Critical Point and Event-by-Event Fluctuations

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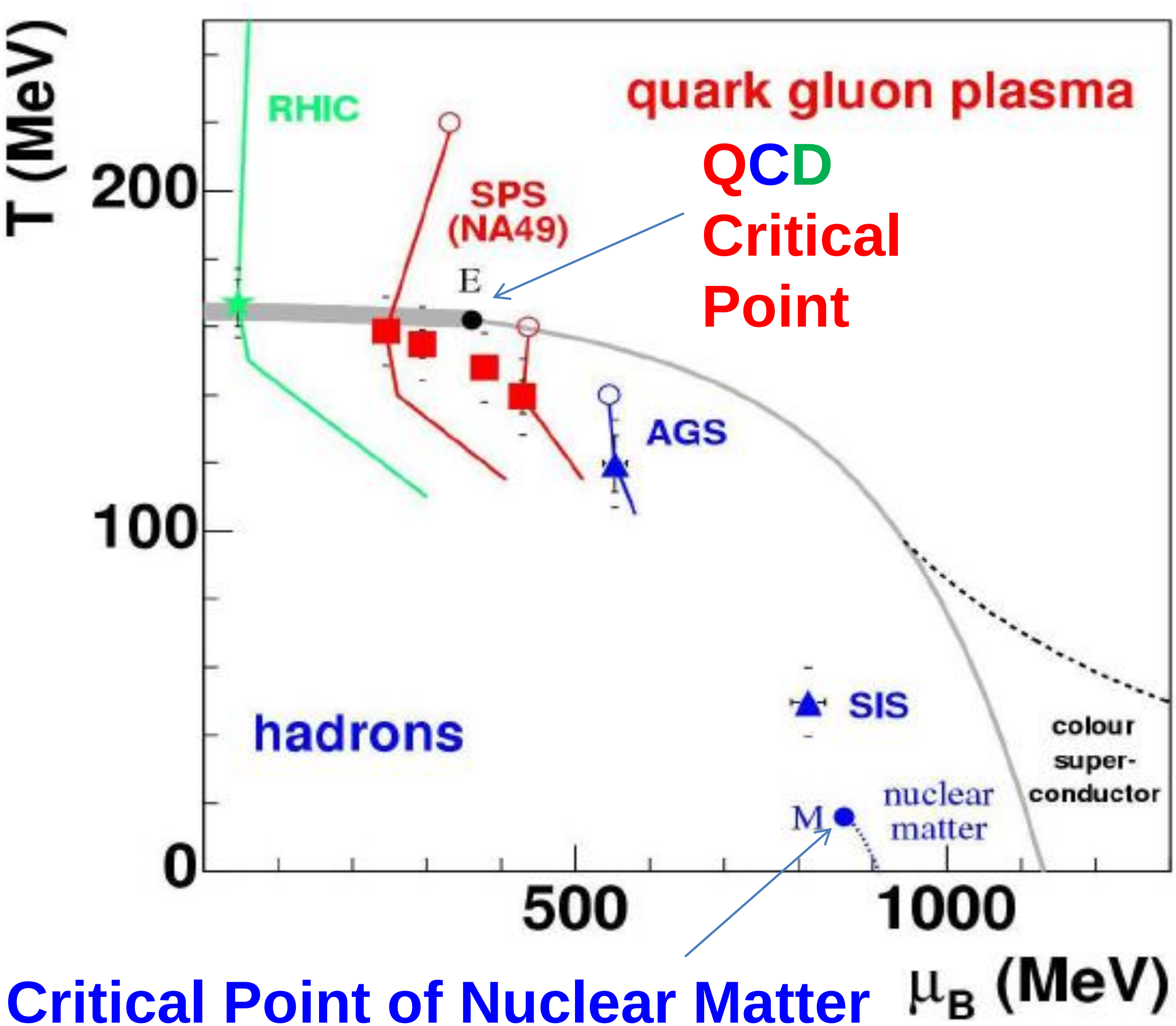
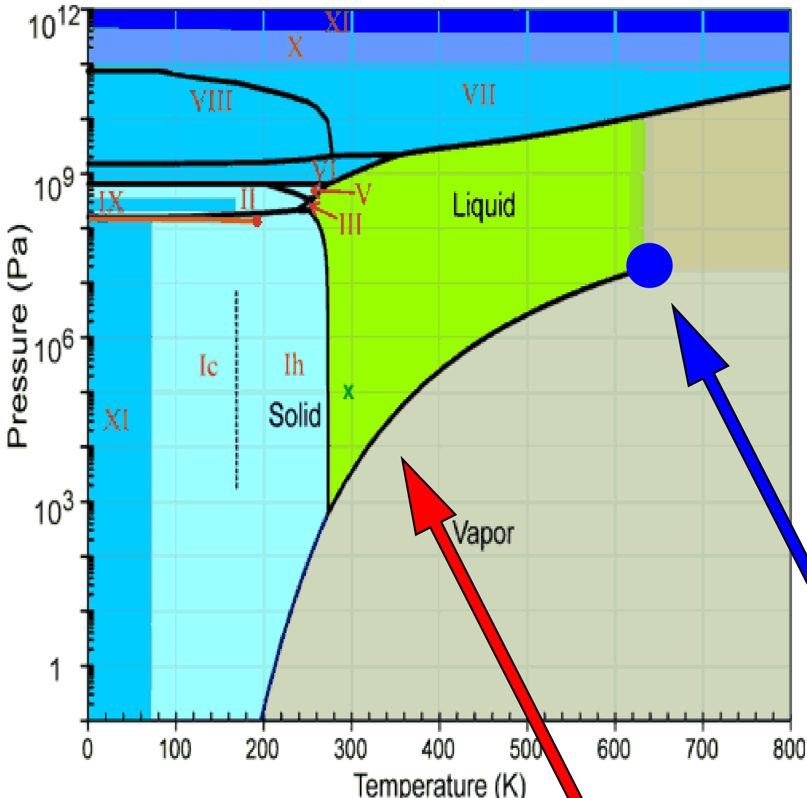
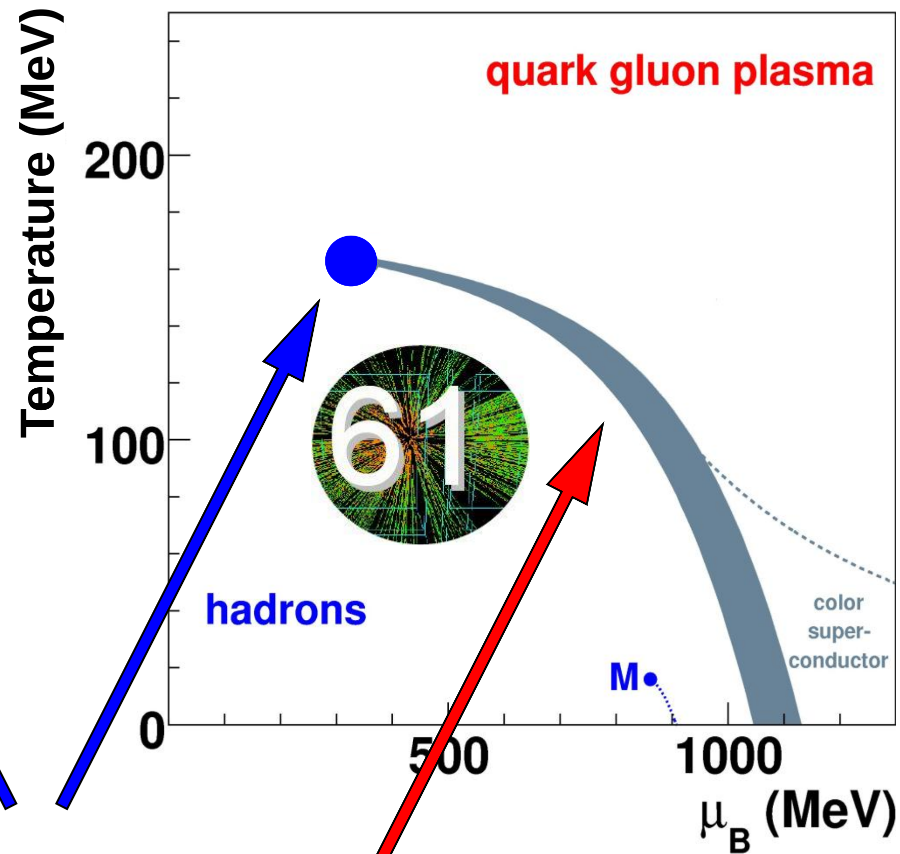

