Relativistic dissipative hydrodynamics for paricles of arbitrary mass

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Outline

1. Motivation: Why study fluids and transport?
2. Goal: Dissipative Hydrodynamics
3. Tool: Kinetic theory
4. Results: Transport equations for hydrodynamics
   - Transport coefficients for arbitrary mases
   - Non-relativistic limit reproduces the Grad equations

Phenomenology of fluid-dynamics: transport coefficients

Slides by Yuuka Kanakubo. *Dynamical evolution and collective dynamics*. Apr. 2025. URL: https://indico.cern.ch/event/1334113/contributions/6369376/.

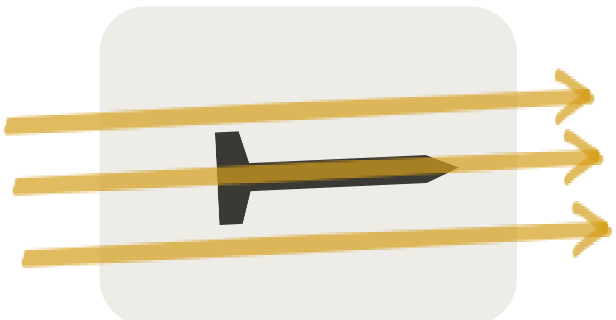
How does it flow?

like water or honey?

Characterised by **transport coefficient** of Apfelwein

Transport coefficients

How it responds when you "hit" it



Electric field

▼

Electric current

$$J = \sigma E$$

Electric conductivity

How easily electric current can flow through the nail

=> Property of the nail!

Transport coefficients

How it responds when you "hit" it

Temperature gradient

Heat current

$$q^i = -\kappa \partial^i T$$

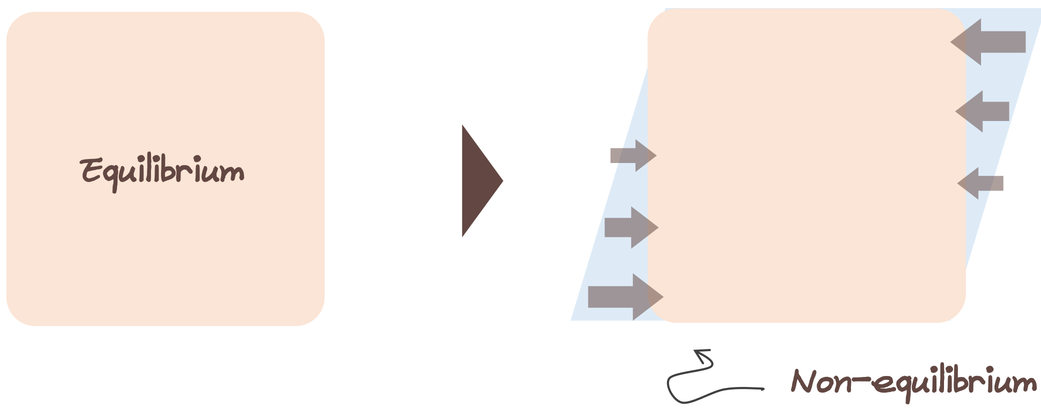
Heat conductivity

How easily the cup can be heated up

=> Property of the cup!

Transport coefficients

Drag the system slightly away from equilibrium



In hydrodynamics, for instance...

η **Shear viscosity**

How it responds when you "hit" it

=> Dynamical property of matter!

*Linear response theory -> Transport coefficients can be derived microscopically

How does it flow?

like water or honey?

Characterised by **transport coefficient** of **QGP**

Second-order kinetic equation for non-uniform gases

- Navier-Stokes (first-order) equations are universal for different gases close to equilibrium, but fail for more complicated flows.
- Going further out-of-equilibrium requires higher-order corrections to hydrodynamical quantities. 🥲
- Grad's equations provide the evolution of dissipative quantities of the fluid at second order:

$$-\frac{2}{3}\frac{P_0}{\eta}q^i = \frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{5}{2}P_0\frac{\partial RT}{\partial x_i} + \frac{7}{5}q^i\frac{\partial u_k}{\partial x_k} + q_k\frac{\partial u^i}{\partial x_k} + \frac{2}{5}q_k\left(\frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i}\right) - RT\pi^{ik}\frac{\partial \ln P_0}{\partial x^k} + RT\frac{\partial \pi^{ik}}{\partial x^k} + \frac{7}{2}\pi^{ik}\frac{\partial RT}{\partial x^k} - \frac{\pi^{ik}}{\rho_0}\frac{\partial \pi_{kl}}{\partial x_l}\,, \tag{1}$$

$$-\frac{P_0}{\eta}\pi^{ij} = \frac{\mathrm{d}\pi^{ij}}{\mathrm{d}t} + 2P_0\frac{\partial u^{\langle i}}{\partial x_{j\rangle}} + \pi^{ij}\frac{\partial u_k}{\partial x_k} + 2\pi^{r\langle i}\frac{\partial u^{j\rangle}}{\partial x^r} + \frac{4}{5}\frac{\partial q^{\langle i}}{\partial x_{j\rangle}}\,, \tag{2}$$

where $\boldsymbol{q}$ is the heat flow, ρ_0 the mass density, and $R \equiv k_B/m \equiv 1/m$ the gas constant per unit mass.

Navier-Stokes fails to predict the correct flow profile for, e.g., force-driven Poiseuille flow.

Agrawal et al. *Microscale Flow and Heat Transfer*

Brief reminder of deriving hydrodynamics from kinetic theory

Based on the talk by David Wagner. *Inverse Reynolds-Dominance approach to transient fluid dynamics*. June 2022. URL: https://itp.uni-frankfurt.de/~hees/transport-meeting/ss22/talk-Wagner.pdf.

Set-up of hydrodynamics

Hydrodynamics: Conservation equations

$$\partial_\mu T^{\mu\nu} = 0 \, , \qquad \partial_\mu N^\mu = 0 \tag{3}$$

- Hydrodynamics from $(4+1=5)$ conservation equations
  - Ideal case: Sufficient (if equation of state is supplied)
    - $\longrightarrow$ Variables: $\epsilon,\ n,\ u^\mu$
  - Dissipative case: Underdetermined
    - $\longrightarrow$ Variables: $\epsilon,\ n,\ u^\mu,\ \Pi,\ V^\mu,\ \pi^{\mu\nu}$
- **Fundamental question**: How to obtain the dissipative components of N^μ and $T^{\mu\nu}$? 😮

Decomposition of conserved currents (Landau frame)

$$N^\mu = n u^\mu + V^\mu \tag{4}$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \tag{5}$$

Projectors: $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu,\ \Delta^{\mu\nu}_{\alpha\beta} \equiv (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha)/2 - \Delta^{\mu\nu}\Delta_{\alpha\beta}/3$

