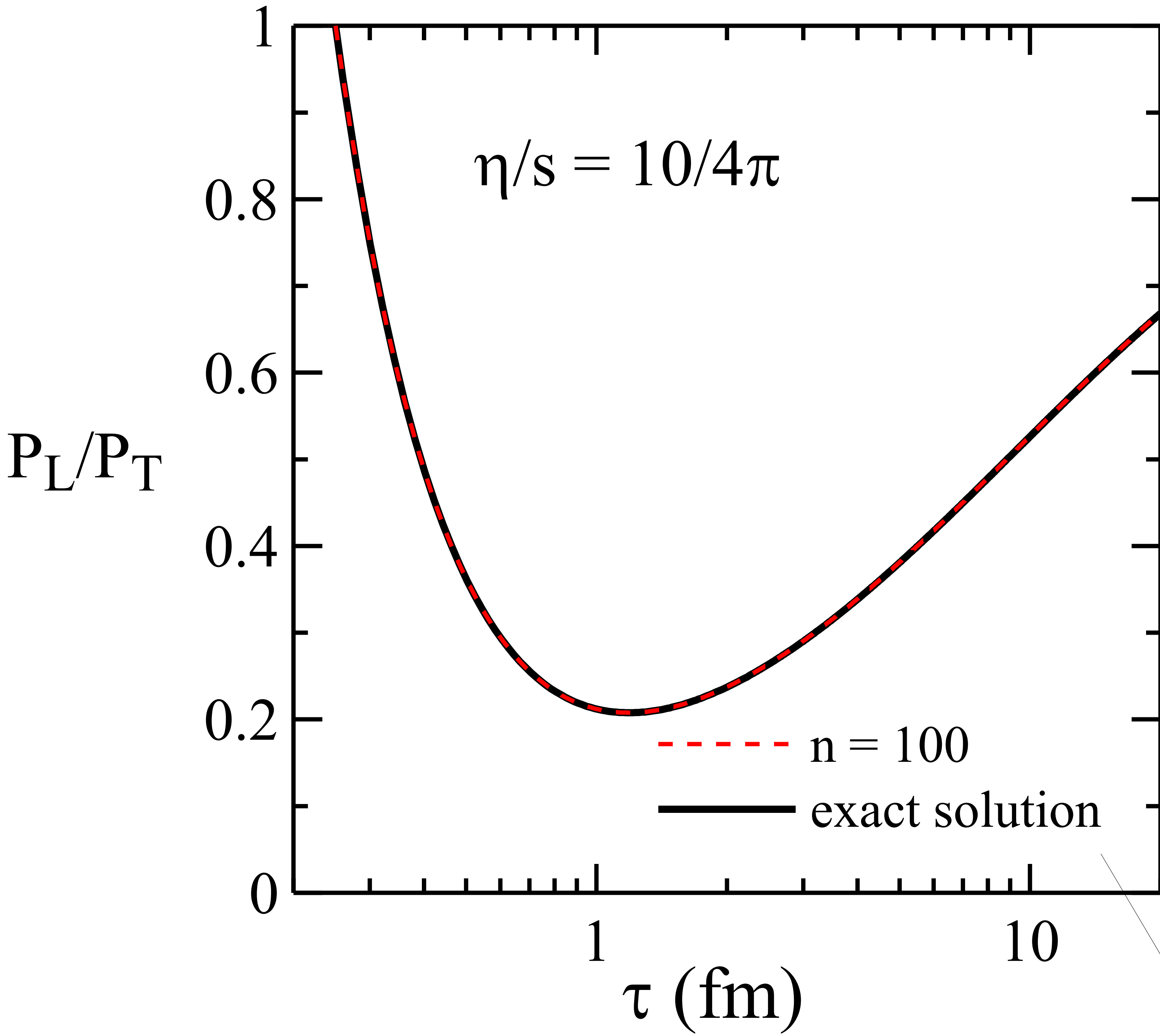
how can these equations be truncated?

does the moment expansion converge?

Convergence of the method of moments



1
0.8
0.6
0.4
0.2
0
0.1
1
10
P_L/P_T
τ (fm)
$\eta/s = 1/4\pi$
$\eta/s = 5/4\pi$
$\eta/s = 10/4\pi$
n=2
n=4
n=8
exact solution

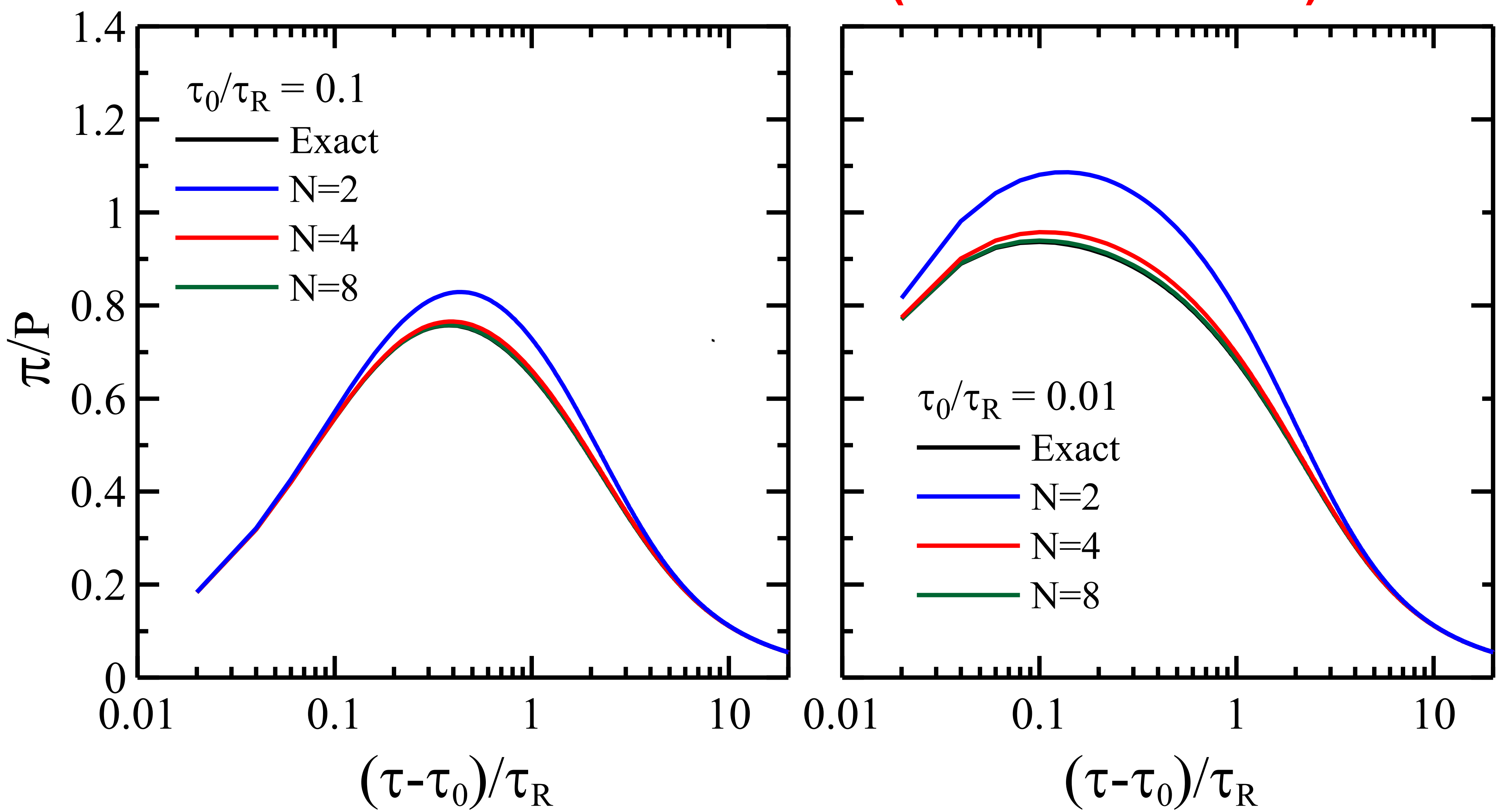


Ryblewski,Florkowski,Strickland
PRC 88 024903, NPA 916, 249

Bjorken scalling + RTA

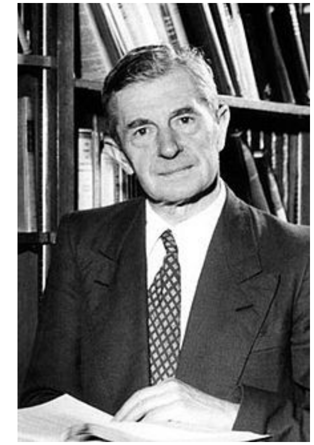
Convergence of the method of moments -

constant relaxation time (rest of this talk!)



Divergence of the Gradient expansion

Chapman-Enskog theory
(1910-1920)


Chapman


Enskog

$$k^\mu \partial_\mu f_{\mathbf{k}} = \frac{1}{\epsilon} C\left[f_{\mathbf{k}}\right]$$

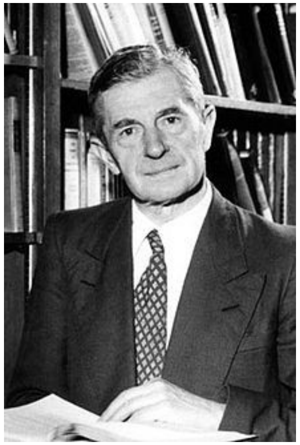
ϵ → **Knudsen number**

Perturbative expansion

$$f_{\mathbf{k}} = f_{\mathbf{k}}^{(0)} + \epsilon f_{\mathbf{k}}^{(1)} + \epsilon^2 f_{\mathbf{k}}^{(2)} + \ldots$$

Result is an expansion in powers of gradients of μ, T, and u^μ (gradient expansion)

Chapman-Enskog theory
(1910-1920)


Chapman


Enskog

Zeroth order truncation → **Ideal hydrodynamics**

First order truncation → **Navier-Stokes theory**

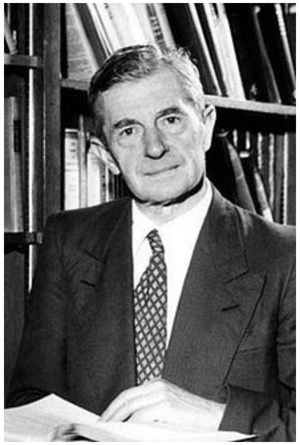
Higher order truncations are **unstable** (Bobylev)

H. Grad: CE is an asymptotic series, Physics of Fluids 6, 147 (1963).

First example of divergence: Couette flow problem (RTA), Santos *et al*, PRL 56, 1571 (1986).

