Anisotropic fluid dynamics from the Boltzmann equation

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with:
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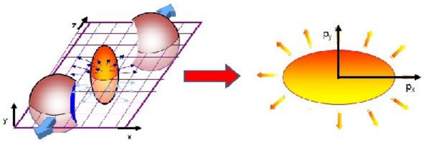
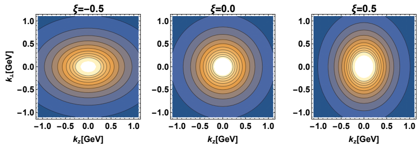
arXiv:1602.00573 (PRD) + new results

Introduction I.

- Isotropy is uniformity in all orientations; it is derived from the Greek isos ("equal") and tropos ("way"). Exceptions, or inequalities, are frequently indicated by the prefix an, hence anisotropy. (wikipedia)
- Anisotropy is the property of being directionally dependent, as opposed to isotropy, which implies identical properties in all directions. It can be defined as a difference, when measured along different axes, in a material's physical or mechanical properties (wikipedia)
- Anisotropy, in physics, the quality of exhibiting properties with different values when measured along axes in different directions. Anisotropy is most easily observed in single crystals of solid elements or compounds, in which atoms, ions, or molecules are arranged in regular lattices. In contrast, the random distribution of particles in liquids, and especially in gases, causes them rarely, if ever, to be anisotropic. (britannica)
- The vast majority of fluids (including air and water) are isotropic. (britannica)

Introduction II.





- Elliptic flow - momentum space anisotropy of particle emission in non-central heavy-ion collisions.

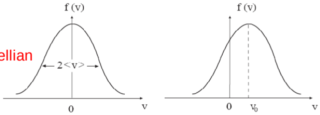
$$\hat{f}_{RS} \equiv \left[\exp\left(-\beta\sqrt{k_\perp^2 + (1+\xi)k_z^2}\right)\right]$$

Spheroidal momentum distribution function. Oblate (left) - Prolate (right).

Introduction III.

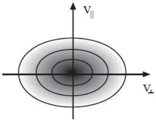
Examples of distribution functions

Maxwellian



$$f(v) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m(\mathbf{v}-\mathbf{V}_0)^2}{2k_B T}\right)$$

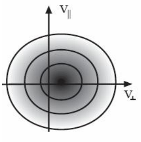
Maxwellian in a frame of reference that moves with velocity $\mathbf{V}_0$



Anisotropic (pancake) distribution ($\mathbf{v}_\parallel \parallel \mathbf{B}$)

$$f(v_\perp, v_\parallel) = \frac{n}{T_\perp T_\parallel^{1/2}}\left(\frac{m}{2\pi k_B}\right)^{3/2} \exp\left(-\frac{m v_\perp^2}{2k_B T_\perp} - \frac{m v_\parallel^2}{2k_B T_\parallel}\right)$$

Can also be cigar-shaped (elongated in the direction of $\mathbf{B}$)



Drifting Maxwellian

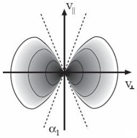
$$f(v_\perp, v_\parallel) = \frac{n}{T_\perp T_\parallel^{1/2}}\left(\frac{m}{2\pi k_B}\right)^{3/2} \exp\left(-\frac{m(\mathbf{v}_\perp - \mathbf{v}_{0\perp})^2}{2k_B T_\perp} - \frac{m v_\parallel^2}{2k_B T_\parallel}\right)$$

Introduction IV.

Magnetic field-aligned beam (e.g., particles causing the aurora):

$$f(v_\perp, v_\|) = \frac{n}{T_\perp T_\|^{1/2}} \left(\frac{m}{2\pi k_B}\right)^{3/2} \exp\left(-\frac{m v_\perp^2}{2k_B T_\perp} - \frac{m(v_\| - v_{0\|})^2}{2k_B T_\|}\right)$$

Loss-cone distribution in a magnetic bottle:



Kappa distribution ~ Maxwellian with high-energy tail

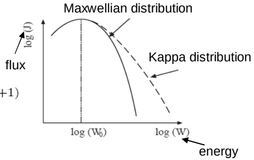
The tail follows a power law $f(W) \propto (W_0/W)^\kappa$

$$f_\kappa(W) = n\left(\frac{m}{2\pi\kappa W_0}\right)^{3/2} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \left(1 + \frac{W}{\kappa W_0}\right)^{-(\kappa+1)}$$

$[f_\kappa] = \mathrm{m}^{-6}\mathrm{s}^3$

Γ-function

energy at the peak
of the distribution



Observed particle distributions often resemble kappa distributions;
a signature that non-thermal acceleration has taken place somewhere

Notation I.

Notation Isotropic

- The normalized (time-like) fluid dynamical 4-velocity is denoted by $u^{\mu}(t, \mathbf{x})$, where $u^{\mu} u_{\mu} \equiv c^2 = 1$.
- The local rest (LR) frame is defined as, $u^{\mu}_{LR} = (1, 0, 0, 0)$.
- The projection tensor $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu} u^{\nu}$ is orthogonal to the 4-flow of matter i.e., $\Delta^{\mu\nu} u_{\mu} = 0$, where the metric is $g^{\mu\nu} \equiv g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- The orthogonal projection of a 4-vector, $A^{\langle\mu\rangle} = \Delta^{\mu\nu} A_{\nu}$, while the symmetric, traceless and orthogonal part of a tensor is denoted by, $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta} = \left[\frac{1}{2}\left(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\nu}_{\alpha}\Delta^{\mu}_{\beta}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] A^{\alpha\beta}$.

$$A^{\mu} = A^{\nu} u_{\nu}\, u^{\mu} + A^{\langle\mu\rangle} \quad \Rightarrow \quad k^{\mu} = E_{\mathbf{k}u}\, u^{\mu} + k^{\langle\mu\rangle}$$

$$\begin{aligned} A^{\mu\nu} &= A^{\alpha\beta} u_{\alpha} u_{\beta}\, u^{\mu} u^{\nu} + \Delta^{\mu\alpha} A_{\alpha\beta} u^{\beta}\, u^{\nu} + \Delta^{\nu\beta} A_{\alpha\beta} u^{\alpha}\, u^{\mu} \\ &+ \frac{1}{3} A^{\alpha\beta} \Delta_{\alpha\beta}\, \Delta^{\mu\nu} + \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta} + \Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} A^{[\alpha\beta]} \end{aligned}$$

Notation II.