Equations of hydrodynamics

Projecting the derivative,

$$\partial_\mu A = (u_\mu u_\nu \partial^\nu + \Delta_{\mu\nu}\partial^\nu)\, A \equiv u_\mu \dot{A} + \nabla_\mu A \,,$$

we project the conservation equations:

$$\partial_\mu N^\mu = 0 \quad \Leftrightarrow \quad \dot{n}_0 + n_0\theta + \partial_\mu V^\mu = 0 \,, \tag{6}$$

$$u_\alpha \partial_\beta T^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \dot{\varepsilon}_0 + (\varepsilon_0 + P_0 + \Pi)\,\theta - \pi^{\alpha\beta}\sigma_{\alpha\beta} = 0 \,, \tag{7}$$

$$\Delta^\mu_\alpha \partial_\beta T^{\alpha\beta} = 0 \quad \Leftrightarrow \quad (\varepsilon_0 + P_0 + \Pi)\,\dot{u}^\mu - \nabla^\mu(P_0 + \Pi) + \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta} = 0 \,, \tag{8}$$

which are generic equations for non-fluctuating dissipative relativistic fluid.

- To solve these fluid equations we need 🔧
  - an equation of state – ideal-gas $P_0 = n_0 T$
  - constitutive relations for $\{\Pi, \boldsymbol{V}, \boldsymbol{\pi}\}$ – TBD

Notation: $A^{\langle\mu}B^{\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}A^\alpha B^\beta$; $\theta \equiv \nabla^\mu u_\mu$, $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}$

First- and second-order hydrodynamics

- First-order hydro: Relate dissipative quantities to fluid-dynamical gradients

$$\Pi = -\zeta\theta\,, \quad V^{\mu} = \kappa I^{\mu}\,, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \tag{9}$$

- (In Eckart or Landau frame): **Acausal**!
- Second-order hydro: Treat dissipative quantitites as dynamical, provide relaxation equations

Relaxation equations

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \text{h.o.t.} \tag{10a}$$

$$\tau_V \dot{V}^{\langle\mu\rangle} + V^{\mu} = \kappa I^{\mu} + \text{h.o.t.} \tag{10b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \text{h.o.t.} \tag{10c}$$

- Needs input from **microscopic theory** 🔬
- This talk: derived from **kinetic theory**

$I^{\mu} \equiv \nabla^{\mu}\alpha$, $\alpha \equiv \mu/T$

Kinetic Theory: Basics

- Describe system in (x,k)-phase space through one-particle distribution function $f_{\mathbf{k}}(x)$
- Connection to hydrodynamics through conserved currents

Conserved quantities (moments of the distribution function)

$$N^\mu = \int \mathrm{d}K\, f_{\mathbf{k}}(x) k^\mu \,, \quad T^{\mu\nu} = \int \mathrm{d}K\, f_{\mathbf{k}}(x) k^\mu k^\nu \tag{11}$$

- Hydrodynamic quantities are distributed through $f_{\mathbf{k}}(x)$
  - Governed by Boltzmann equation $k^\mu \partial_\mu f_{\mathbf{k}}(x) = C[f]$
  - Only elastic $2 \leftrightarrow 2$ scattering
- Separate into equilibrium part $f_{0\mathbf{k}}(x)$ and deviation $\delta f_{\mathbf{k}}(x)$
  - $f_{0\mathbf{k}}(x)$ determined by $C[f_0] = 0$

$\mathrm{d}K \equiv \mathrm{d}^3k/[(2\pi)^3 k^0]$

Equilibrium: Thermodynamics from the distribution

Equilibrium distribution: $f_{0\mathbf{k}}(x) = \left[\exp\left(-\alpha(x) + \beta(x)E_{\mathbf{k}}\right) + a\right]^{-1}$

- $a \in \{-1, 0, 1\}$ determined by statistics of particles
- α, β, u^μ: Lagrange multipliers (fields)

Thermodynamic quantities can be determined from one integral:

Thermodynamic integral

$$I_{nq} \equiv \frac{(-1)^q}{(2q+1)!!} \int \mathrm{d}K \, f_{0\mathbf{k}} E_{\mathbf{k}}^{n-2q} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q \ , \qquad (12)$$

for example, density $n_0 \equiv I_{10}$, pressure $P_0 \equiv I_{21}$, etc.

$E_{\mathbf{k}} \equiv u^\mu(x) k_\mu$

Non-equilibrium: Moment expansion

- Question: Which parts of $\delta f_{\mathbf{k}}(x)$ in momentum space are important for hydrodynamics?
- Expand in terms of complete and orthogonal basis $\{1, k^{\langle\mu\rangle}, k^{\langle\mu}k^{\nu\rangle}, \dots\}$
  - Equivalent to spherical harmonics (angular part) and a radial part

Expansion of δf

$$\delta f(x,k) = f_0 \tilde{f}_0 \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \rho_{n,\mu_1\cdots\mu_\ell}(x) \tag{13}$$

- Irreducible moments $\rho_n^{\mu_1\cdots\mu_\ell}$ carry all information

Irreducible moments

$$\rho_r^{\mu_1\cdots\mu_\ell}(x) \equiv \int \mathrm{d}K\, E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f_{\mathbf{k}}(x) \tag{14}$$

$\tilde{f}_0 \equiv 1 - a f_0$

Equations of motion

Boltzmann equation

$$\delta\dot{f} = E_{\mathbf{k}}^{-1}C[f] - \dot{f}_0 - E_{\mathbf{k}}^{-1}k^{\mu}\nabla_{\mu}(f_0 + \delta f) \tag{15}$$

- Boltzmann equation determines evolution of all moments
  - Infinite set of coupled differential equations

Moment equations

$$(\ell = 0) \quad \dot{\rho}_r + \sum_{n=0,\neq 1,2}^{N_0} \mathcal{A}_{rn}^{(0)}\rho_n = \alpha_r^{(0)}\theta + \text{h.o.t.} \tag{16a}$$

$$(\ell = 1) \quad \dot{\rho}_r^{\langle\mu\rangle} + \sum_{n=0,\neq 1}^{N_1} \mathcal{A}_{rn}^{(1)}\rho_V^{\mu} = \alpha_r^{(1)} I^{\mu} + \text{h.o.t.} \tag{16b}$$

$$(\ell = 2) \quad \dot{\rho}_r^{\langle\mu\nu\rangle} + \sum_{n=0}^{N_2} \mathcal{A}_{rn}^{(2)}\rho_n^{\mu\nu} = 2\alpha_r^{(2)}\sigma^{\mu\nu} + \text{h.o.t.} \tag{16c}$$

$$(\ell > 2) \quad \dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} + \sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)}\rho_n^{\mu_1\cdots\mu_\ell} = \text{h.o.t.} \tag{16d}$$