Heller et al: Holography+Bjorken scaling, PRL 110, 211602 (2013).

Chapman-Enskog theory
(1910-1920)



Chapman

Enskog

1st order truncation: Navier-Stokes theory

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$$

$$\sigma^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\Delta^{\mu\nu}\theta.$$

2nd order truncation: Burnett theory

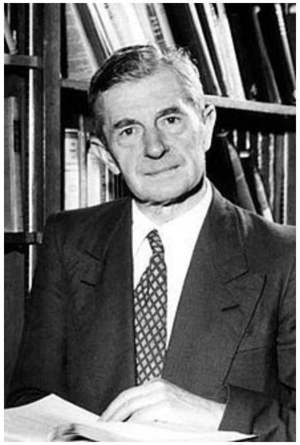
$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \eta_1 \omega_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \eta_4\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu}I^{\nu\rangle} + \eta_6 J^{\langle\mu}J^{\nu\rangle} + \eta_7 I^{\langle\mu}J^{\nu\rangle} + \eta_8\nabla^{\langle\mu}I^{\nu\rangle} + \eta_9\nabla^{\langle\mu}J^{\nu\rangle}.$$

$$\omega^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu).$$

$$\theta = \nabla_\mu u^\mu,$$

$$I^\mu \equiv \nabla^\mu \alpha_0, \quad J^\mu \equiv \nabla^\mu \beta_0,$$

Chapman-Enskog theory
(1910-1920)



Chapman

Enskog

Second-order truncation: Burnett theory

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \eta_1\omega_{\lambda}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\sigma_{\lambda}^{\nu\rangle} + \eta_4\sigma_{\lambda}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu}I^{\nu\rangle} + \eta_6 J^{\langle\mu}J^{\nu\rangle} + \eta_7 I^{\langle\mu}J^{\nu\rangle} + \eta_8\nabla^{\langle\mu}I^{\nu\rangle} + \eta_9\nabla^{\langle\mu}J^{\nu\rangle}.$$

Hydrodynamical constitutive equations are usually derived by *truncating* this series.

Effective theory: can be systematically corrected

Convergence is assumed!

Bjorken scalling + RTA

Moment equations: $M_{n,\ell} \equiv \frac{\rho_{n,\ell} - \rho_{n,\ell}^{\text{eq}}}{\rho_{n,\ell}^{\text{eq}}}$ $M_{1,1} = -\pi/P$

$$\partial_\tau M_{n,\ell} + \frac{1}{\tau_R} M_{n,\ell} + \frac{6\ell - n}{3\tau} M_{n,\ell} - \frac{n+3}{12\tau} M_{1,1}\left(1 + M_{n,\ell}\right) + \frac{1}{\tau}\frac{(n-2\ell)(1+2\ell)}{2\ell+3} M_{n,\ell+1} = -\frac{1}{\tau}\frac{4\ell(n+3)}{3(2\ell+3)}$$

Chapman-Enskog series: $K_N \sim \hat{\tau}^{-1} \equiv \tau_R/\tau$

$$M_{n,\ell} = \sum_{p=0}^{\infty} \frac{\alpha_p^{(n,\ell)}}{\hat{\tau}^p}$$

Taylor series in Knudsen number

Bjorken scalling + RTA

Series expansion:

$$M_{n,\ell} = \sum_{p=0}^{\infty} \frac{\alpha_p^{(n,\ell)}}{\hat{\tau}^p}$$

Zeroth and first order solution

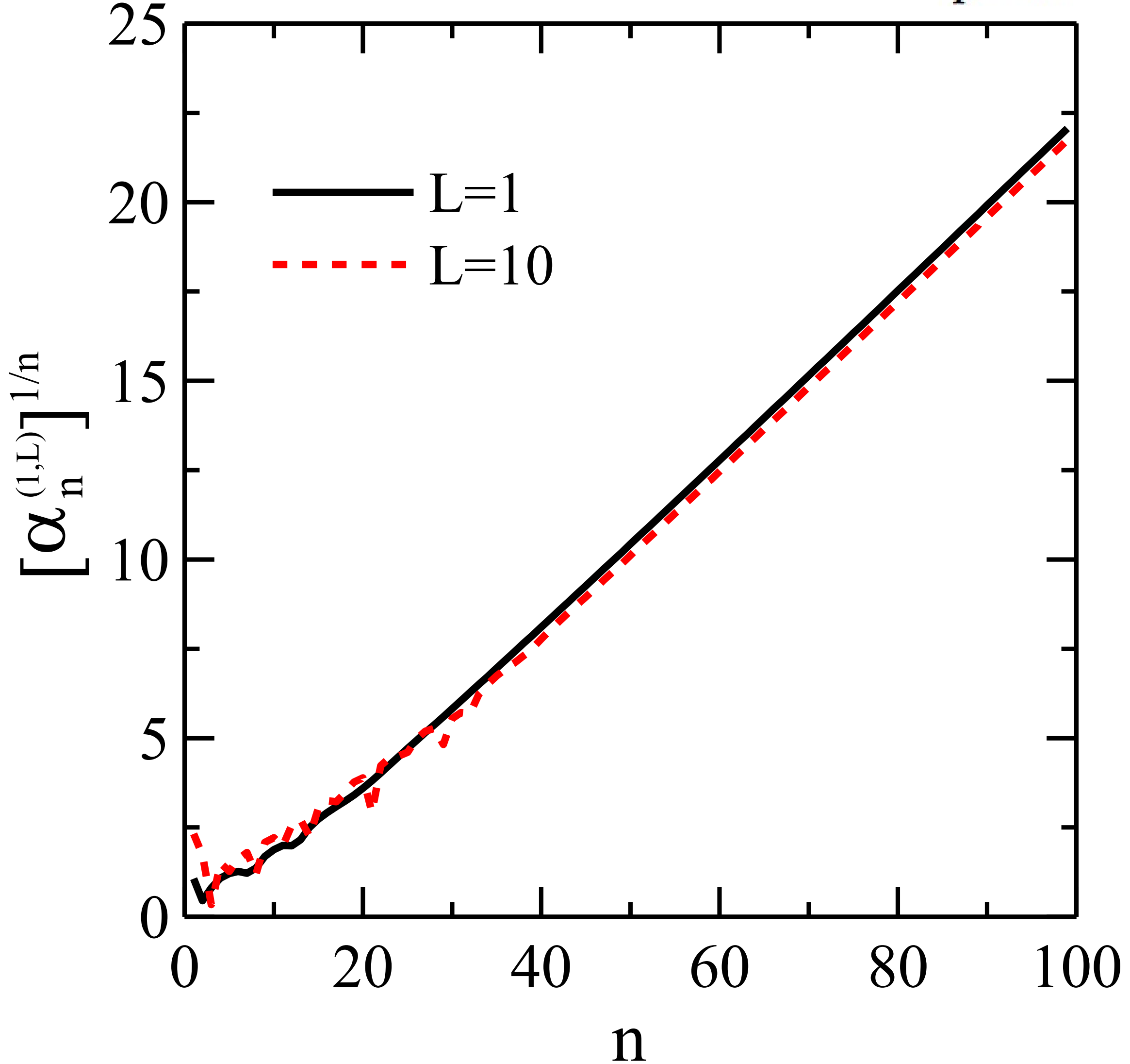
$$\alpha_0^{(n,\ell)} = 0 \qquad \alpha_1^{(n,\ell)} = -\frac{4\ell\,(n+3)}{3\,(2\ell+3)}$$

Higher order solutions

$$\alpha_{m+1}^{(n,\ell)} = -\frac{6\ell - n - 3m}{3}\alpha_m^{(n,\ell)} + \frac{n+3}{12}\alpha_m^{(1,1)} - \frac{(n-2\ell)\,(1+2\ell)}{2\ell+3}\alpha_m^{(n,\ell+1)} + \frac{n+3}{12}\sum_{p=0}^{m}\alpha_p^{(1,1)}\alpha_{m-p}^{(n,\ell)}$$