Notation Anisotropic

- The normalized (space-like) anisotropy 4-vector is denoted by $l^\mu(t,\mathbf{x})$, where $l^\mu l_\mu \equiv -l^2 = -1$.
- The anistropy 4-vector is orthogonal to the flow, $l^\mu u_\mu = 0$, e.g., $l^\mu_{LR} = (0,0,0,1)$ or $u^\mu = (\cosh\eta, 0, 0, \sinh\eta)$ and $l^\mu = (\sinh\eta, 0, 0, \cosh\eta)$.
- The projection tensor $\Xi^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu + l^\mu l^\mu = \Delta^{\mu\nu} + l^\mu l^\mu$ is orthogonal to **both** u^μ and l^μ i.e., $\Xi^{\mu\nu} u_\mu = \Xi^{\mu\nu} l_\mu = 0$.
- The orthogonal projection of a 4-vector, $A^{\{\mu\}} = \Xi^{\mu\nu} A_\nu$, while the symmetric, traceless and orthogonal part of any tensor is denoted by, $A^{\{\mu\nu\}} \equiv \Xi^{\mu\nu}_{\alpha\beta} A^{\alpha\beta} = \left[\frac{1}{2}\left(\Xi^\mu_\alpha \Xi^\nu_\beta + \Xi^\nu_\alpha \Xi^\mu_\beta\right) - \frac{1}{2}\Xi^{\mu\nu}\Xi_{\alpha\beta}\right] A^{\alpha\beta}$.

$$A^\mu = A^\nu u_\nu\, u^\mu - A^\nu l_\nu\, l^\mu + A^{\{\mu\}} \quad \Rightarrow \quad k^\mu = E_{\mathbf{k}u}\, u^\mu - E_{\mathbf{k}l}\, l^\mu + k^{\{\mu\}}$$

$$\begin{aligned} A^{\mu\nu} &= A^{\alpha\beta} u_\alpha u_\beta\, u^\mu u^\nu + A^{\alpha\beta} l_\alpha l_\beta\, l^\mu l^\nu - A^{\alpha\beta} u_\alpha l_\beta\, u^\mu l^\nu - A^{\alpha\beta} u_\beta l_\alpha\, u^\nu l^\mu \\ &+ \Xi^{\mu\alpha} A_{\alpha\beta} u^\beta\, u^\nu + \Xi^{\nu\beta} A_{\alpha\beta} u^\alpha\, u^\mu - \Xi^{\mu\alpha} A_{\alpha\beta} l^\beta\, l^\nu - \Xi^{\nu\beta} A_{\alpha\beta} l^\alpha\, l^\mu \\ &+ \frac{1}{2} A^{\alpha\beta} \Xi_{\alpha\beta}\, \Xi^{\mu\nu} + \Xi^{\mu\nu}_{\alpha\beta} A^{\alpha\beta} + \Xi^\mu_\alpha \Xi^\nu_\beta A^{[\alpha\beta]} \end{aligned}$$

The relativistic Boltzmann equation and balance equations

The relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} \equiv C\left[f_{\mathbf{k}}\right] = \frac{1}{2}\int dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}\left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'}\right)$$

Here $k^\mu = (k^0, \mathbf{k})$ is the four-momenta of particles with mass m and energy $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. $\tilde{f}_{\mathbf{k}} = 1 - a f_{\mathbf{k}}$, with $a = 0/a = 1/a = -1$ for Boltzmann/Fermi/Bose statistics. The inv. phase-space element is, $dK = g d^3\mathbf{k}/\left[(2\pi)^3 k^0\right]$. The transition rate satisfies $W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} = W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}'\mathbf{p}} = W_{\mathbf{p}\mathbf{p}'\to\mathbf{k}\mathbf{k}'}$.

Conservation and balance equations

$$\partial_\mu N^\mu \equiv \partial_\mu \langle k^\mu \rangle = \int dK k^\mu \partial_\mu f_{\mathbf{k}} = \int dK C\left[f_{\mathbf{k}}\right] = 0 \quad \textbf{charge cons.}$$

$$\partial_\mu T^{\mu\nu} \equiv \partial_\mu \langle k^\mu k^\nu \rangle = \int dK k^\nu k^\mu \partial_\mu f_{\mathbf{k}} = \int dK k^\nu C\left[f_{\mathbf{k}}\right] = 0 \quad \textbf{energy-momentum cons.}$$

$$\ldots$$

$$\partial_\mu \langle k^{\mu_1} \ldots k^{\mu_n} \rangle = \int dK\, k^{\mu_1} \ldots k^{\mu_{n-1}}\, C\left[f_{\mathbf{k}}\right] \quad \textbf{balance eqs.}$$

Conservation laws are obtained, 5-collisional invariants, but we still need the solution of the Boltzmann equation $f_{\mathbf{k}}$! These balance equations are never closed!

Ideal Fluid

In Equilibrium

$$f_{\mathbf{k}} = f_{0\mathbf{k}}$$

Local thermal equilibrium

Local thermal equilibrium $f_{\mathbf{k}} \to f_{0\mathbf{k}}$, where $f_{0\mathbf{k}} = f(\alpha_0, \beta_0, E_{\mathbf{k}})$

$$f_{0\mathbf{k}} \equiv \left[\exp\left(-\alpha_0 + \beta_0 E_{\mathbf{k}u}\right) + a\right]^{-1}$$ **Jüttner distribution**

$\alpha_0 = \mu_0/T_0$, $\beta_0 = 1/T_0$ is the inverse temperature and $E_{\mathbf{k}u} = k^{\mu} u_{\mu}$

Moments of the equilibrium distribution function

$$N_0^{\mu}(t,\mathbf{x}) \equiv \int dK\, k^{\mu} f_{0\mathbf{k}} = \langle k^{\mu} \rangle_0$$ **charge current**

$$T_0^{\mu\nu}(t,\mathbf{x}) \equiv \int dK\, k^{\mu} k^{\nu} f_{0\mathbf{k}} = \langle k^{\mu} k^{\nu} \rangle_0$$ **energy-momentum tensor**

$$\cdots \qquad \cdots$$

$$\mathcal{I}_i^{\mu_1 \cdots \mu_n} = \left\langle E_{\mathbf{k}u}^i\, k^{\mu_1} \cdots k^{\mu_n} \right\rangle_0$$ **rank-n tensor moment**

Ideal Fluids I.

Conservation laws for a simple (single component) ideal fluid (no dissipation)

$$\partial_\mu N_0^\mu = 0 \quad \textbf{charge conservation} \quad \Rightarrow \textbf{1 eq.}$$

$$\partial_\mu T_0^{\mu\nu} = 0 \quad \textbf{energy-momentum conservation} \quad \Rightarrow \textbf{4 eqs.}$$

Ideal fluid decomposition with respect to u^μ

$$\mathcal{I}_i^{\mu_1\cdots\mu_n} = \sum_{q=0}^{[n/2]} (-1)^q \, b_{nq} \, I_{i+n,q} \, \Delta^{(\mu_1\mu_2} \cdots \Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}} \cdots u^{\mu_n)}$$

$$I_{nq}(\alpha_0, \beta_0) = \frac{(-1)^q}{(2q+1)!!} \left\langle E_{\mathbf{k}u}^{n-2q} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q \right\rangle_0 \quad \textbf{thermodynamic integral}$$

$$N_0^\mu = n_0 u^\mu$$

$$T_0^{\mu\nu} = e_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}$$

$$n_0 \equiv \langle E_{\mathbf{k}u} \rangle_0 = N_0^\mu u_\mu \quad \textbf{(net)charge density}$$

$$e_0 \equiv \langle E_{\mathbf{k}u}^2 \rangle_0 = T_0^{\mu\nu} u_\mu u_\nu \quad \textbf{energy density}$$

$$P_0 \equiv -\frac{1}{3} \langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle_0 = -\frac{1}{3} \Delta_{\mu\nu} T_0^{\mu\nu} \quad \textbf{isotropic pressure}$$

- We have 5 equations for 6 unknowns *not closed*: $n_0(1)$, $e_0(1)$, $P_0(1)$ and $u^\mu(3)$.