- **Problem?**

Notation: $\dot{f} \equiv u^{\mu}\partial_{\mu}f$; Matching conditions: $\rho_1 = \rho_2 = \rho_1^{\mu} = 0$

☠️ Irreducible moment equation of rank two

Just to give an impression:

$$\begin{aligned}
\dot{\rho}_r^{\langle\mu\nu\rangle} + \sum_{n=0}^{N_2} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} &= 2\left[I_{r+2,1} + (r-1)I_{r+2,2}\right]\sigma^{\mu\nu} \\
&- \frac{2}{7}\left[(2r+5)\rho_r^{\lambda\langle\mu} - 2m^2(r-1)\rho_{r-2}^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle} \\
&+ 2\rho_r^{\lambda\langle\mu}\omega_\lambda^{\nu\rangle} + \frac{2}{15}\left[(r+4)\rho_{r+2} - (2r+3)m^2\rho_r + (r-1)m^4\rho_{r-2}\right]\sigma^{\mu\nu} \\
&+ \frac{2}{5}\nabla^{\langle\mu}\left(\rho_{r+1}^{\nu\rangle} - m^2\rho_{r-1}^{\nu\rangle}\right) - \frac{2}{5}\left[(r+5)\rho_{r+1}^{\langle\mu} - m^2 r\rho_{r-1}^{\langle\mu}\right]\dot{u}^{\nu\rangle} \\
&- \frac{1}{3}\left[(r+4)\rho_r^{\mu\nu} - m^2(r-1)\rho_{r-2}^{\mu\nu}\right]\theta \\
&+ (r-1)\rho_{r-2}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_r^{\lambda\alpha\beta} + r\rho_{r-1}^{\lambda\mu\nu}\dot{u}_\lambda \,,
\end{aligned}$$

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)}\int \mathrm{d}K\,\mathrm{d}K'\,\mathrm{d}P\,\mathrm{d}P'\,W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}$$

$$\times\left(\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle\mu_1}\cdots k'_{\mu_\ell\rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle\mu_1}\cdots p_{\mu_\ell\rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle\mu_1}\cdots p'_{\mu_\ell\rangle}\right).$$

Truncation and power counting

- Basic idea: Power-counting scheme to **second order** in two small quantities:
  1. Knudsen number $\mathrm{Kn} \sim \lambda_{\mathrm{mfp}}/\lambda_{\mathrm{hydro}}$, and
  2. inverse Reynolds numbers $\mathrm{IRe} \sim \delta f / f_0$
- Interested in the evolution of $T^{\mu\nu}$ and N^{μ} as moments of δf
  - → Benchmark: Evolution equations for $\Pi = -(m^2/3)\rho_0$, $V^{\mu} = \rho_0^{\mu}$, $\pi^{\mu\nu} = \rho_0^{\mu\nu}$
  - → Only interested in moments with $\ell \leqslant 2$
- $\rho_r^{\mu_1 \cdots \mu_{\ell>2}} = 0$, corrections of order $\mathcal{O}\big(\mathrm{Kn}^2\mathrm{IRe}, \mathrm{Kn}^3\big)$

Moment equations

$$\sum_{n=0,\neq 1,2}^{N_0} \tau_{rn}^{(0)} \dot{\rho}_n + \rho_r = -\zeta_r \theta + \text{h.o.t.} \tag{17a}$$

$$\sum_{n=0,\neq 1}^{N_1} \tau_{rn}^{(1)} \dot{\rho}_n^{\langle\mu\rangle} + \rho_r^{\mu} = \kappa_r I^{\mu} + \text{h.o.t.} \tag{17b}$$

$$\sum_{n=0}^{N_2} \tau_{rn}^{(2)} \dot{\rho}_n^{\langle\mu\nu\rangle} + \rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \text{h.o.t.} \tag{17c}$$

- Still coupled system of $N_0 + 3N_1 + 5N_2$ equations
- **How to decouple the remaining equations?**

$\tau^{(\ell)} \equiv (\mathcal{A}^{(\ell)})^{-1}$

Obtaining hydrodynamics

- Use asymptotic matching conditions to Navier-Stokes relations
- Express all irreducible moments through dissipative quantities
- Re-sum the relaxation modes into τ and discard h.o.t

Hydrodynamic transport equations (IReD)

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \mathcal{J} + \mathcal{R} \tag{18a}$$

$$\tau_V \dot{V}^{\langle\mu\rangle} + V^\mu = \kappa V^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu \tag{18b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} \tag{18c}$$

- Only terms $\sim \mathcal{O}(\mathrm{IRe})$, $\sim \mathcal{O}(\mathrm{Kn})$, $\sim \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe})$, $\sim \mathcal{O}\left(\mathrm{IRe}^2\right)$ appear

Example of couplings at second order: bulk viscosity

- In the bulk relaxation equation,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \mathcal{J} + \mathcal{R}$$

the second-order dissipative corrections to bulk-viscous pressure are

$$\mathcal{J} \equiv -\ell_{\Pi V} \nabla_\mu V^\mu - \tau_{\Pi V} V^\mu F_\mu - \delta_{\Pi\Pi} \Pi\theta - \lambda_{\Pi V} V^\mu I_\mu + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu} \ ,$$
$$\mathcal{R} \equiv \varphi_1 \Pi^2 + \varphi_2 V^\mu V_\mu + \varphi_3 \pi^{\mu\nu} \pi_{\mu\nu} \ .$$

The transport coefficients can be computed analytically as thermodynamic functions! 🥳

- We computed all transport coefficients of bulk-viscosity, particle-diffusion, and shear-viscosity.