Bjorken scalling + RTA

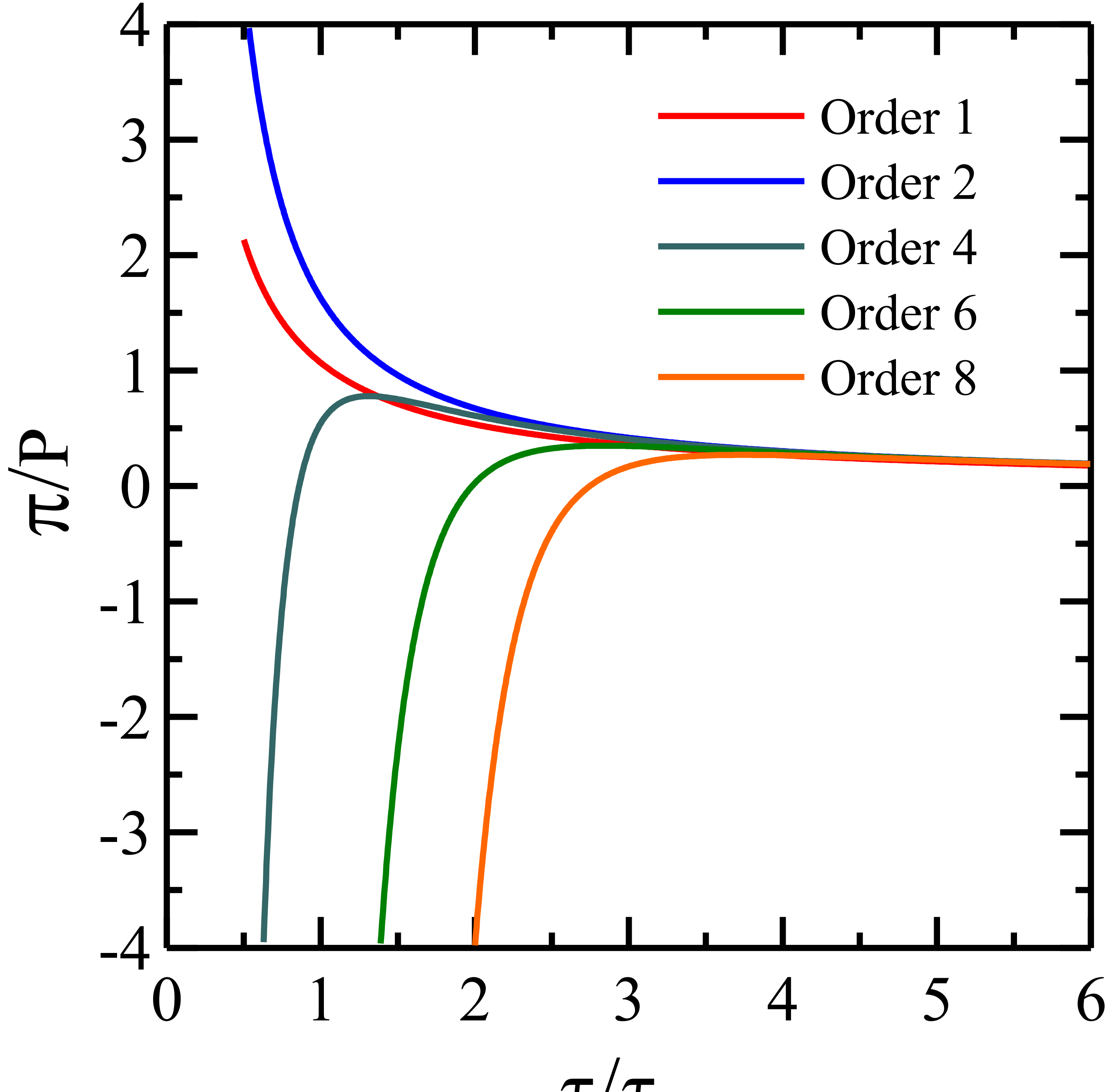
Series expansion: $$M_{n,\ell} = \sum_{p=0}^{\infty} \frac{\alpha_p^{(n,\ell)}}{\hat{\tau}^p}$$



$\alpha_p \sim p!$

Series has a **zero radius of convergence**

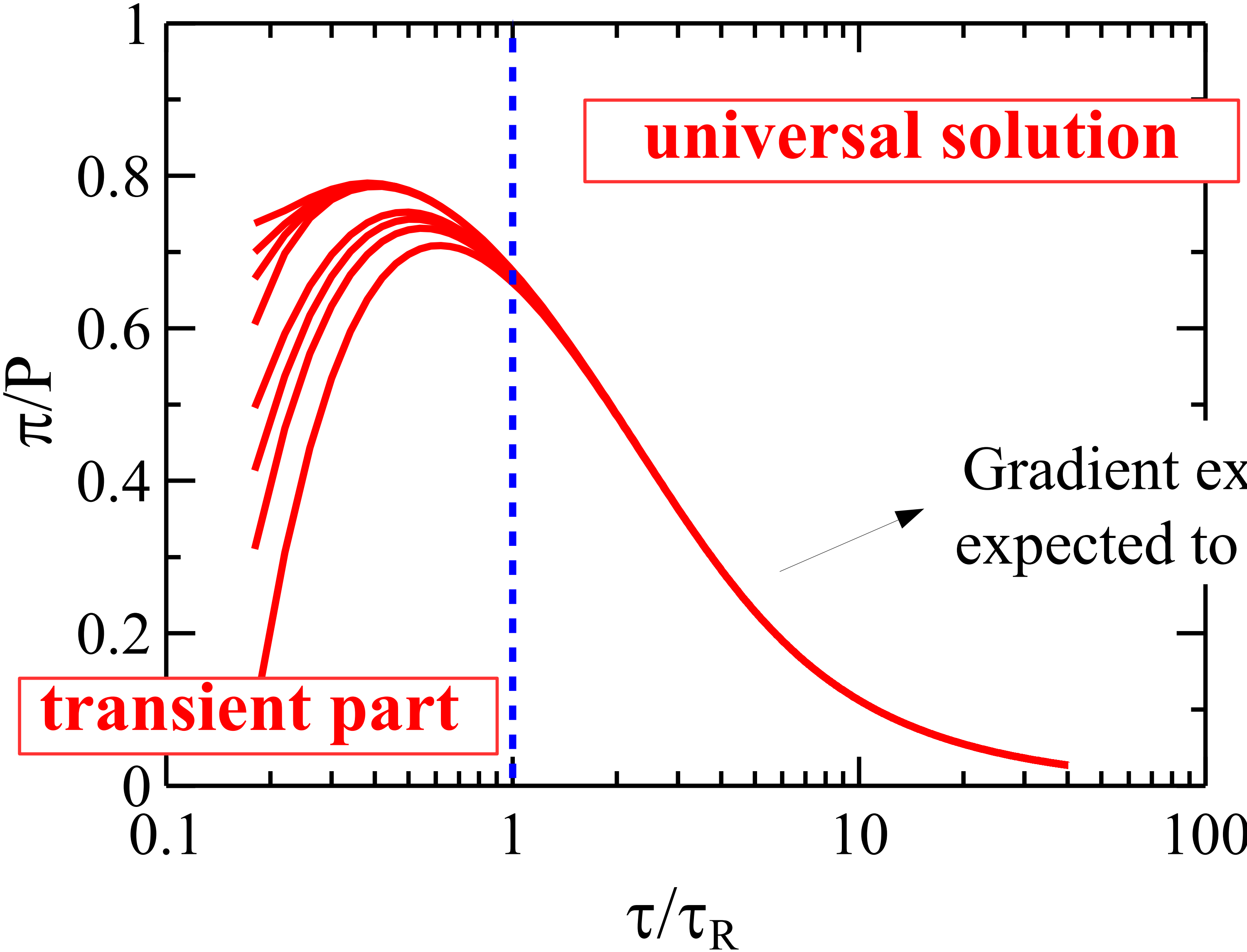
This is valid for all n and L



Order 1
Order 2
Order 4
Order 6
Order 8
π/P
τ/τ_R
4
3
2
1
0
-1
-2
-3
-4
0
1
2
3
4
5
6

Fluid-dynamical regime

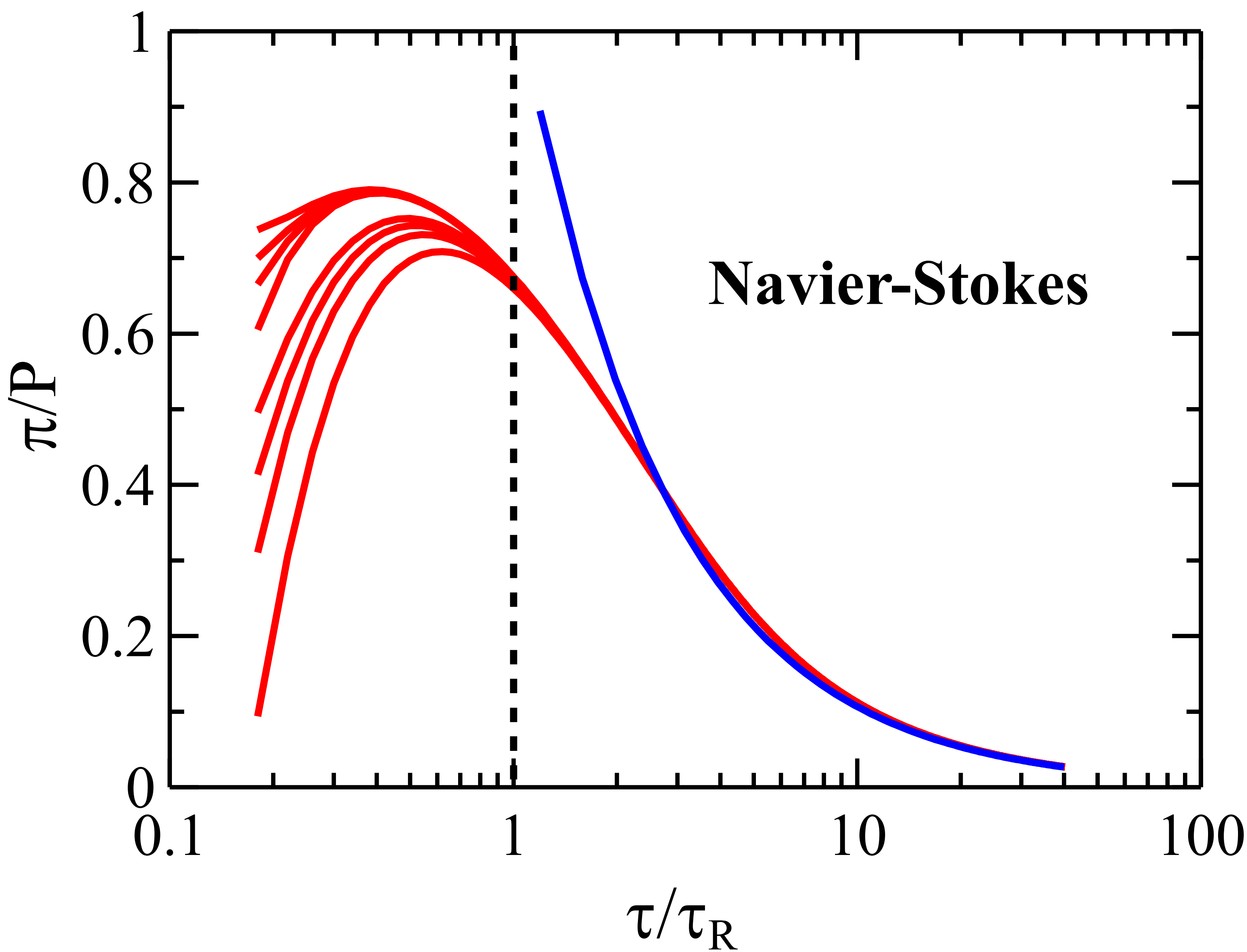
Late-time solution of the Boltzmann equation appears to be universal – constitutive relations?