Ideal Fluids II.

- The assumption of local thermal equilibrium provides closure

Equation of State (EoS)

$$P_0 = P_0(e_0, n_0) \qquad \text{EoS} \Rightarrow \textbf{1 additional eq.}$$

- Auxiliary, $S_0^\mu = s_0 u^\mu$, where $s_0 = S_0^\mu u_\mu$,

$$\partial_\mu S_0^\mu \equiv \partial_\mu \int dK\, k^\mu\, f_{0\mathbf{k}} \left(\ln f_{0\mathbf{k}} - 1\right) \geq 0$$

entropy is maximum in local thermal equilibrium.

Thermodynamics

$$\begin{aligned} Ts &= e + P - \mu n \\ T\dot{s} &= \dot{e} - \mu\dot{n} \\ \dot{P} &= s\dot{T} + n\dot{\mu} \end{aligned}$$

- The fundamental thermodynamic relations are derived from, $T\partial_\mu(su^\mu) = \partial_\mu(eu^\mu) + P(\partial_\mu u^\mu) - \mu\partial_\mu(nu^\mu)$

Dissipative Fluid

Out of equilibrium

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$$

Dissipative Fluids I.

General decomposition with respect to u^μ

$$
\begin{aligned}
N^\mu &\equiv N_0^\mu + \delta N^\mu = n u^\mu + V^\mu \\
T^{\mu\nu} &\equiv T_0^{\mu\nu} + \delta T^{\mu\nu} = e u^\mu u^\nu - (P_0 + \Pi)\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu} \\
n &\equiv \langle E_{\mathbf{k}u} \rangle = N^\mu u_\mu \qquad \textbf{charge density} \\
e &\equiv \langle E_{\mathbf{k}u}^2 \rangle = T^{\mu\nu} u_\mu u_\nu \qquad \textbf{energy density in LRF} \\
P &\equiv -\frac{1}{3}\langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle = P_0 + \Pi \equiv -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} \qquad \textbf{isotropic (eq. + bulk) pressure} \\
V^\mu &\equiv \langle k^{\langle\mu\rangle} \rangle = \Delta^{\mu\alpha} N_\alpha \qquad \textbf{charge flow} \\
W^\mu &\equiv \langle E_{\mathbf{k}u} k^{\langle\mu\rangle} \rangle = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta \qquad \textbf{energy-momentum flow} \\
q^\mu &= W^\mu - \frac{e + P_0}{n} V^\mu \qquad \textbf{heat flow} \qquad \Rightarrow \mathbf{1\ eq.} \\
\pi^{\mu\nu} &\equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle = T^{\langle\mu\nu\rangle} \qquad \textbf{stress tensor}
\end{aligned}
$$

- We only have 5 equations for 18 unknowns, $n(1), e(1), P_0(1), u^\mu(3)$ and $\Pi(1), V^\mu(3), W^\mu(3), \pi^{\mu\nu}(5)$.

Dissipative Fluids II.

Simplifications (I): Matching to equilibrium and the EOS

$$\begin{aligned} n &= n_0 + \delta n \quad \Rightarrow \quad (N^\mu - N_0^\mu)u_\mu \equiv \delta N^\mu u_\mu = 0 \\ e &= e_0 + \delta e \quad \Rightarrow \quad (T^\mu - T_0^\mu)u_\mu u_\nu \equiv \delta T^{\mu\nu} u_\mu u_\nu = 0 \\ P(e,n) &= P_0(e_0, n_0) + \delta P \end{aligned}$$

- Convenient choice, $\delta n = 0$, $\delta e = 0$, $\delta p = \Pi$ hence, $P_0 = P_0(e_0, n_0)$, such that $P(e,n) = P_0 + \Pi$, while $T = T_0$ and $\mu = \mu_0$, but $s = s_0 + \delta s$!
- Still with $14 + 3$ unknowns! $n(1), e(1), \Pi(1)$, $V^\mu(3), W^\mu(3), \pi^{\mu\nu}(5)$ and $u^\mu(3)$.

Simplifications (II): Fixing the Local Rest Frame

$$u_E^\mu = N^\mu / n \quad \Leftrightarrow V^\mu = 0 \quad \Rightarrow q^\mu = W^\mu \qquad \textbf{Eckart}$$

$$u_L^\mu = T^{\mu\nu} u_{\nu,L} / e \quad \Leftrightarrow W^\mu = 0 \quad \Rightarrow q^\mu = -\frac{e + P_0}{n} V^\mu \qquad \textbf{Landau \& Lifsitz}$$

- We eliminated (3) unknowns, such that $u_L^\mu = u_E^\mu + \delta u^\mu$.
- We are left with 14 unknowns! $n(1), e(1), u^\mu(3)$ and $\Pi(1), q^\mu(3), \pi^{\mu\nu}(5)$.

Dissipative Fluids III. - The relativistic Navier-Stokes theory

- The entropy current is also modified $S^\mu \equiv S_0^\mu + \delta S^\mu = (s_0 + \delta s)u^\mu + \Phi^\mu$, where $s \equiv S^\mu u_\mu = (s_0 + \delta s)$ and $\Phi^\mu = \Delta^{\mu\nu} S_\nu$.

2nd law of thermodynamics (Eckart's frame)

$$\partial_\mu S^\mu = \partial_\mu \left[\Phi^\mu - \frac{q^\mu}{T}\right] - \frac{q^\mu}{T}\left(\frac{1}{T}\partial_\mu T - \dot{u}_\mu\right) - \frac{\Pi}{T}\partial_\mu u^\mu + \frac{\pi^{\mu\nu}}{T}\partial_\mu u_\nu \geq 0$$

- Assuming that $s = s_0$, i.e., $\delta s = 0$, while $\Phi^\mu = q^\mu / T$
- For small gradients, linear relations between thermodynamical forces and fluxes!

The relativistic Navier-Stokes relations

$$\Pi_{NS} = -\zeta \nabla_\mu u^\mu$$

$$\pi^{\mu\nu}_{NS} = 2\eta \nabla^{\langle\mu} u^{\nu\rangle}$$

$$q^\mu_{NS} = -\kappa T \frac{T n}{e + p} \nabla^\mu \left(\frac{\mu}{T}\right)$$

- $\zeta \geq 0$, $\eta \geq 0$ bulk and shear viscosity coefficients, $\kappa \geq 0$ coefficient of thermal conductivity.
- Now, the equations of fluid dynamics are closed, but the relativistic Navier-Stokes theory leads to accausal signal propagation and stability issues.