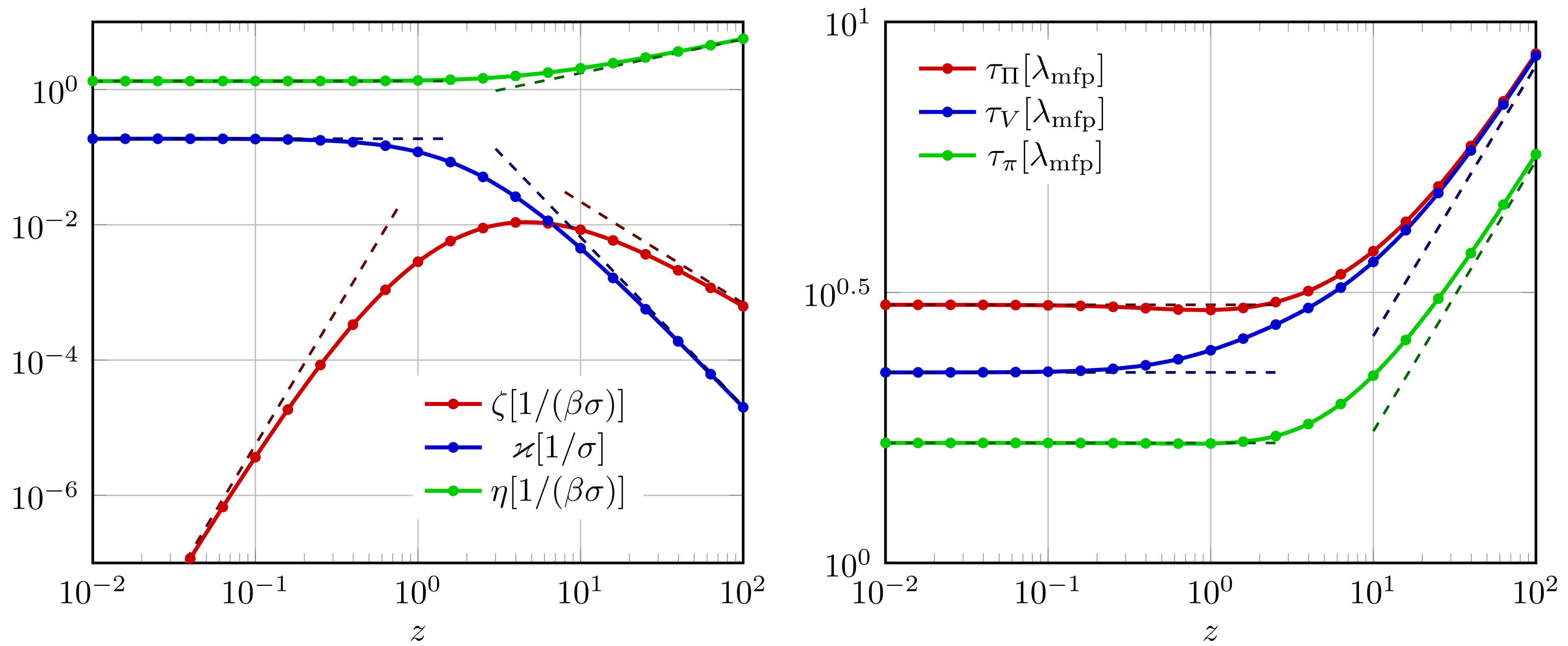
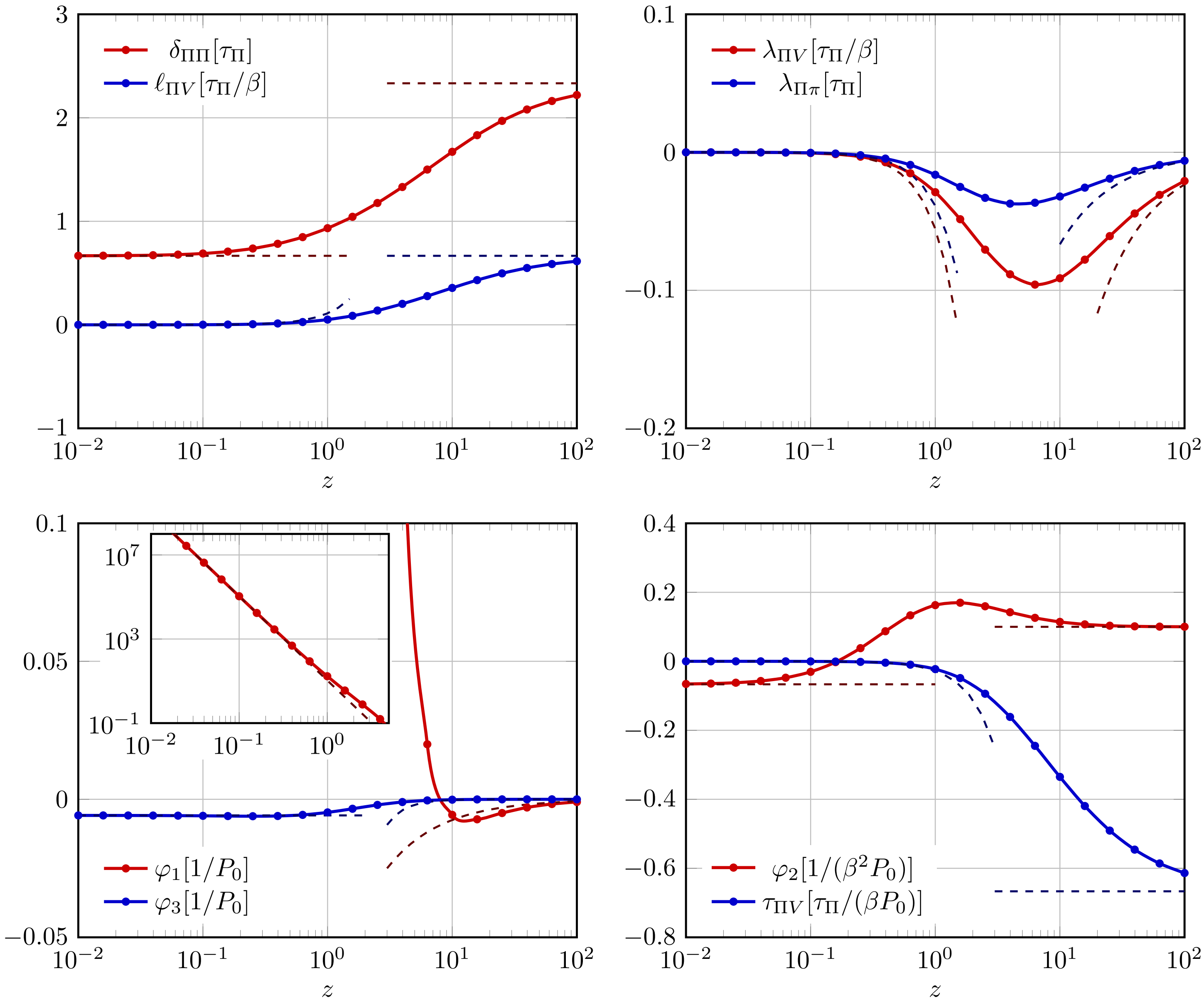