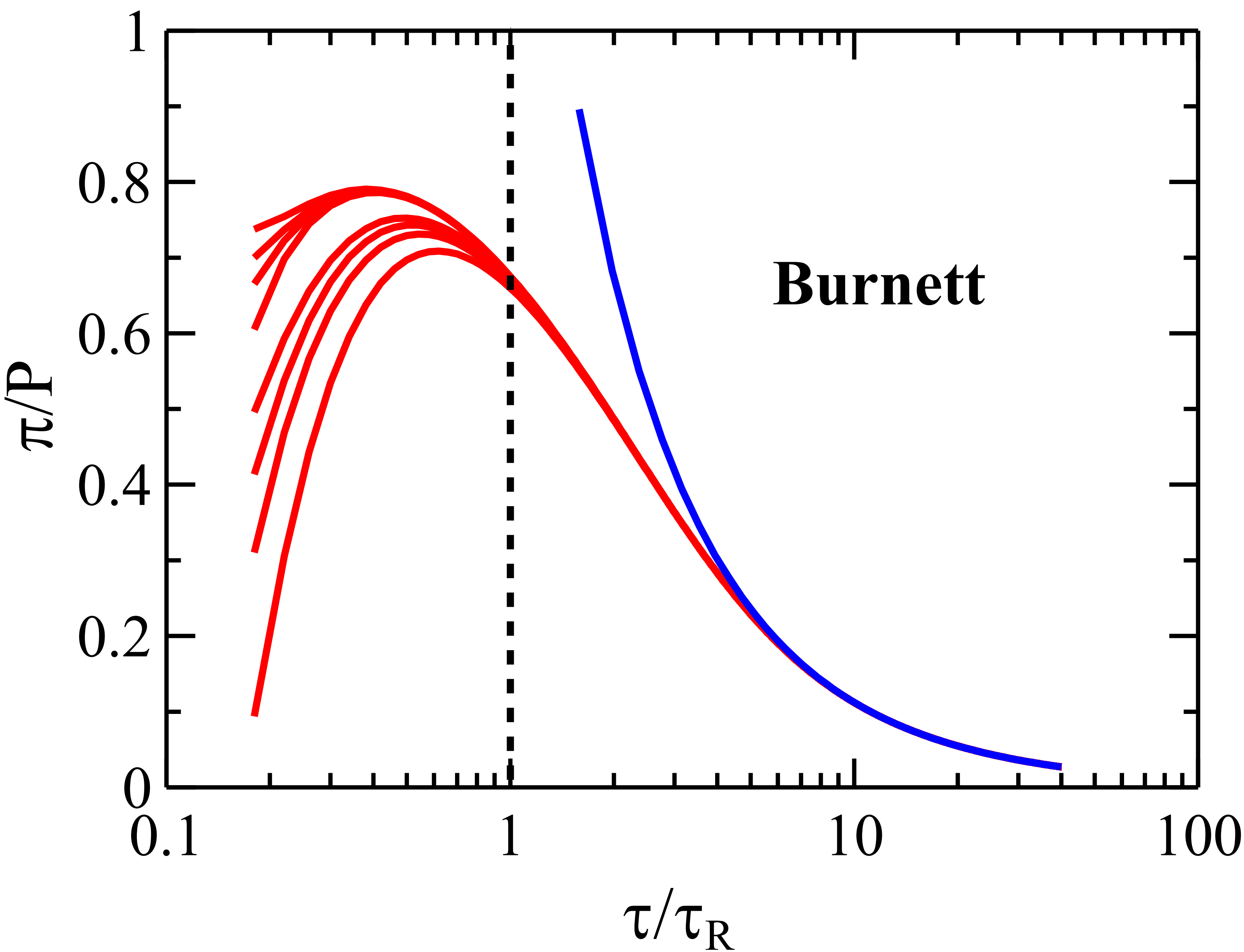
M. Heller et al, arxiv:1103.3452

Gradient expansion was expected to describe this

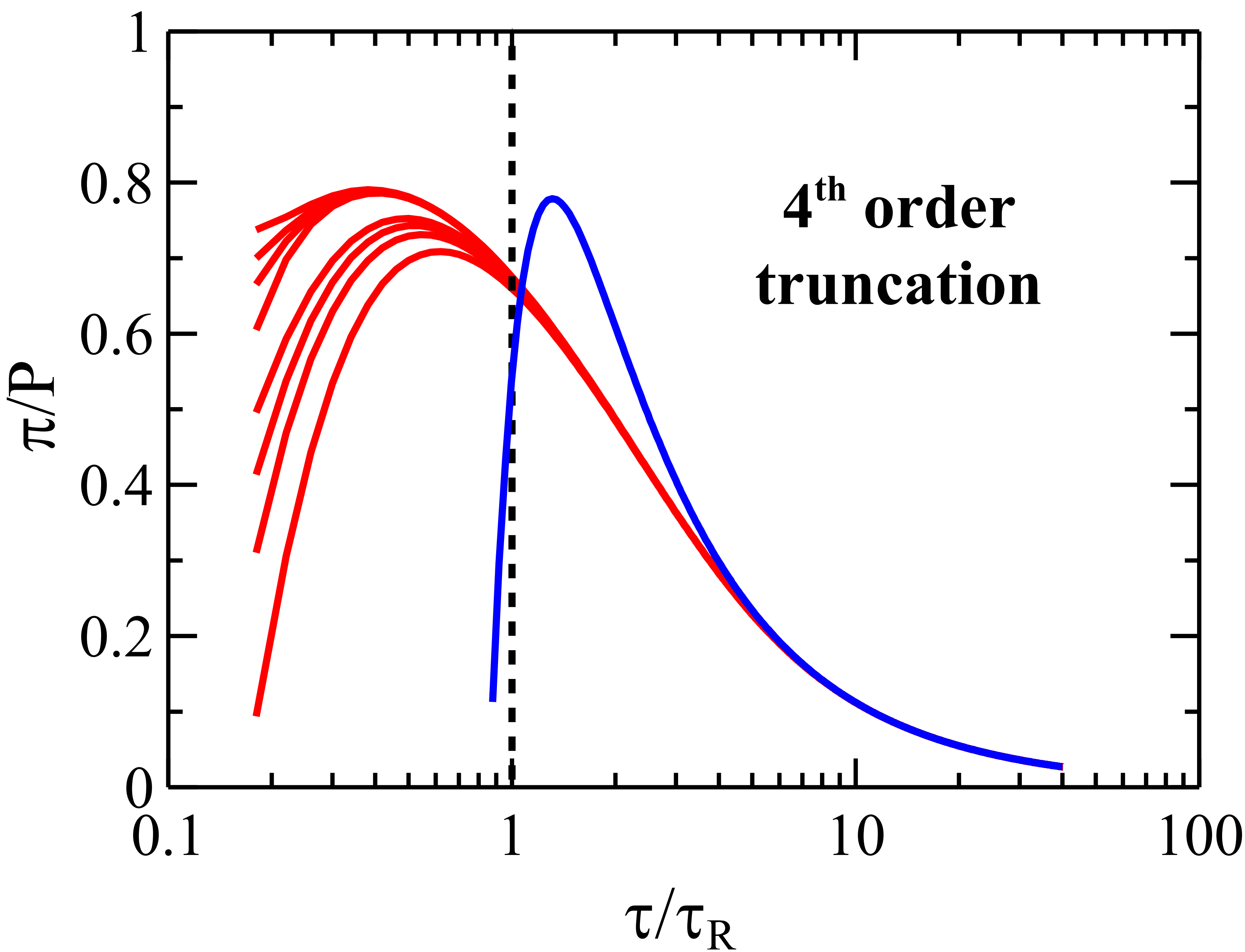
What now?



1
0.8
0.6
0.4
0.2
0
0.1
1
10
100
π/P
τ/τ_R
Navier-Stokes



Burnett
π/P
τ/τ_R
0
0.2
0.4
0.6
0.8
1
0.1
1
10
100



1
0.8
0.6
0.4
0.2
0
0.1
1
10
100
π/P
τ/τ_R
4th order
truncation

Generalized Chapman-Enskog series

Series expansion:

$$M_{n,\ell}(\hat{\tau}) = \sum_{p=0}^{\infty} \frac{\beta_p^{(n,\ell)}(\hat{\tau})}{\hat{\tau}^p}$$

Non-perturbative dependence in Kn

$$K_N \sim \hat{\tau}^{-1}$$

$$f(\xi, \mathbf{x}, t; \epsilon) = \sum_{0}^{\infty} \epsilon^n f^n(\xi, \mathbf{x}, t).$$

$$f = \sum_m \exp\left(-\frac{\lambda_m t}{\epsilon}\right) f_m, \qquad f_m = \sum_n \epsilon^n f_m^n.$$

H. Grad

Physics of Fluids 6, 147 (1963).