General equations of motion

Using $\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 - a f_{0\mathbf{k}}\right) \phi_{\mathbf{k}}$ let us define the following **irreducible moment**

$$\rho_i^{\mu_1 \cdots \mu_\ell} \equiv \left\langle E_{\mathbf{k}u}^i k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \right\rangle_\delta$$

$\langle \cdots \rangle_\delta \equiv \langle \cdots \rangle - \langle \cdots \rangle_0 = \int dK (\cdots) \delta f_{\mathbf{k}}$ and $k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} = \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} k^{\nu_1} \cdots k^{\nu_\ell}$

$$\begin{aligned} \rho_1 &\equiv \delta n = 0 , \qquad \rho_2 \equiv \delta e = 0 , \qquad \rho_0 \equiv -\frac{3}{m^2} \Pi , \\ \rho_0^\mu &\equiv V^\mu , \qquad \rho_0^\mu \equiv W^\mu , \qquad \rho_0^{\mu\nu} \equiv \pi^{\mu\nu} . \end{aligned}$$

Now, using the Boltzmann equation $k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$ in the following form

$$D\delta f_{\mathbf{k}} = -D f_{0\mathbf{k}} - E_{\mathbf{k}u}^{-1} k_\nu \nabla^\nu f_{0\mathbf{k}} - E_{\mathbf{k}u}^{-1} k_\nu \nabla^\nu \delta f_{\mathbf{k}} + E_{\mathbf{k}u}^{-1} C\left[f_{0\mathbf{k}} + \delta f_{\mathbf{k}}\right]$$

where $D = u^\mu \partial_\mu$ and $\nabla_\mu = \Delta^\nu_\mu \partial_\nu$, we obtain the equations for $\rho_i^{\mu_1 \cdots \mu_\ell}$

$$D\rho_i^{\langle \mu_1 \cdots \mu_\ell \rangle} = \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} \frac{d}{d\tau} \int dK E_{\mathbf{k}u}^i k^{\langle \nu_1} \cdots k^{\nu_\ell \rangle} \delta f_{\mathbf{k}}$$

Anisotropic Fluid

$$f_{\mathbf{k}} = \hat{f}_{0\mathbf{k}} \left(\hat{\alpha}, \hat{\beta}_u E_{\mathbf{k}u}, \hat{\beta}_l E_{\mathbf{k}l} \right)$$

Local anisotropic distribution

Local anisotropic distribution $f_{\mathbf{k}} \to \hat{f}_{0\mathbf{k}}$, where

$$\lim_{\hat{\beta}_l \to 0} \hat{f}_{0\mathbf{k}} \left(\hat{\alpha}, \hat{\beta}_u E_{\mathbf{k}u}, \hat{\beta}_l E_{\mathbf{k}l} \right) = f_{0\mathbf{k}} \left(\hat{\alpha}, \hat{\beta}_u E_{\mathbf{k}u} \right)$$

Moments of the equilibrium distribution function

$$\hat{N}^{\mu} \equiv \int dK\, k^{\mu}\, \hat{f}_{0\mathbf{k}} = \langle k^{\mu} \rangle_{\hat{0}}$$ **charge current**

$$\hat{T}^{\mu\nu} \equiv \int dK\, k^{\mu} k^{\nu}\, \hat{f}_{0\mathbf{k}} = \langle k^{\mu} k^{\nu} \rangle_{\hat{0}}$$ **energy-momentum tensor**

$$\ldots \qquad \ldots$$

$$\hat{\mathcal{I}}^{\mu_1 \cdots \mu_n}_{ij} \equiv \left\langle E^{i}_{\mathbf{k}u}\, E^{j}_{\mathbf{k}l}\, k^{\mu_1} \cdots k^{\mu_n} \right\rangle_{\hat{0}}$$ **rank-n tensor moment**

Anisotropic Fluids I.

General tensorial decomposition with respect to u^μ and l^μ

$$\hat{\mathcal{I}}_{ij}^{\mu_1\cdots\mu_n} = \sum_{q=0}^{[n/2]}\sum_{r=0}^{n-2q}(-1)^q\ b_{nrq}\ \hat{I}_{i+j+n,j+r,q} \times \Xi^{(\mu_1\mu_2}\ldots\Xi^{\mu_{2q-1}\mu_{2q}}l^{\mu_{2q+1}}\ldots l^{\mu_{2q+r}}u^{\mu_{2q+r+1}}\ldots u^{\mu_n)}$$

$$\hat{I}_{nrq}\left(\hat{\alpha},\hat{\beta}_u,\hat{\beta}_l\right) = \frac{(-1)^q}{(2q)!!}\left\langle E_{\mathbf{k}u}^{n-r-2q}\,E_{\mathbf{k}l}^r\left(\Xi^{\mu\nu}k_\mu k_\nu\right)^q\right\rangle_{\hat{0}}$$

$$\hat{N}^\mu = \hat{n}u^\mu + \hat{n}_l l^\mu$$

$$\hat{T}^{\mu\nu} = \hat{e}u^\mu u^\nu + 2\hat{M}u^{(\mu}l^{\nu)} + \hat{P}_l l^\mu l^\nu - \hat{P}_\perp \Xi^{\mu\nu}$$

$\hat{n} \equiv \langle E_{\mathbf{k}u}\rangle_{\hat{0}} = \hat{N}^\mu u_\mu$ **charge density**

$\hat{n}_l \equiv \langle E_{\mathbf{k}l}\rangle_{\hat{0}} = -\hat{N}^\mu l_\mu$ **charge current l^μ-direction**

$\hat{e} \equiv \langle E_{\mathbf{k}u}^2\rangle_{\hat{0}} = \hat{T}^{\mu\nu}u_\mu u_\nu$ **energy density**

$\hat{M} \equiv \langle E_{\mathbf{k}u}E_{\mathbf{k}l}\rangle_{\hat{0}} = -\hat{T}^{\mu\nu}u_\mu l_\nu$ **energy-momentum current**

$\hat{P}_l \equiv \langle E_{\mathbf{k}l}^2\rangle_{\hat{0}} = \hat{T}^{\mu\nu}l_\mu l_\nu$ **longitudinal pressure**

$\hat{P}_\perp \equiv -\frac{1}{2}\langle \Xi_{\mu\nu}k^\mu k^\nu\rangle_{\hat{0}} = -\frac{1}{2}\Xi_{\mu\nu}\hat{T}^{\mu\nu}$ **transverse pressure**

$\hat{P} \equiv \frac{1}{3}(\hat{P}_l + 2\hat{P}_\perp) = -\frac{1}{3}\Delta_{\mu\nu}\hat{T}^{\mu\nu}$ **isotropic pressure**