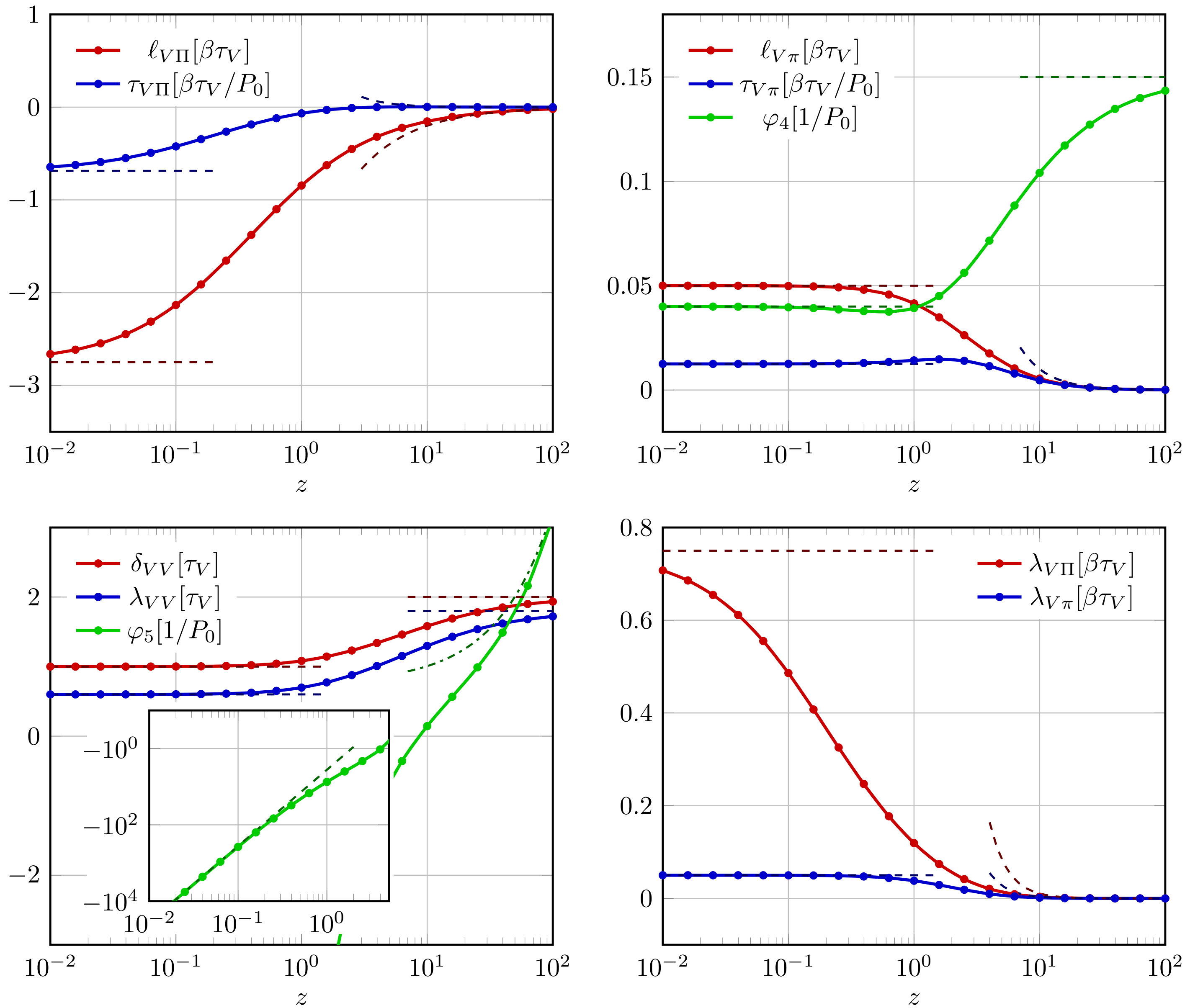
Figure: The first-order coefficients ζ, $\varkappa$, and η, as well as the relaxation times τ_Π, τ_V, and τ_π.

$z \equiv m/T$

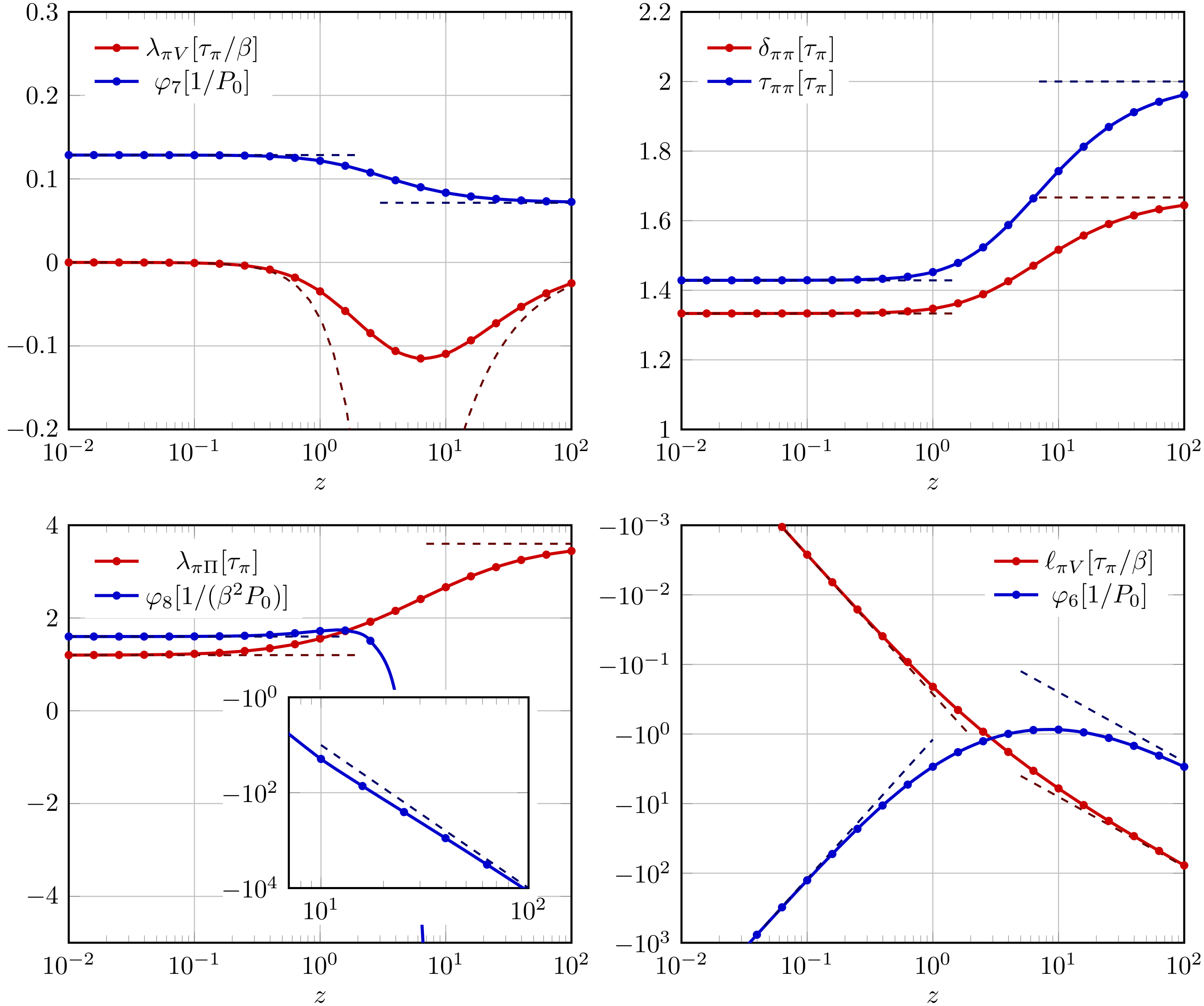
Second-order coefficients: bulk



Second-order coefficients: diffusion



Second-order coefficients: shear



Non-relativistic limit of relativistic transport equations

Asymptotic thermodynamics

- Thermodynamic integral becomes

$$I_{nq} \rightsquigarrow \frac{e^{\alpha - z}}{(2\pi)^{3/2}\beta^{n+2}} z^{n-q+1/2} \ . \tag{19}$$