Another Chapman-Enskog series

Series expansion:

$$M_{n,\ell}(\hat{\tau}) = \sum_{p=0}^{\infty} \frac{\beta_p^{(n,\ell)}(\hat{\tau})}{\hat{\tau}^p}$$

Non-perturbative dependence in Kn

$$K_N \sim \hat{\tau}^{-1}$$

Zeroth order:

$$\partial_{\hat{\tau}} \beta_0^{(n,\ell)} + \beta_0^{(n,\ell)} = 0$$

Higher orders:

$$\partial_{\hat{\tau}} \beta_{m+1}^{(n,\ell)} + \beta_{m+1}^{(n,\ell)} = -\frac{4\ell(n+3)}{3(2\ell+3)}\delta_{m,0}$$

$$-\frac{(n-2\ell)(1+2\ell)}{2\ell+3}\beta_m^{(n,\ell+1)} - \frac{6\ell-n-3m}{3}\beta_m^{(n,\ell)}$$

$$+\frac{n+3}{12}\beta_m^{(1,1)} + \frac{n+3}{12}\sum_{p=0}^{m} \beta_{m-p}^{(1,1)}\beta_p^{(n,\ell)}$$

Solutions of the expansion: eq. initial state

New zeroth order solution

$$\Delta\hat{\tau} \equiv \hat{\tau} - \hat{\tau}_0$$

$$\beta_0^{(n,\ell)}(\hat{\tau}) = \beta_0^{(n,\ell)} \underbrace{(\hat{\tau}_0)\exp(-\Delta\hat{\tau})}_{\text{purely transient}}$$

purely transient

It is not zero, but tends to zero

New first order solution

$$\Delta\hat{\tau} \equiv \hat{\tau} - \hat{\tau}_0$$

$$\beta_1^{(n,\ell)} = \underbrace{\alpha_1^{(n,\ell)}}_{\text{asymptotic part}} \underbrace{[1 - \exp(-\Delta\hat{\tau})]}_{\text{transient part}}$$

asymptotic part transient part

Solutions of the expansion: eq. initial state

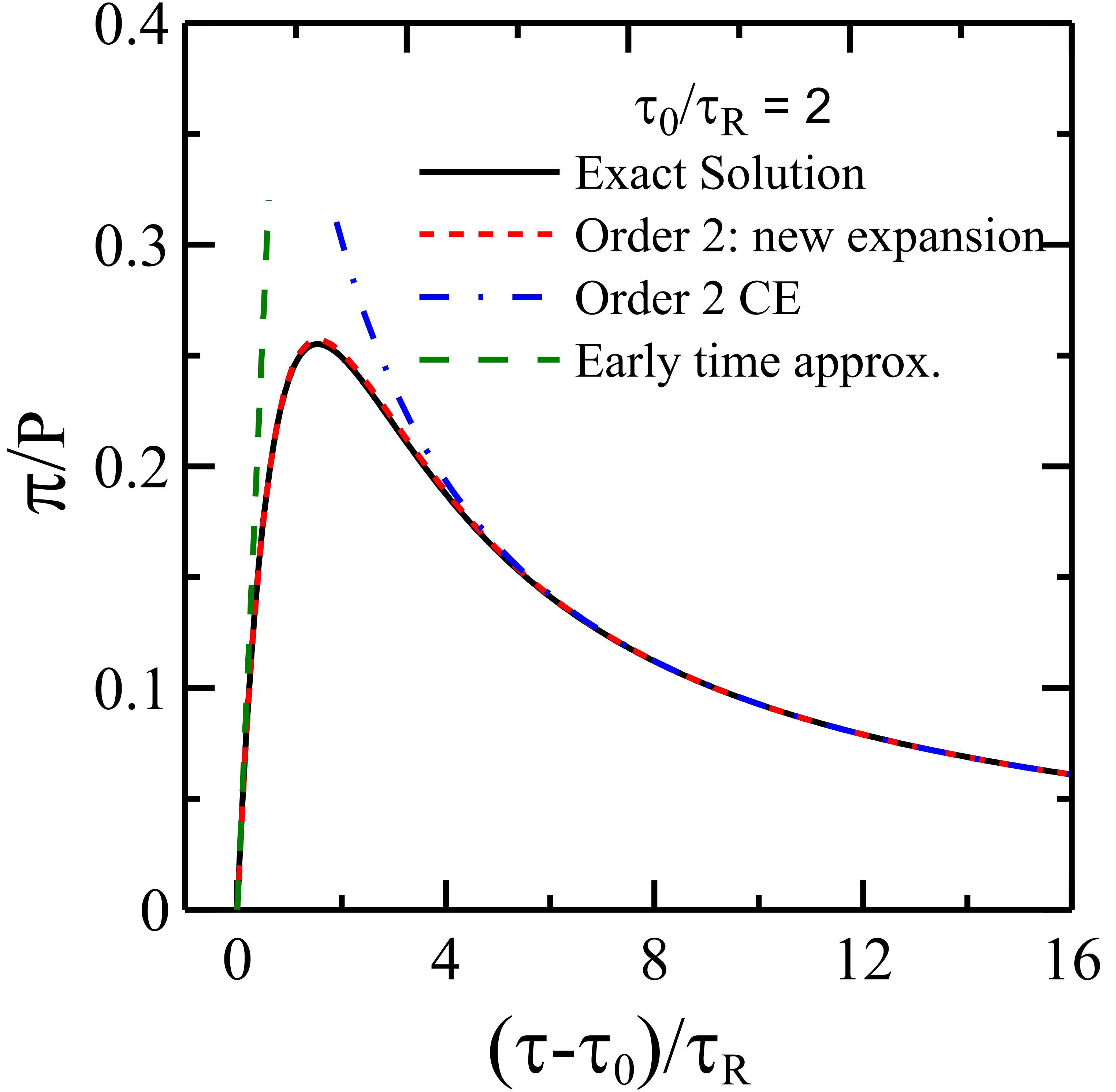
New second order solution $\Delta\hat{\tau} \equiv \hat{\tau} - \hat{\tau}_0$

$$\beta_2^{(n,\ell)}(\hat{\tau}) = \underbrace{\alpha_2^{(n,\ell)}}_{\text{asymptotic part}} \underbrace{\exp(-\Delta\hat{\tau})\left[\exp(\Delta\hat{\tau}) - \Delta\hat{\tau} - 1\right]}_{\text{transient part}}$$

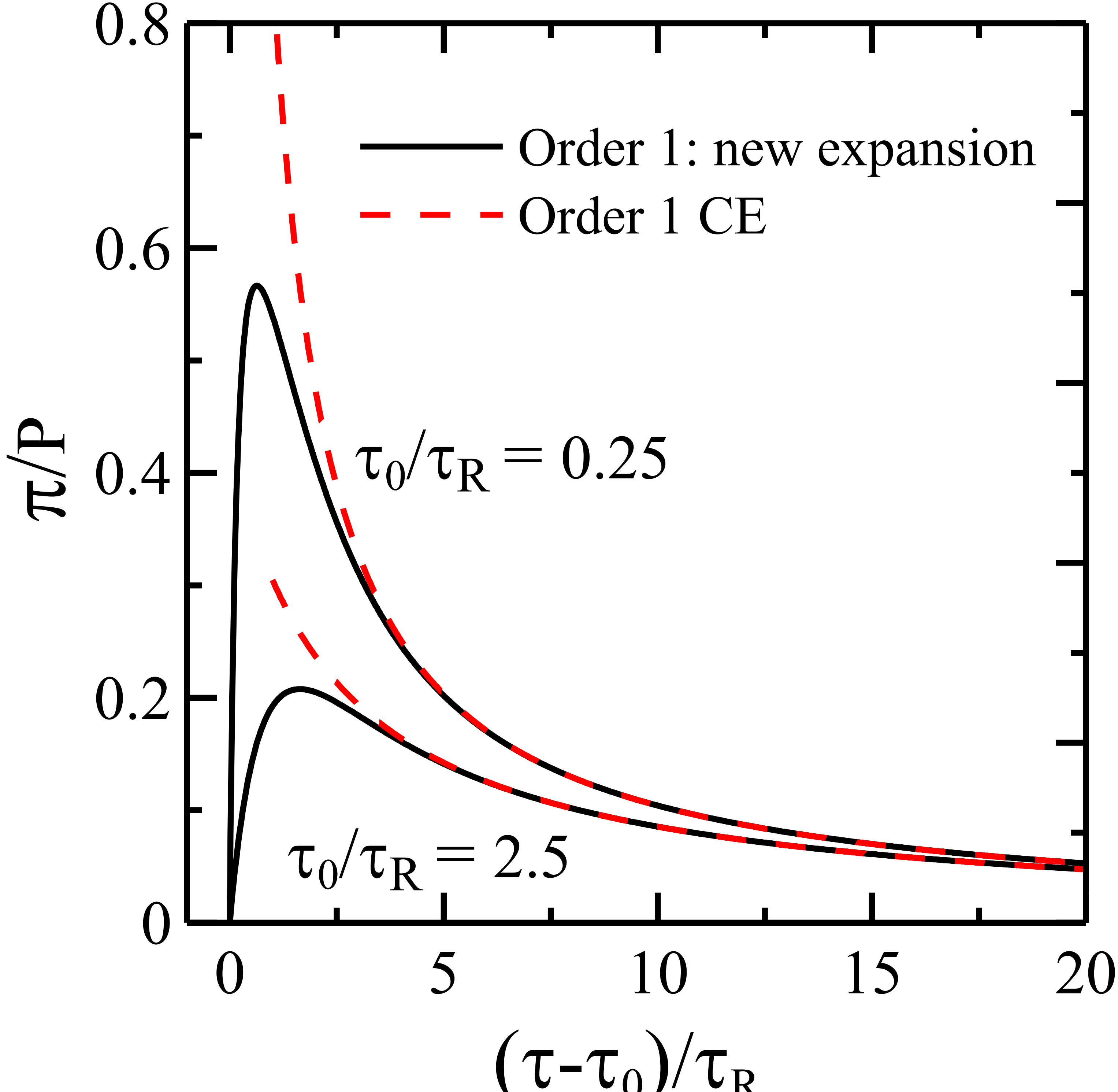
Transient terms appear naturally. Israel-Stewart theory arises as a natural consequence of this expansion scheme.