Equations of motion for anisotropic fluids

Conservation laws

$$0 = \partial_\mu \hat{N}^\mu \equiv D\hat{n} - D_l \hat{n}_l + \hat{n}\tilde{\theta} + \hat{n}_l \tilde{\theta}_l + \hat{n}\left(l_\mu D_l u^\mu\right) - \hat{n}_l \left(l_\mu D u^\mu\right), \quad (1)$$

$$0 = u_\nu \partial_\mu \hat{T}^{\mu\nu} \equiv D\hat{e} - D_l \hat{M} + \left(\hat{e} + \hat{P}_\perp\right)\tilde{\theta} + \hat{M}\tilde{\theta}_l + \left(\hat{e} + \hat{P}_l\right)\left(l_\mu D_l u^\mu\right) - 2\hat{M}\left(l_\mu D u^\mu\right), \quad (2)$$

$$0 = l_\nu \partial_\mu \hat{T}^{\mu\nu} \equiv -D\hat{M} + D_l \hat{P}_l - \hat{M}\tilde{\theta} + \left(\hat{P}_\perp - \hat{P}_l\right)\tilde{\theta}_l - 2\hat{M}\left(l_\mu D_l u^\mu\right) + \left(\hat{e} + \hat{P}_l\right)\left(l_\mu D u^\mu\right), \quad (3)$$

$$\begin{aligned} 0 &= \Xi^\alpha_\nu \partial_\mu \hat{T}^{\mu\nu} \equiv \left(\hat{e} + \hat{P}_\perp\right)\left[D u^\alpha + l^\alpha \left(l_\nu D u^\nu\right)\right] \\ &- \tilde{\nabla}^\alpha \hat{P}_\perp + \left(\hat{P}_\perp - \hat{P}_l\right)\left[D_l l^\alpha + u^\alpha \left(l_\nu D_l u^\nu\right)\right] \\ &+ \hat{M}\left[D l^\alpha + u^\alpha \left(l_\nu D u^\nu\right)\right] - \hat{M}\left[D_l u^\alpha + l^\alpha \left(l_\nu D_l u^\nu\right)\right]. \quad (4) \end{aligned}$$

- Only 5 eqs., for 11 unknowns, $\hat{n}, \hat{n}_l, \hat{e}, \hat{M}, \hat{P}_l, \hat{P}_\perp$ and $l^\mu(2), u^\mu(3)$.

Anisotropic Fluids II.

Simplifications (I): Matching to equilibrium and the EOS

$$\hat{n}(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l) = n_0(\alpha_0, \beta_0) \Rightarrow (\hat{N}^\mu - N_0^\mu)u_\mu = 0$$
$$\hat{e}(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l) = e_0(\alpha_0, \beta_0) \Rightarrow (\hat{T}^\mu - T_0^\mu)u_\mu u_\nu = 0$$
$$\hat{P}(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l) = P_0(\alpha_0, \beta_0) + \hat{\Pi}(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l) \equiv \frac{1}{3}(\hat{P}_l + 2\hat{P}_\perp)$$

- This leads, $\alpha_0(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l)$ and $\beta_0(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l)$, but there is no "matching" for $\hat{\beta}_l$.

Simplifications (II): Fixing the Local Rest Frame

$$u_E^\mu = N^\mu / n \Leftrightarrow \hat{n}_l = 0 \quad \textbf{Eckart}$$
$$u_L^\mu = T^{\mu\nu} u_{\nu,L} / e \Leftrightarrow \hat{M} = 0 \quad \textbf{Landau \& Lifsitz}$$

- Since all quantities are function $\hat{\alpha}$, $\hat{\beta}_u$ and $\hat{\beta}_l$ we are left with 6 unknowns! $\hat{n}(1), \hat{e}(1), \hat{P}_l(1)$ and $u^\mu(3)$.

Anisotropic Dissipative Fluids I.

Out of equilibrium

$$f_{\mathbf{k}} \equiv \hat{f}_{0\mathbf{k}} + \delta\hat{f}_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$$

Conservation laws for a simple (single component) anisotropic dissipative fluid

$\partial_\mu N^\mu = 0$ **charge conservation** $\Rightarrow$ **1 eq.**

$\partial_\mu T^{\mu\nu} = 0$ **energy-momentum conservation** $\Rightarrow$ **4 eqs.**

General decomposition with respect to u^μ and l^μ

$$N^\mu \equiv \hat{N}^\mu + \delta\hat{N}^\mu = n u^\mu + n_l u^\mu + V_\perp^\mu$$

$n \equiv \langle E_{\mathbf{k}u} \rangle = N^\mu u_\mu$ **charge density**

$n_l \equiv \langle E_{\mathbf{k}l} \rangle = -N^\mu l_\mu$ **charge current l^μ-direction**

$V_\perp^\mu \equiv \langle k^{\{\mu\}} \rangle = \Xi^{\mu\alpha} N_\alpha$ **charge flow in the transverse direction**

$V^\mu \equiv n_l l^\mu + V_\perp^\mu = \Delta^{\mu\alpha} N_\alpha$

Anisotropic Dissipative Fluids II.

General decomposition with respect to u^μ and l^μ

$$
\begin{aligned}
T^{\mu\nu} &\equiv \hat{T}^{\mu\nu} + \delta\hat{T}^{\mu\nu} \\
&= e u^\mu u^\nu + 2M u^{(\mu} l^{\nu)} + P_l l^\mu l^\nu - P_\perp \Xi^{\mu\nu} + 2W_{\perp u}^{(\mu} u^{\nu)} + 2W_{\perp l}^{(\mu} l^{\nu)} + \pi_\perp^{\mu\nu}
\end{aligned}
$$

$$e \equiv \langle E_{\mathbf{k}u}^2 \rangle = T^{\mu\nu} u_\mu u_\nu \quad \textbf{energy density in LRF}$$

$$M \equiv \langle E_{\mathbf{k}u} E_{\mathbf{k}l} \rangle = -T^{\mu\nu} u_\mu l_\nu \quad \textbf{energy-momentum current}$$

$$P_l \equiv \langle E_{\mathbf{k}_l}^2 \rangle = T^{\mu\nu} l_\mu l_\nu \quad \textbf{longitudinal pressure}$$

$$P_\perp \equiv -\frac{1}{2}\langle \Xi^{\mu\nu} k_\mu k_\nu \rangle = -\frac{1}{2}\Xi_{\mu\nu} T^{\mu\nu} \quad \textbf{transverse pressure}$$

$$W_{\perp u}^\mu \equiv \langle E_{\mathbf{k}u} k^{\{\mu\}} \rangle = \Xi^{\mu\alpha} T_{\alpha\beta} u^\beta \quad \textbf{energy-momentum flow } u^\mu\textbf{-direction}$$

$$W_{\perp l}^\mu \equiv \langle E_{\mathbf{k}l} k^{\{\mu\}} \rangle = -\Xi^{\mu\alpha} T_{\alpha\beta} l^\beta \quad \textbf{energy-momentum flow } l^\mu\textbf{-direction}$$

$$\pi_\perp^{\mu\nu} \equiv \langle k^{\{\mu} k^{\nu\}} \rangle = T^{\{\mu\nu\}} \quad \textbf{stress tensor}$$

$$W^\mu \equiv M l^\mu + W_{\perp u}^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$$

$$\pi^{\mu\nu} \equiv \pi_\perp^{\mu\nu} + 2\, W_{\perp l}^{(\mu} l^{\nu)} + \frac{1}{3}(P_l - P_\perp)(2\, l^\mu l^\nu + \Xi^{\mu\nu})$$