  - For example, the pressure is

$$I_{21} \rightsquigarrow \frac{e^{\alpha - z}}{(2\pi)^{3/2}\beta^{4}} z^{3/2} \equiv P_0^{(\infty)} \ . \tag{20}$$

- Specific enthalphy reads

$$h_0 = mc^2 + m\left(e_0 + \frac{P_0}{\rho_0}\right) \rightsquigarrow mc^2 \equiv h_0^{(\infty)} \ , \tag{21}$$

where $\rho_0 \equiv n_0 m$ the rest-mass density and e_0 the kinetic energy density.

Particle-heat correspondence

- Combining the Gibbs-Duhem relation and the Euler relation,

$$0 = s\,\mathrm{d}T - \mathrm{d}P + n_0\,\mathrm{d}\mu \ , \quad \epsilon = Ts - P + n_0\mu \ ,$$

respectively, yields

$$\mathrm{d}\beta = h_0^{-1}\,\mathrm{d}\alpha - \frac{\beta}{\epsilon_0 + P_0}\,\mathrm{d}P_0 \ .$$

In terms of a covariant derivative, it holds

Gibbs-Duhem relation for particle-heat correspondence

$$I_\mu - h_0 J_\mu = \nabla_\mu \ln P_0 \ .$$

This equation connects the particle-change to heat-change at variable pressure and temperature. ❗

- Heat-flow (in Landau frame),

$$q_\mu \equiv -h_0 V_\mu \ .$$

is a more natural variable than particle-diffusion in non-relativistic gases.

$h_0 \equiv (\epsilon_0 + P_0)/(\beta P_0)$, $I_\mu \equiv \nabla_\mu \alpha$, $J_\mu \equiv \nabla_\mu \beta$

Deriving the Grad equations from IReD

Grad's transport equations can be derived from the relativistic transport equations of IReD as follows ☑️:

- Rewrite the relativistic equations at $\mathcal{O}\left(\mathrm{Kn}\,\mathrm{IRe}, \mathrm{IRe}^2\right)$ in terms of heat variables
- Substitute the irreducible tensors θ, $\boldsymbol{\sigma}$, $\boldsymbol{\omega}$, as well as other characteristic thermodynamic variables
- Insert the asymptotic values of the transport coefficients
- (Take the spatial part)

Comparing asymptotic IReD to Grad: diffusion

- Comparing the transport equations for heat-flow:
  - IReD (hard-spheres):

$$-\frac{2}{3}\frac{P_0^{(\infty)}}{\eta^{(\infty)}}q^\mu = \Delta^\mu_\nu \frac{\mathrm{d}q^\nu}{\mathrm{d}\tau} - \frac{5}{2}P_0^{(\infty)} R\nabla^\mu T + q_\nu \nabla^\nu u^\mu + \frac{2}{5}q_\nu\left(\nabla^\nu u^\mu + \nabla^\mu u^\nu\right) + \frac{7}{5}q^\mu \nabla_\lambda u^\lambda - RT\pi^{\mu\nu}\nabla_\nu \ln P_0^{(\infty)} + RT\Delta^{\mu\nu}\nabla_\lambda \pi^\lambda_\nu + \frac{7}{2}R\pi^{\mu\nu}\nabla_\nu T - \underbrace{\frac{1}{10\eta^{(\infty)}}\pi^{\mu\nu}q_\nu}_{\text{interaction-specific}}, \tag{22}$$

  - Grad (Maxwell-molecules):

$$-\frac{2}{3}\frac{P_0}{\eta}q^i = \frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{5}{2}P_0\frac{\partial RT}{\partial x_i} + \frac{7}{5}q^i\frac{\partial u_k}{\partial x_k} + q_k\frac{\partial u^i}{\partial x_k} + \frac{2}{5}q_k\left(\frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i}\right) - RT\pi^{ik}\frac{\partial \ln P_0}{\partial x^k} + RT\frac{\partial \pi^{ik}}{\partial x^k} + \frac{7}{2}\pi^{ik}\frac{\partial RT}{\partial x^k} - \underbrace{\frac{\pi^{ik}}{\rho_0}\frac{\partial \pi_{kl}}{\partial x_l}}_{\sim\mathcal{O}(\mathrm{KnIRe}^2)}. \tag{23}$$

Comparing asymptotic IReD to Grad: shear

- Comparing the transport equations for heat-flow:
  - IReD (hard-spheres):

$$-\frac{P_0^{(\infty)}}{\eta^{(\infty)}}\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\frac{\mathrm{d}\pi^{\alpha\beta}}{\mathrm{d}\tau} - 2P_0^{(\infty)}\nabla^{\langle\mu}u^{\nu\rangle} + \pi^{\mu\nu}\nabla_\lambda u^\lambda + 2\pi^{\lambda\langle\mu}\nabla_\lambda u^{\nu\rangle} - \frac{4}{5}\nabla^{\langle\mu}q^{\nu\rangle} - \underbrace{\frac{1}{14\eta^{(\infty)}}\pi^{\lambda\langle\mu}\pi^{\nu\rangle}{}_\lambda}_{\text{interaction-specific}} + \underbrace{\frac{1}{100RT\eta^{(\infty)}}q^{\langle\mu}q^{\nu\rangle}}_{\text{interaction-specific}} \,. \tag{24}$$

  - Grad (Maxwell-molecules):

$$-\frac{P_0}{\eta}\pi^{ij} = \frac{\mathrm{d}\pi^{ij}}{\mathrm{d}t} + 2P_0\frac{\partial u^{\langle i}}{\partial x_{j\rangle}} + \pi^{ij}\frac{\partial u_k}{\partial x_k} + 2\pi^{r\langle i}\frac{\partial u^{j\rangle}}{\partial x^r} + \frac{4}{5}\frac{\partial q^{\langle i}}{\partial x_{j\rangle}} \,, \tag{25}$$