$\longrightarrow$ $\underbrace{\exp(-\Delta\hat{\tau}) \sim \exp(-\mathrm{Kn}^{-1})}$

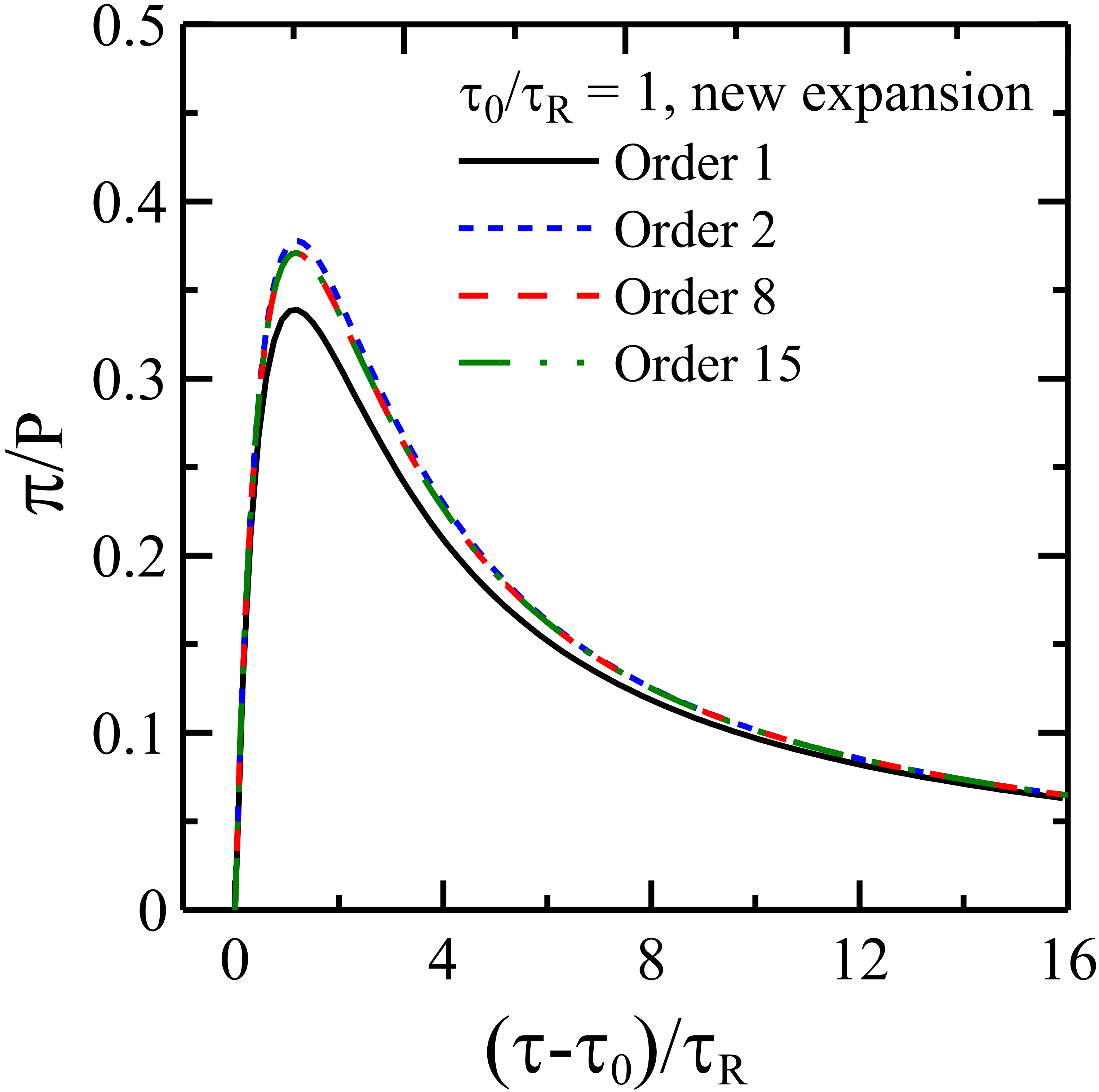
Expansion in powers of Knudsen number impossible



0.4
0.3
0.2
0.1
0
π/P
0
4
8
12
16
(τ-τ0)/τR
τ0/τR = 2
Exact Solution
Order 2: new expansion
Order 2 CE
Early time approx.



0.8
0.6
0.4
0.2
0
0
5
10
15
20
π/P
(τ-τ0)/τR
Order 1: new expansion
Order 1 CE
τ0/τR = 0.25
τ0/τR = 2.5

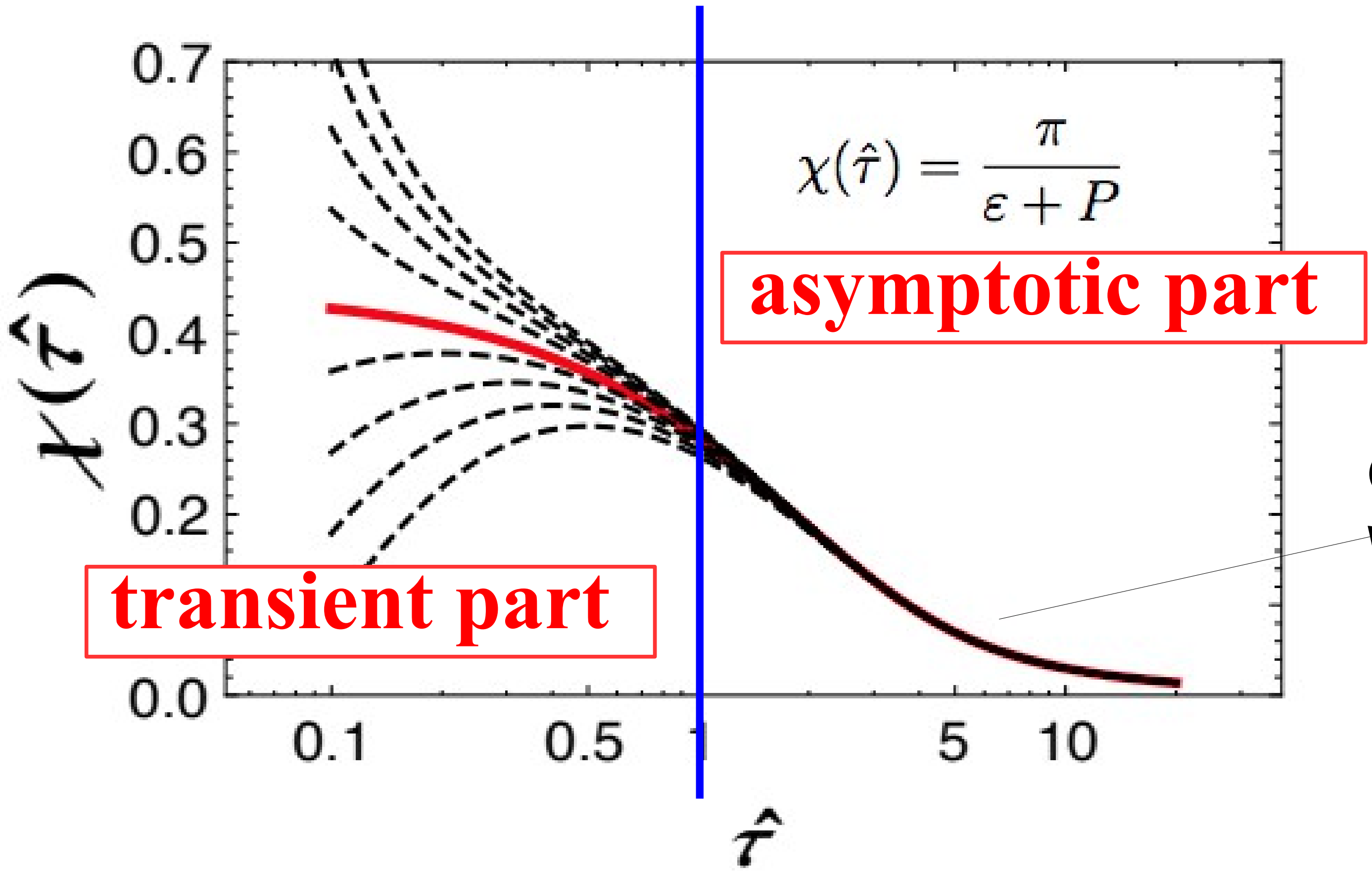


0.5
0.4
0.3
0.2
0.1
0
π/P
τ0/τR = 1, new expansion
Order 1
Order 2
Order 8
Order 15
0
4
8
12
16
(τ-τ0)/τR

Israel-Stewart theory
and
the gradient expansion

Attractor: Israel-Stewart theory

$$\tau_R \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \delta_{\pi\pi}\,\theta\pi^{\mu\nu} + \tau_{\pi\pi}\,\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\lambda}\sigma^{\beta}_{\lambda} - 2\,\tau_R \Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha}_{\lambda}\omega^{\beta\lambda} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$



Constitutive relation?

Gradient expansion?