General equations of motion

Using $\delta \hat{f}_{\mathbf{k}} = \hat{f}_{0\mathbf{k}} \left(1 - a\hat{f}_{0\mathbf{k}}\right) \hat{\phi}_{\mathbf{k}}$ let us define the following **irreducible moment**

$$\hat{\rho}_{ij}^{\mu_1 \cdots \mu_\ell} \equiv \left\langle E_{\mathbf{k}u}^{i} E_{\mathbf{k}l}^{j} k^{\{\mu_1} \cdots k^{\mu_\ell\}} \right\rangle_{\hat{\delta}}$$

$\langle \cdots \rangle_{\hat{\delta}} \equiv \langle \cdots \rangle - \langle \cdots \rangle_{\hat{0}} = \int dK (\cdots) \delta f_{\mathbf{k}}$ and $k^{\{\mu_1} \cdots k^{\mu_\ell\}} = \Xi^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} k^{\nu_1} \cdots k^{\nu_\ell}$

$$\begin{aligned} \hat{\rho}_{10} &\equiv n - \hat{n}\,, \quad \hat{\rho}_{20} \equiv e - \hat{e}\,, \quad \hat{\rho}_{01} \equiv n_l - \hat{n}_l\,, \quad \hat{\rho}_{11} \equiv M - \hat{M}\,, \quad \hat{\rho}_{02} \equiv P_l - \hat{P}_l \\ \hat{\rho}^{\mu}_{00} &\equiv V^{\mu}_{\perp}\,, \quad \hat{\rho}^{\mu}_{10} \equiv W^{\mu}_{\perp u}\,, \quad \hat{\rho}^{\mu}_{01} \equiv W^{\mu}_{\perp l}\,, \quad \hat{\rho}^{\mu\nu}_{00} \equiv \pi^{\mu\nu}_{\perp}\,. \end{aligned}$$

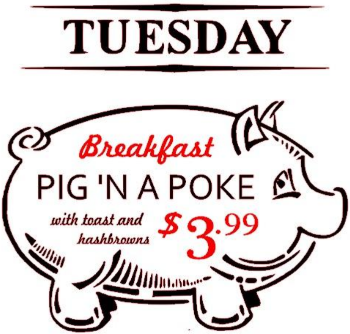
Now, using the Boltzmann equation $k^{\mu}\partial_{\mu} f_{\mathbf{k}} = C[f]$ in the following form

$$D\delta \hat{f}_{\mathbf{k}} = -D\hat{f}_{0\mathbf{k}} + E_{\mathbf{k}u}^{-1} \left(E_{\mathbf{k}l} D_l \hat{f}_{0\mathbf{k}} + E_{\mathbf{k}l} D_l \delta \hat{f}_{\mathbf{k}} - k^{\mu} \tilde{\nabla}_{\mu} \hat{f}_{0\mathbf{k}} - k^{\mu} \tilde{\nabla}_{\mu} \delta \hat{f}_{\mathbf{k}} \right) + E_{\mathbf{k}u}^{-1} C \left[\hat{f}_{0\mathbf{k}} + \delta \hat{f}_{\mathbf{k}} \right]$$

where $D = u^{\mu}\partial_{\mu}$ and $\nabla_{\mu} = \Delta^{\nu}_{\mu}\partial_{\nu}$, we obtain the equations for $\rho_i^{\mu_1 \cdots \mu_\ell}$

$$D\hat{\rho}_{ij}^{\{\mu_1 \cdots \mu_\ell\}} \equiv \Xi^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} D\hat{\rho}_{ij}^{\nu_1 \cdots \nu_\ell}$$

Results


TUESDAY
Breakfast
PIG 'N A POKE
with toast and hashbrowns
$3.99

The Romatschke-Strickland distribution function I.

The RS-distribution function

$$\hat{f}_{RS} \equiv \left[\exp\left(-\alpha_{RS} + \beta_{RS}\sqrt{E_{\mathbf{k}u} + \xi E_{\mathbf{k}l}}\right) + a\right]^{-1},$$

ξ is the so-called anisotropy parameter

The conserved quantities

$$\begin{aligned}
\hat{N}^{\mu}_{RS} &= \hat{n} u^{\mu}, \\
\hat{T}^{\mu\nu}_{RS} &= \hat{e} u^{\mu} u^{\nu} + \hat{P}_l l^{\mu} l^{\nu} - \hat{P}_{\perp} \Xi^{\mu\nu}, \\
\hat{n} &\equiv \hat{I}^{RS}_{100} = n_0(\alpha_{RS}, \beta_{RS}) R_{100}(\xi), \qquad \hat{e} \equiv \hat{I}^{RS}_{200} = e_0(\alpha_{RS}, \beta_{RS}) R_{200}(\xi), \\
\hat{P}_l &\equiv \hat{I}^{RS}_{220} = e_0(\alpha_{RS}, \beta_{RS}) R_{220}(\xi), \qquad \hat{P}_{\perp} \equiv \hat{I}^{RS}_{201} = P_0(\alpha_{RS}, \beta_{RS}) R_{201}(\xi).
\end{aligned}$$

where

The thermodynamic integrals

$$\begin{aligned}
\hat{I}^{RS}_{nrq} &\equiv \frac{(-1)^q}{(2q)!!} \int dK E^{n-r-2q}_{\mathbf{k}u} E^{r}_{\mathbf{k}l} \left(\Xi^{\mu\nu} k_{\mu} k_{\nu}\right)^q \;\hat{f}_{RS} \\
\lim_{m\rightarrow 0} \hat{I}^{RS}_{nrq} &= I_{nq}(\alpha_{RS}, \beta_{RS}) \times R_{nrq}(\alpha_{RS}, \beta_{RS}, \xi).
\end{aligned}$$

The Romatschke-Strickland distribution function II.