- Indeed, further away from equilibrium the geometry of gas particles becomes even more important. 💡

Conclusions

- Second-order transport coefficients were computed for arbitrary mases
  - Finally, all IRe^2-coefficients are available for arbitrary z ⭐
  - Many coefficients vanish for $z \rightsquigarrow 0$ and/or $z \rightsquigarrow \infty$, but not in-between!
- We have shown how to derive Grad's by taking the non-relativistic limit of the transport equations
  - IRe^2-terms depend on the geometry of the gas,
  - however, $\tau_V^{(\infty)}/\tau_\pi^{(\infty)} = 3/2$ stays the same for Maxwell- and hard-sphere molecules

thank you. ✌️

Appendix

DNMR vs. IReD

DNMR: Obtaining hydrodynamics

- Use asymptotic matching to express all irreducible moments through dissipative quantities and fluid-dynamical gradients
- Discard h.o.t

Hydrodynamic transport equations (DNMR)

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta_0 \theta + \mathcal{J} + \mathcal{K} + \mathcal{R} \tag{26}$$

$$\tau_V \dot{V}^{\langle\mu\rangle} + V^\mu = \kappa_0 V^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu \tag{27}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta_0 \sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu} \tag{28}$$

- First-order contributions $\sim \mathcal{O}(\mathrm{IRe})$ and $\sim \mathcal{O}(\mathrm{Kn})$
- Second-order contributions $\sim \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe})$ and $\sim \mathcal{O}\left(\mathrm{Kn}^2, \mathrm{IRe}^2\right)$
- Contributions of order $\mathcal{O}\left(\mathrm{Kn}^2\right)$ result directly from asymptotic matching
  - Example: $\theta \rho_r \rightarrow \theta \Pi$, θ^2

The issue: Parabolic terms

- Consider the second-order terms of tensor-rank two:

$$\mathcal{K}^{\mu\nu} = \widetilde{\eta}_1 \omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda + \widetilde{\eta}_2 \theta\sigma^{\mu\nu} + \widetilde{\eta}_3 \sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \widetilde{\eta}_4 \sigma^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda} + \widetilde{\eta}_5 I^{\langle\mu} I^{\nu\rangle}$$
$$+ \widetilde{\eta}_6 F^{\langle\mu} F^{\nu\rangle} + \widetilde{\eta}_7 I^{\langle\mu} F^{\nu\rangle} + \widetilde{\eta}_8 \nabla^{\langle\mu} I^{\nu\rangle} + \widetilde{\eta}_9 \nabla^{\langle\mu} F^{\nu\rangle} \quad (29)$$

- Second derivatives of fluid-dynamical quantities appear
  - $\longrightarrow$ Equations become **parabolic**!
  - $\longrightarrow$ Theory becomes acausal and unstable
- Conventional remedy: **Ignore** terms of order $\mathcal{O}\left(\mathrm{Kn}^2\right)$ (tDNMR)
  - $\longrightarrow$ Equations are hyperbolic again (but ad hoc)
- Is there a way to ensure $\mathcal{K} = \mathcal{K}^\mu = \mathcal{K}^{\mu\nu} = 0$ from the beginning?

$F^\mu \equiv \nabla^\mu P_0$, $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$

IReD: the way-out

DW, A. Palermo, V. E. Ambruș, arXiv:2203.12608

- General idea: Relate moments through their Navier-Stokes solutions

IReD: Asymptotic matching

$$\rho_r = -\zeta_r \theta + \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe}) \quad \mapsto \quad \rho_r = \frac{\zeta_r}{\zeta_n}\rho_n + \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe}) \tag{30}$$

$$\rho_r^\mu = \kappa_r I^\mu + \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe}) \quad \mapsto \quad \rho_r^\mu = \frac{\kappa_r}{\kappa_n}\rho_V^\mu + \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe}) \tag{31}$$

$$\rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe}) \quad \mapsto \quad \rho_r^{\mu\nu} = \frac{\eta_r}{\eta_n}\rho_n^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe}) \tag{32}$$

- ❗ No terms $\sim \mathcal{O}(\mathrm{Kn})$ appear in asymptotic matching (IRe dominance 😏)
- Equations of motion can be closed in terms of any set of moments $\rho_n,\ \rho_n^\mu,\ \rho_n^{\mu\nu}$
- Choose $n = 0$ to obtain closure in terms of hydrodynamic quantities

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Also known as "order-of-magnitude approximation" J. A. Fotakis, E. Molnár, H. Niemi, C. Greiner, D. H. Rischke arXiv: 2203.11549

IReD: Obtaining hydrodynamics

- Procedure analogous: use new asymptotic matching conditions to express all irreducible moments through dissipative quantities and fluid-dynamical gradients
- Diagonalize and discard h.o.t

Hydrodynamic transport equations (IReD)

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta_0 \theta + \mathcal{J} + \mathcal{R} \tag{33a}$$

$$\tau_V \dot{V}^{\langle\mu\rangle} + V^\mu = \kappa_0 V^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu \tag{33b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta_0 \sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} \tag{33c}$$

- Structure is similar, but transport coefficients different for $N_0 > 2$, $N_1 > 1$, $N_2 > 0$ (> 14 moments)
- Only terms $\sim \mathcal{O}(\mathrm{IRe})$, $\sim \mathcal{O}(\mathrm{Kn})$, $\sim \mathcal{O}(\mathrm{Kn\,IRe})$, $\sim \mathcal{O}\left(\mathrm{IRe}^2\right)$ appear
  - $\rightarrow$ Equations stay **hyperbolic**, no need to discard terms
- Absence of parabolic terms due to modified asymptotic matching

A different path to hyperbolicity

- Basic idea of IReD and DNMR: Relate quantities up to order $\mathcal{O}(\mathrm{Kn}\,\mathrm{IRe})$
- 🤔 Ambiguities in second-order terms since to first order

$$\Pi \simeq -\zeta\theta\,,\ V^\mu \simeq \kappa_0 I^\mu\,,\ \pi^{\mu\nu} \simeq 2\eta_0\sigma^{\mu\nu} \tag{34}$$

- For example:

$$\langle\mathcal{K}\rangle \ni \theta^2 = -\frac{\Pi\theta}{\zeta_0} \in \langle\mathcal{J} + \mathcal{O}(\mathrm{Kn}\,\mathrm{IRe})\rangle\ ,$$

- 👀 IReD way: "Trade one power of Kn for one power of IRe"
- To eliminate the parabolic terms:
  1. Start with the DNMR approach
  2. Use prescription to absorb coefficients in $\mathcal{K}, \mathcal{K}^\mu, \mathcal{K}^{\mu\nu}$ into $\mathcal{J}, \mathcal{J}^\mu, \mathcal{J}^{\mu\nu}$
- Allows to relate transport coefficients in the two approaches
- **IReD and DNMR equivalent up to (and including) order** $\mathcal{O}\left(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{IRe}, \mathrm{IRe}^2\right)$