So we can study these expansions here as well

Analytical Solution for constant relaxation times

$$\partial_{\hat{\tau}}\chi + \chi + \frac{4}{3\hat{\tau}}\chi^2 = \frac{3a}{4\hat{\tau}},$$

$$a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

$$\chi(\hat{\tau}) = \frac{3\sqrt{a}}{4}\left[\frac{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}\right]$$

First analytical expression for a hydrodynamic attractor

$$\chi(\hat{\tau}) \to \chi_{att}(\hat{\tau}) = \frac{3\sqrt{a}}{4}\left[\frac{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}\right]$$

Analytical Solution for constant relaxation times

$$\partial_{\hat\tau}\chi + \chi + \frac{4}{3\hat\tau}\chi^2 = \frac{3a}{4\hat\tau},$$

$$a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

Trans-series can be easily generated (a=1)

$$\chi(\hat\tau) = \frac{3}{4}\sum_{n=0}^{\infty}\frac{1}{\hat\tau^{n+1}} - \frac{3}{4}\alpha\, e^{-\hat\tau}\sum_{n=0}^{\infty}\frac{1}{\hat\tau^{n}}\left(1 + \frac{1}{\hat\tau} + \frac{n+1}{\hat\tau^2}\right) + \mathcal{O}(\alpha^2 e^{-2\hat\tau})$$

$$= \frac{3}{4}\frac{1}{(\hat\tau - 1)} - \frac{3}{4}\alpha\, e^{-\hat\tau}\frac{\hat\tau^2}{(\hat\tau - 1)^2} + \mathcal{O}(\alpha^2 e^{-2\hat\tau}),$$

non-perturbative

Ressumed
Gradient expansion (finite radius of convergence)

Analytical Solution for constant relaxation times

For physical case, gradient expansion does not converge

a=16/45

Trans-series can be easily generated (a=1)

$$\chi(\hat{\tau}) = \frac{3}{4}\sum_{n=0}^{\infty}\frac{1}{\hat{\tau}^{n+1}} - \frac{3}{4}\alpha\, e^{-\hat{\tau}}\sum_{n=0}^{\infty}\frac{1}{\hat{\tau}^{n}}\left(1+\frac{1}{\hat{\tau}}+\frac{n+1}{\hat{\tau}^2}\right) + \mathcal{O}(\alpha^2 e^{-2\hat{\tau}})$$

$$= \frac{3}{4}\frac{1}{(\hat{\tau}-1)} - \frac{3}{4}\alpha\, e^{-\hat{\tau}}\frac{\hat{\tau}^2}{(\hat{\tau}-1)^2} + \mathcal{O}(\alpha^2 e^{-2\hat{\tau}}),$$

non-perturbative

Ressumed
Gradient expansion (finite radius of convergence)

Slow-Roll expansion

$$\epsilon\,\hat{\tau}\frac{d\chi}{d\hat{\tau}} + \frac{4}{3}\chi^2 + \hat{\tau}\chi - \frac{3a}{4} = 0 \qquad a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

$$\chi(\hat{\tau};\epsilon) = \sum_{n=0}^{\infty}\chi_n(\hat{\tau})\epsilon^n$$

Solution of the form:

$$\chi_0(\hat{\tau}) = \frac{3}{8}\left(\sqrt{\hat{\tau}^2 + 4a} - \hat{\tau}\right)$$

$$\chi_n(\hat{\tau}) = -\frac{1}{\sqrt{\hat{\tau}^2 + 4a}}\left(\hat{\tau}\frac{d\chi_{n-1}}{d\hat{\tau}} + \frac{4}{3}\sum_{m=1}^{n-1}\chi_{n-m}\chi_m\right)$$

Slow-Roll expansion

$$\epsilon\,\hat{\tau}\frac{d\chi}{d\hat{\tau}} + \frac{4}{3}\chi^2 + \hat{\tau}\chi - \frac{3a}{4} = 0$$

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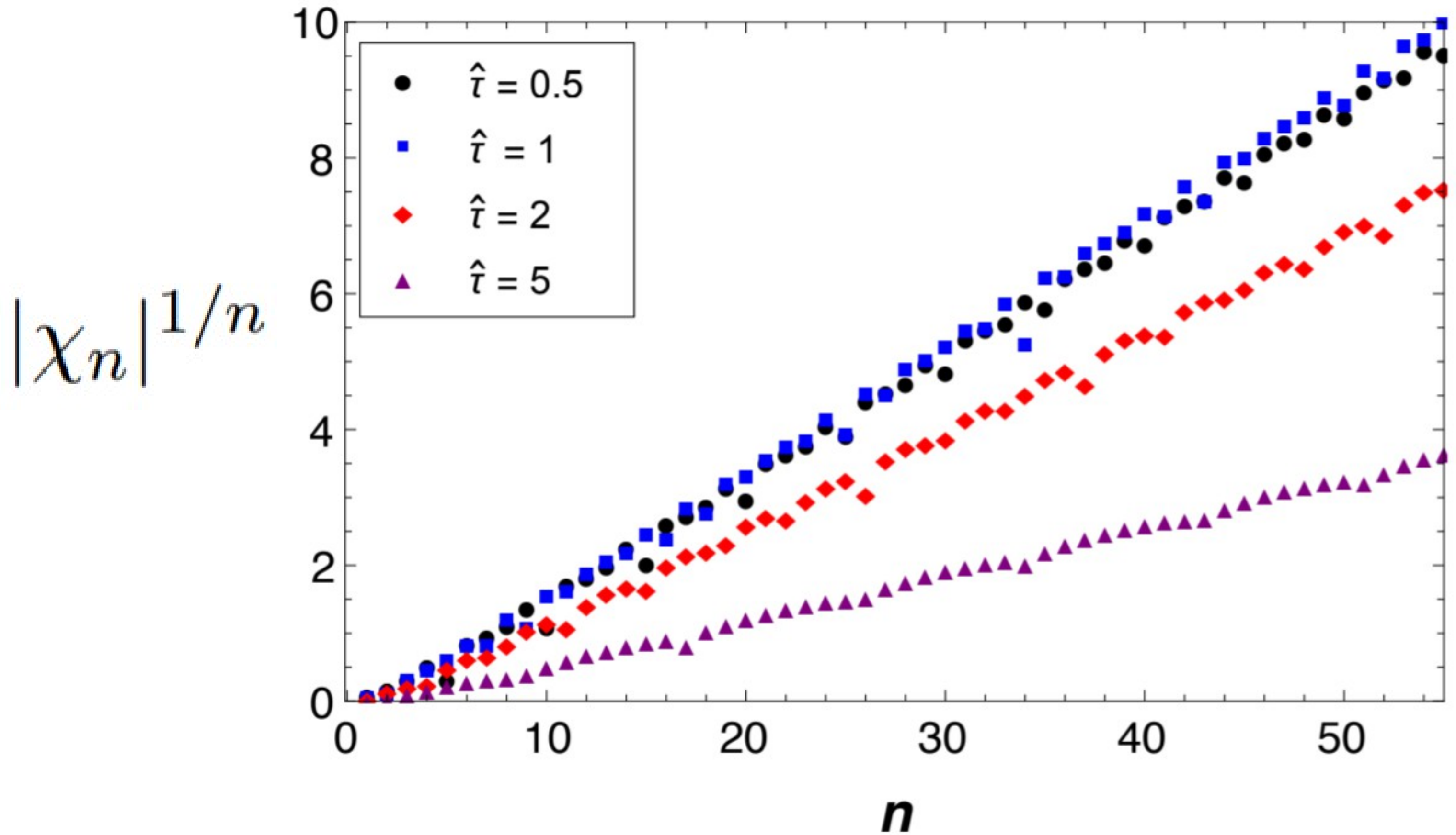