Landau matching

$$(\hat{N}^{\mu}_{RS} - N^{\mu}_{0})u_{\mu} = 0 \quad \Rightarrow \quad \hat{n}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = n_0\left(\alpha_0, \beta_0\right),$$
$$(\hat{T}^{\mu\nu}_{RS} - T^{\mu\nu}_{0})u_{\mu}u_{\nu} = 0 \quad \Rightarrow \quad \hat{e}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = e_0\left(\alpha_0, \beta_0\right).$$

Conservation of particles

$$\beta_0 = \beta_{RS}\frac{R_{100}(\xi)}{R_{200}(\xi)}, \qquad \lambda_0 = \lambda_{RS}\frac{\left[R_{100}(\xi)\right]^4}{\left[R_{200}(\xi)\right]^3},$$
$$\hat{I}^{RS}_{nrq}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = I_{nq}\left(\alpha_0, \beta_0\right) R_{nrq}(\xi)\,\frac{\left[R_{200}(\xi)\right]^{1-n}}{\left[R_{100}(\xi)\right]^{2-n}}.$$

Non-Conservation of particles

$$\beta_0 = \frac{\beta_{RS}}{\left[R_{200}(\xi)\right]^{1/4}}$$
$$\hat{I}^{RS}_{nrq}\left(\beta_{RS}, \xi\right) = I_{nq}\left(\beta_0\right)\frac{R_{nrq}(\xi)}{\left[R_{200}(\xi)\right]^{(n+2)/4}}.$$

0+1D Bjorken expansion I.

$$u^{\mu} \equiv \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right) = (\cosh\eta, 0, 0, \sinh\eta)\,,$$
$$l^{\mu} \equiv \left(\frac{z}{\tau}, 0, 0, \frac{t}{\tau}\right) = (\sinh\eta, 0, 0, \cosh\eta)\,.$$

General equation of motion

$$\frac{\partial \hat{I}_{i+j,j,0}}{\partial \tau} + \frac{1}{\tau}\left[(j+1)\,\hat{I}_{i+j,j,0} + (i-1)\,\hat{I}_{i+j,j+2,0}\right] = \hat{\mathcal{C}}_{i-1,j},$$
$$\hat{\mathcal{C}}_{i-1,j} = -\frac{1}{\tau_{eq}}\left(\hat{I}_{i+j,j,0} - I_{i+j,j,0}\right)\,, \qquad \tau_{eq}(\tau) = 5\beta_0(\tau)\frac{\eta}{s}.$$

Conservation equations

$$\frac{\partial n_0\left(\alpha_0, \beta_0\right)}{\partial \tau} + \frac{1}{\tau} n_0\left(\alpha_0, \beta_0\right) = 0,$$
$$\frac{\partial e_0\left(\alpha_0, \beta_0\right)}{\partial \tau} + \frac{1}{\tau}\left[e_0\left(\alpha_0, \beta_0\right) + \hat{P}_l\left(\alpha_{RS}, \beta_{RS}, \xi\right)\right] = 0.$$

0+1D Bjorken expansion I.

Balance equations

$$\frac{\partial \hat{P}_l}{\partial \tau} + \frac{1}{\tau}\left(3\hat{P}_l - \hat{I}^{RS}_{240}\right) = -\frac{1}{\tau_{eq}}\left(\hat{P}_l - P_0\right),$$

$$\frac{\partial \hat{n}}{\partial \tau} + \frac{1}{\tau}\hat{n} = -\frac{1}{\tau_{eq}}\left(\hat{n} - n_0\right),$$

$$\frac{\partial \hat{I}^{RS}_{000}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}^{RS}_{000} - \hat{I}^{RS}_{020}\right) = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{000} - I_{000}\right),$$

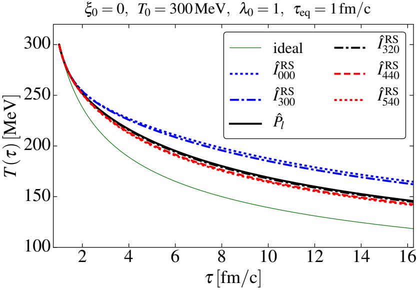
$$\frac{\partial \hat{I}^{RS}_{300}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}^{RS}_{300} + 2\hat{I}^{RS}_{320}\right) = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{300} - I_{300}\right),$$

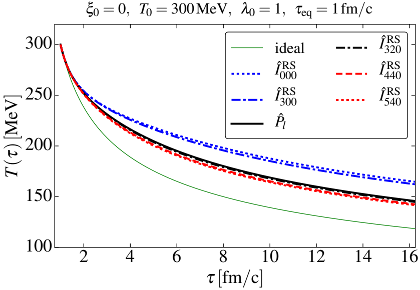
$$\frac{\partial \hat{I}^{RS}_{320}}{\partial \tau} + \frac{3}{\tau}\hat{I}^{RS}_{320} = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{320} - I_{320}\right)$$

$$\frac{\partial \hat{I}^{RS}_{440}}{\partial \tau} + \frac{1}{\tau}\left(5\hat{I}^{RS}_{440} - \hat{I}^{RS}_{460}\right) = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{440} - I_{440}\right),$$

$$\frac{\partial \hat{I}^{RS}_{540}}{\partial \tau} + \frac{5}{\tau}\hat{I}^{RS}_{540} = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{540} - I_{540}\right).$$

0+1D Bjorken expansion I. - Temperature

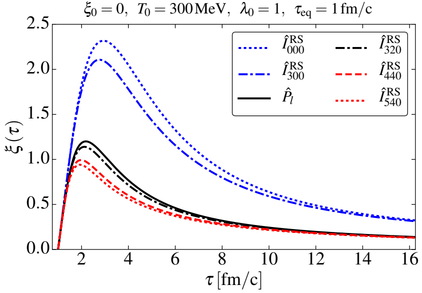


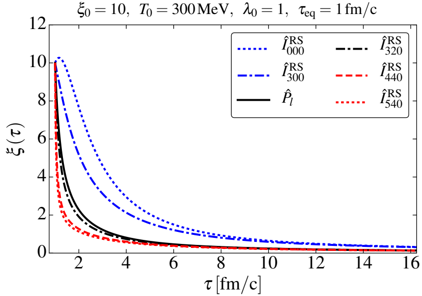


- The evolution of temperature for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of temperature for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion I. - Anisotropy

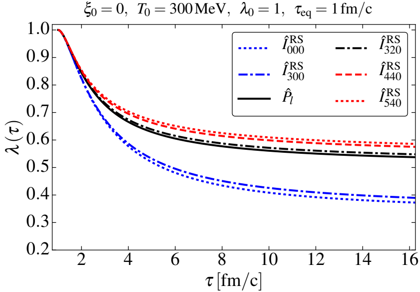


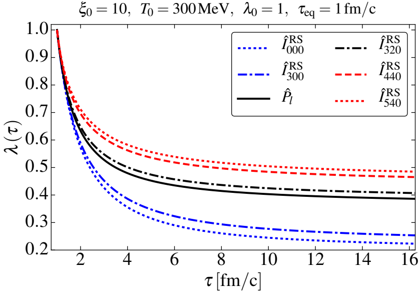


- The evolution of the anisotropy parameter for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of anisotropy parameter for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion I. - Fugacity