Non-relativistic gas dynamics

Thermodynamic integral for large masses

The integral

$$I_{nq} = \frac{(-1)^q}{(2q+1)!!} \int \mathrm{d}K \, f_{0\mathbf{k}} E_{\mathbf{k}}^{n-2q} \left(m^2 - E_{\mathbf{k}}^2\right)^q \ ,$$

can be rewritten for $|\mathbf{k}| < m$ to

$$\begin{aligned} I_{nq} = \frac{e^{\alpha - z}}{(2\pi)^{3/2}\beta^{n+2}} z^{n-q+1/2} \sum_{\ell=0}^{\infty} \binom{\frac{n-1}{2} - q}{\ell} z^{-\ell} \frac{(2\ell+2q+1)!!}{(2q+1)!!} \\ \times \left\{ 1 + \sum_{j=1}^{\infty} \frac{z^{-j}}{4^j j!} \sum_{k=1}^{j} (-1)^{j+k} \frac{(2j+2k+2\ell+2q+1)!!}{2^k (2\ell+2q+1)!!} \right. \\ \left. \times B_{j,k}\left[1, \cdots, \frac{(2j-2k+2)!}{(j-k+2)!}\right] \right\} , \end{aligned} \tag{35}$$

where $B_{j,k}$ denotes the incomplete exponential Bell polynomials 😵‍💫.

Brief detour on special functions: Bell theory

I will follow a nice exposition of Bell polynomials by Nicholas Wheeler. *Bell Polynomials & Related Constructs*. 2020. URL: https://www.reed.edu/physics/faculty/wheeler/documents/Miscellaneous%20Math/Bell%20Polynomials,%20Pochhammer%20Symbols,%20Etc/Bell%20Polynomials.pdf.

Theory of Bell polynomials

For example, we want to expand the composite function

$$F(x) = f\left(g(x)\right) \ ,$$

for a generic $g(x) \equiv \sum_k a_k x^k$. Then,

$$f(x) \equiv \frac{1}{1-x} \implies F(x) = \sum_k D_k x^k \ ,$$

with the new coefficients determined by the Toeplitz matrix

$$D_n = \det \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_{n-1} & a_n \\ -1 & a_1 & a_2 & \ddots & \ddots & a_{n-1} \\ 0 & -1 & a_1 & a_2 & \ddots & \vdots \\ 0 & 0 & -1 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_2 \\ 0 & \cdots & \cdots & \cdots & -1 & a_1 \end{pmatrix} = \sum_{k=1}^{n} a_k D_{n-k} \ .$$

Recursion formula for coefficients. 😎

Theory of Bell polynomials

Suppose now $F(x) = \exp(g(x))$. Then, $F(x) = \sum_k E_k x^k$, where

$$E_n = \frac{1}{n!} \det \begin{pmatrix} a_1 & 2a_2 & \cdots & \cdots & (n-1)a_{n-1} & na_n \\ -1 & a_1 & 2a_2 & \ddots & \ddots & (n-1)a_{n-1} \\ 0 & -2 & a_1 & 2a_2 & \ddots & \vdots \\ 0 & 0 & -3 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2a_2 \\ 0 & \cdots & \cdots & \cdots & -n+1 & a_1 \end{pmatrix}$$

$$= \sum_{k=1}^{n} \frac{(n-1)!\,k}{(n-k)!} a_k E_{n-k} \,.$$

Arising connection to binomial coefficient? 🤔

Theory of Bell polynomials

Finally, suppose the generating function is $F(x) = \exp\left(\sum_k \frac{b^k}{k!} x_k\right)$. Then,

$$B_n(b_1, \ldots, b_n) = \det \begin{pmatrix} b_1 & \frac{1}{1!}b_2 & \cdots & \cdots & \frac{1}{(n-2)!}b_{n-1} & \frac{1}{(n-1)!}b_n \\ -1 & b_1 & \frac{1}{1!}b_2 & \ddots & \ddots & (n-2)!\,b_{n-1} \\ 0 & -2 & b_1 & \frac{1}{1!}b_2 & \ddots & \vdots \\ 0 & 0 & -3 & b_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{1}{1!}b_2 \\ 0 & \cdots & \cdots & \cdots & -n+1 & b_1 \end{pmatrix}.$$

Indeed, we get a binomial coefficient:

$$B_{n+1}\left(b_1, \ldots, b_{n+1}\right) = \sum_{m=0}^{n} \binom{n}{m} b_{m+1} B_{n-m}\left(b_1, \ldots, b_{n-m}\right) \;.$$

where $B_n\left(b_1, \ldots, b_n\right)$ are the so-called Bell polynomials. 😌

Theory of Bell polynomials

To recap, the Bell polynomials, generated by

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} B_n\left(x_1, \ldots, x_n\right) \frac{t^n}{n!} \ ,$$

are actually multinomials. The Bell numers,

$$B_n \equiv B_n\left(1, \ldots, 1\right) \hookleftarrow B_n\left(x_1, \ldots, x_n\right) \ ,$$

count set partitions. For example, $\{a, b, c\}$ can be partioned $(B_3 = 5)$-many ways:

$$\begin{gathered} \{a\}, \{b\}, \{c\} \quad \{a, b, c\}, \{\}, \{\} \\ \{\}, \{b\}, \{a, c\} \quad \{a, b\}, \{\}, \{c\} \\ \{a\}, \{b, c\}, \{\} \ . \end{gathered}$$

Theory of Bell polynomials

$$\underbrace{B_j(x_1, \cdots, x_j)}_{\text{Complete}} = \sum_{k=1}^{j} \underbrace{B_{j,k}(x_1, \cdots, x_{j-k+1})}_{\text{Incomplete}} \,. \tag{36}$$

Incomplete Bell polynomials are homogeneous functions,

$$B_{j,k}(\gamma\delta x_1, \cdots, \gamma\delta^{j-k+1} x_{j-k+1}) = \gamma^k \delta^j B_{j,k}(x_1, \cdots, x_{j-k+1}) \,. \tag{37}$$

Bell polynomials are connected to Hermite polynomials by

$$\sum_{n=0}^{\infty} B_n(x, -1, 0, \ldots, 0) \frac{t^n}{n!} = \exp\left(xt - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} \mathrm{He}_n(x) \frac{t^n}{n!} \,.$$

Notably, H. Grad used Hermite polynomials to construct moment equations.

Theory of Bell polynomials

Combinatorial nature of polynomials gives rise to recursive structures like

1				
1	2			
2	3	5		
5	7	10	15	
15	20	27	37	52