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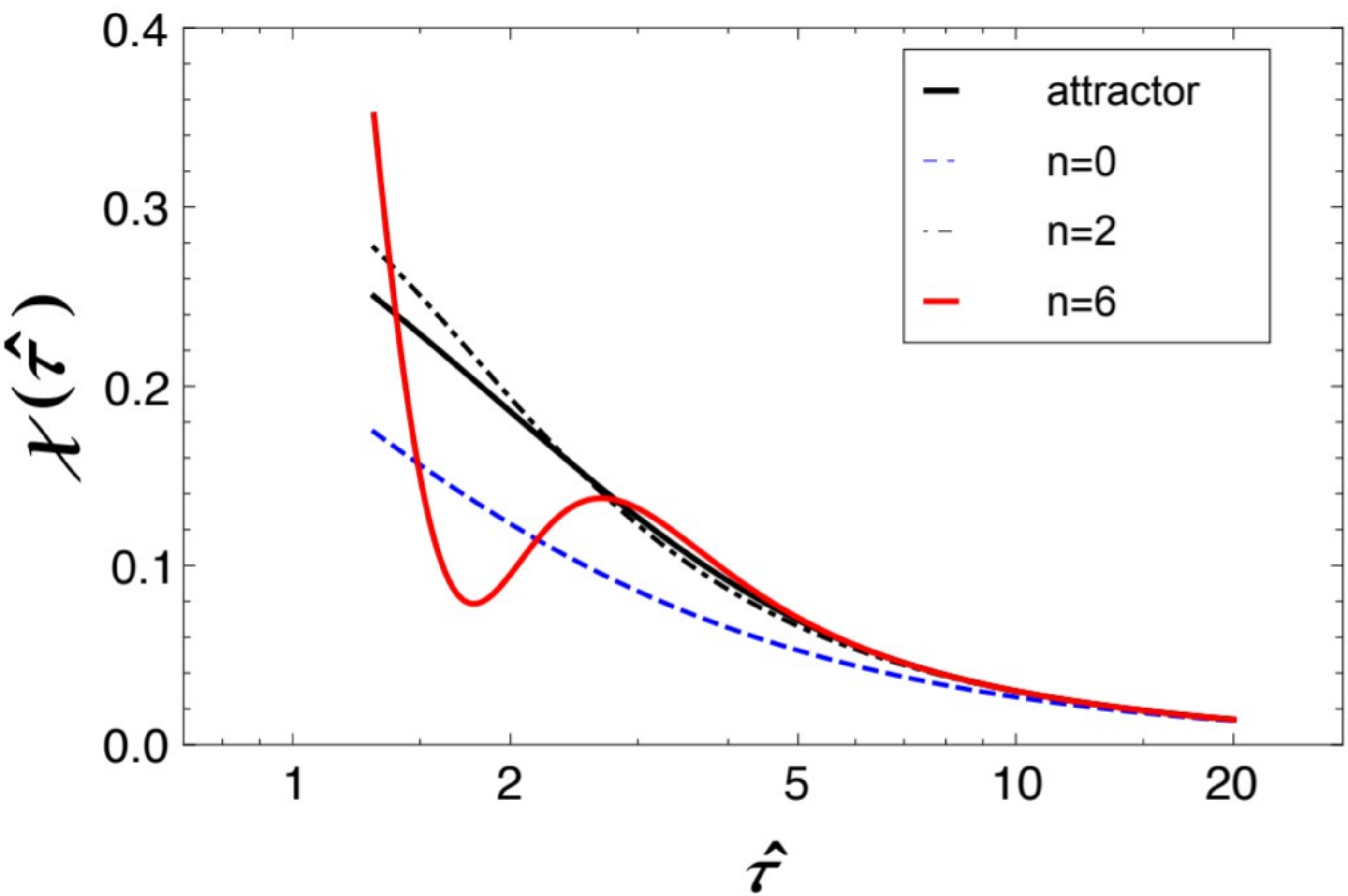
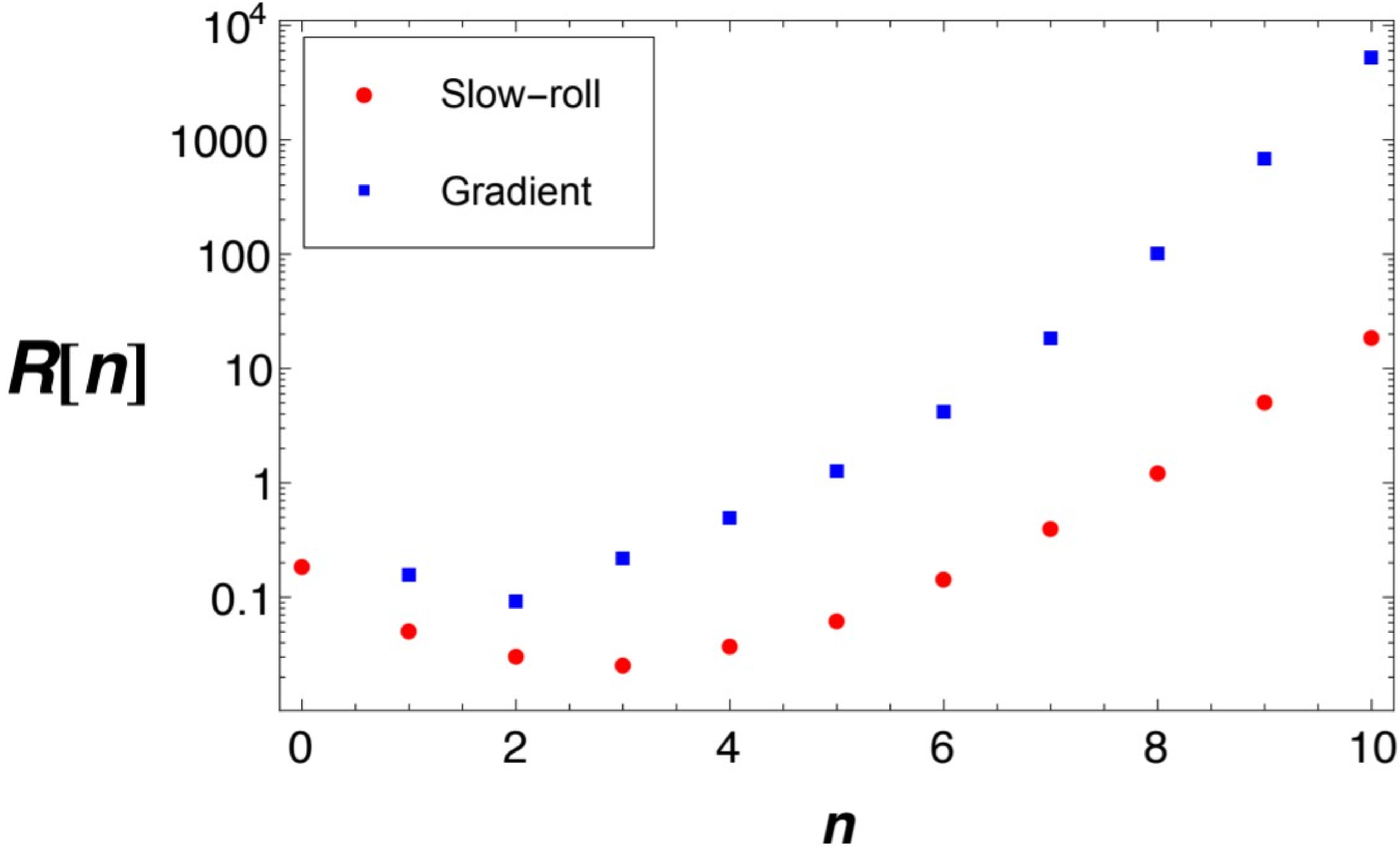


Optimal truncation: $R[n] = \frac{\int_{\hat{\tau}_0}^{\hat{\tau}_f} d\hat{\tau}\, \left| \chi_{att}(\hat{\tau}) - \sum_{m=0}^{n} \chi_n(\hat{\tau}) \right|}{\int_{\hat{\tau}_0}^{\hat{\tau}_f} d\hat{\tau}\, \chi_{att}(\hat{\tau})}$



Generalized gradient expansion

$$\partial_{\hat{\tau}}\chi + \chi + \frac{4}{3\hat{\tau}}\chi^2 = \frac{3a}{4\hat{\tau}},$$

$$\chi(\hat{\tau}) = \frac{3a}{4}\sum_{n=0}^{\infty}\frac{c_n(\hat{\tau})}{\hat{\tau}^n}$$

$$\begin{cases} c_0(\hat{\tau}) = \dfrac{4\chi_0}{3a}e^{-(\hat{\tau}-\hat{\tau}_0)} \\ c_1(\hat{\tau}) = \left(1 - e^{-(\hat{\tau}-\hat{\tau}_0)}\right)\left(1 - \dfrac{16\chi_0^2}{9a}e^{-(\hat{\tau}-\hat{\tau}_0)}\right) \\ \dfrac{dc_{n+1}}{d\hat{\tau}} + c_{n+1} = n\,c_n - a\displaystyle\sum_{m=0}^{\infty}c_{n-m}c_m. \end{cases}$$

Generalized gradient expansion

$$\partial_{\hat{\tau}}\chi + \chi + \frac{4}{3\hat{\tau}}\chi^2 = \frac{3a}{4\hat{\tau}},$$

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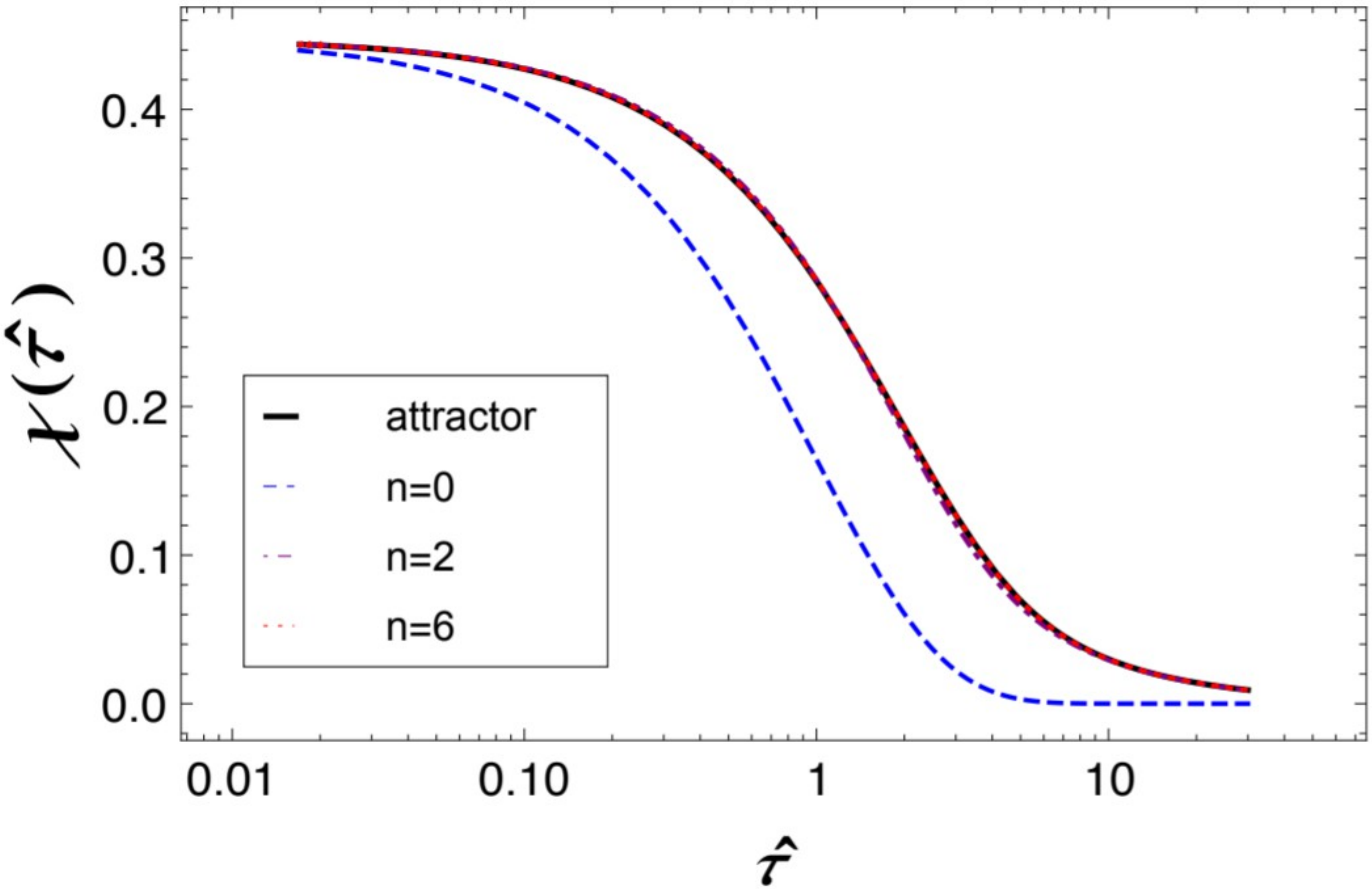

Conclusions

We studied the convergence of CE and method of moments

Assumptions: kinetic theory + RTA + Bjorken scaling

➔CE series diverges, just like in holography.

➔Method of moments converges (fast) to exact solution

➔We proposed a new expansion that considers non-perturbative corrections in Knudsen number

Conclusions

We studied the convergence of CE and method of moments

Assumptions: kinetic theory + RTA + Bjorken scaling

- Lack of convergence is not necessarily a problem – divergent series can capture some features of solution

- This is why NS and Burnett are not that bad

- How can the theory be systematically improved? What is the domain of applicability?