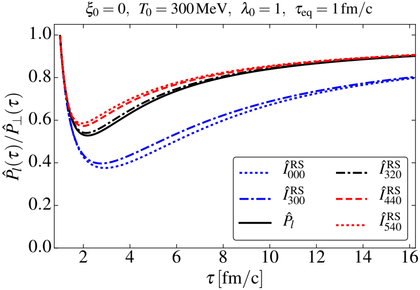


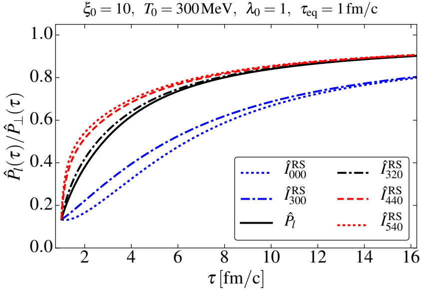


- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion I. - Pressure

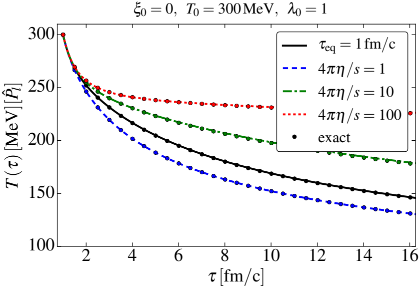


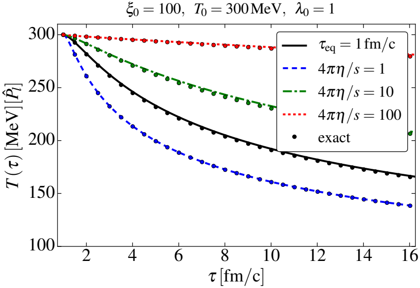


- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion - Comparisons to the exact solution I.

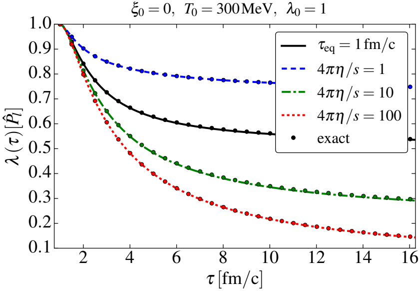


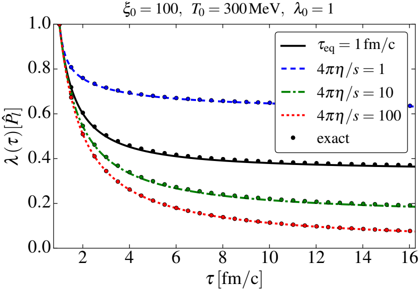


- The evolution of the temperature for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the temperature for $\xi(\tau_0) = \xi_0 = 100$.

0+1D Bjorken expansion - Comparisons to the exact solution II.

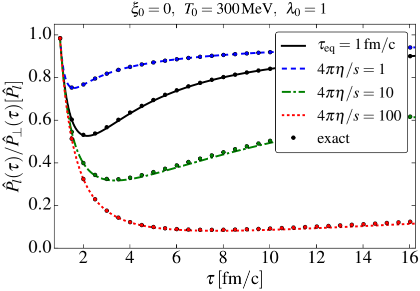


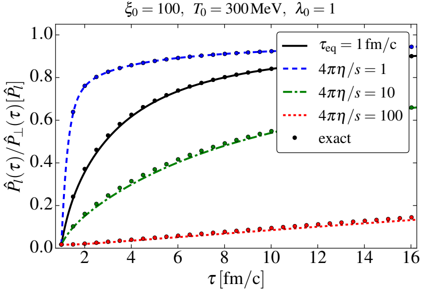


- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 100$.

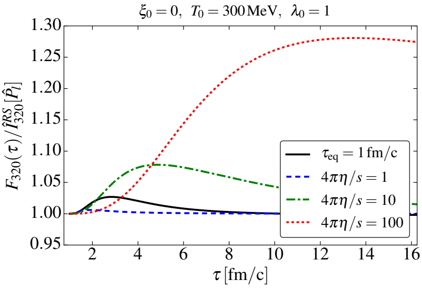
0+1D Bjorken expansion - Comparisons to the exact solution III.

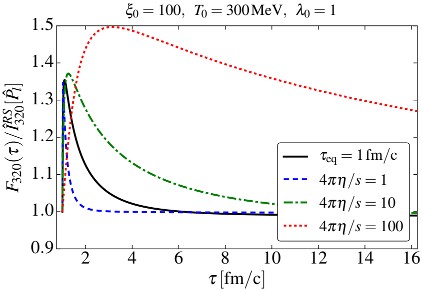




- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 0$.
- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 100$.

0+1D Bjorken expansion - Comparisons to the exact solution IV.



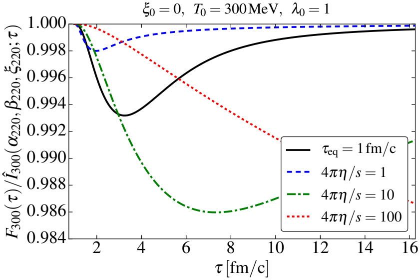


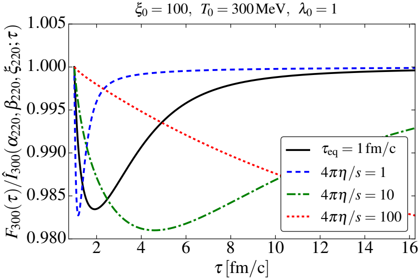
- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 100$.

$$F_{nrq} = \frac{(-1)^q}{(2q)!!} \int dK E_{\mathbf{k}u}^{n-r-2q} E_{\mathbf{k}l}^{r} \left(\Xi^{\mu\nu} k_\mu k_\nu\right)^q f_{\mathbf{k}}$$

0+1D Bjorken expansion - Comparisons to the exact solution V.

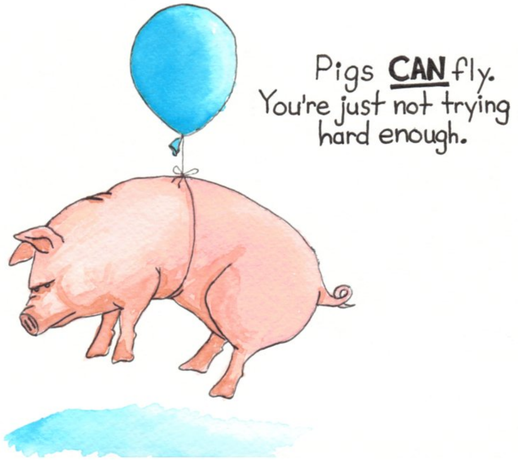




- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the pressure for $\xi(\tau_0) = \xi_0 = 100$.

$$\hat{I}_{nrq}(\alpha_{220}, \beta_{220}, \xi_{220}) \neq \hat{I}_{nrq}(\alpha_{300}, \beta_{300}, \xi_{300})$$


Pigs CAN fly.
You're just not trying hard enough.