T (MeV)
200
100
0
500
1000
μB (MeV)
quark gluon plasma
RHIC
SPS
(NA49)
E
QCD
Critical
Point
AGS
SIS
hadrons
colour
super-
conductor
nuclear
matter
M
Critical Point of Nuclear Matter

Critical Point

Water



Strongly Interacting Matter



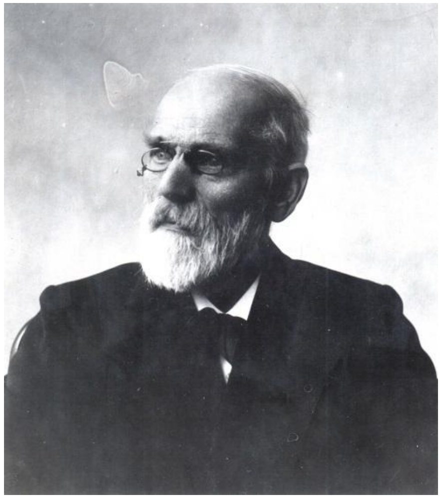
QCD Critical Point

1st Order Phase Transition

II. Van der Waals Equation of State

1873, Ph. D. Thesis

1910, Nobel Prize in Physics



$$p(V,T,N) = \frac{NT}{V-bN} - a\frac{N^2}{V^2} = \frac{nT}{1-bn} - an^2,$$

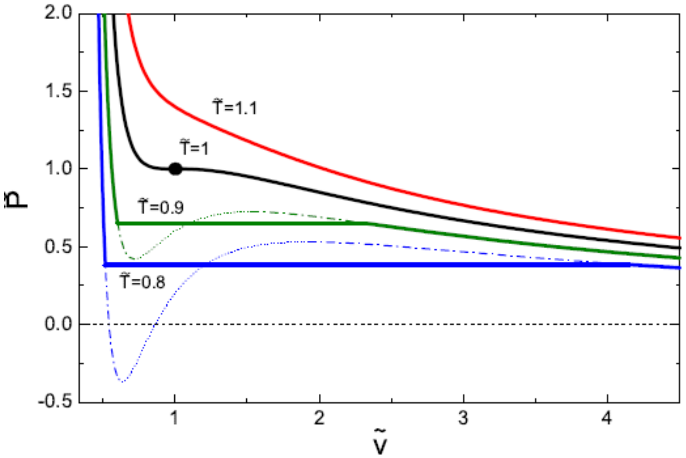
$$\frac{\partial p(T,n)}{\partial n} = 0\,, \qquad \frac{\partial p^2(T,n)}{\partial n^2} = 0$$

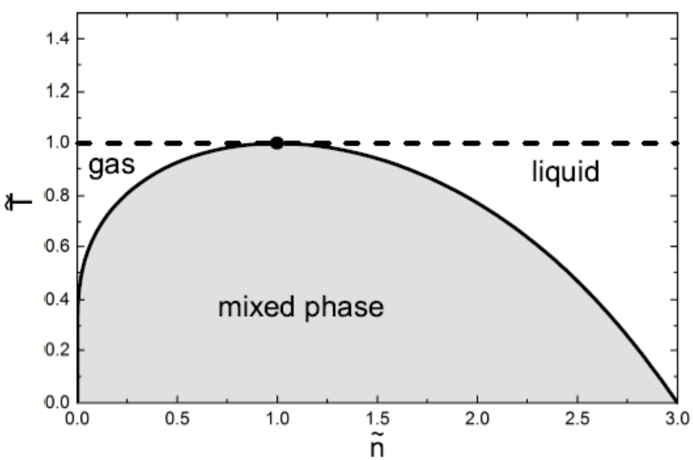
$$T_c = \frac{8a}{27b}, \quad n_c = \frac{1}{3b}, \quad p_c = \frac{a}{27b^2}$$

$$\tilde{n} = n/n_c, \quad \tilde{p} = p/p_c, \quad \tilde{T} = T/T_c,$$

$$\tilde{p} = \frac{8\tilde{T}\tilde{n}}{3-\tilde{n}} - 3\tilde{n}^2$$

$$\tilde{\nu} \equiv 1/\tilde{n}$$





CE GCE

$$p = \frac{nT}{1 - bn} \quad \rightarrow \quad p(T,\mu) = p_{\rm id}(T,\mu^*) ~,$$

$$\mu^* = \mu - bp(T,\mu)$$

Rischke, M.I.G.,Stocker,
W.Greiner,
Z. Phys. C (1991)

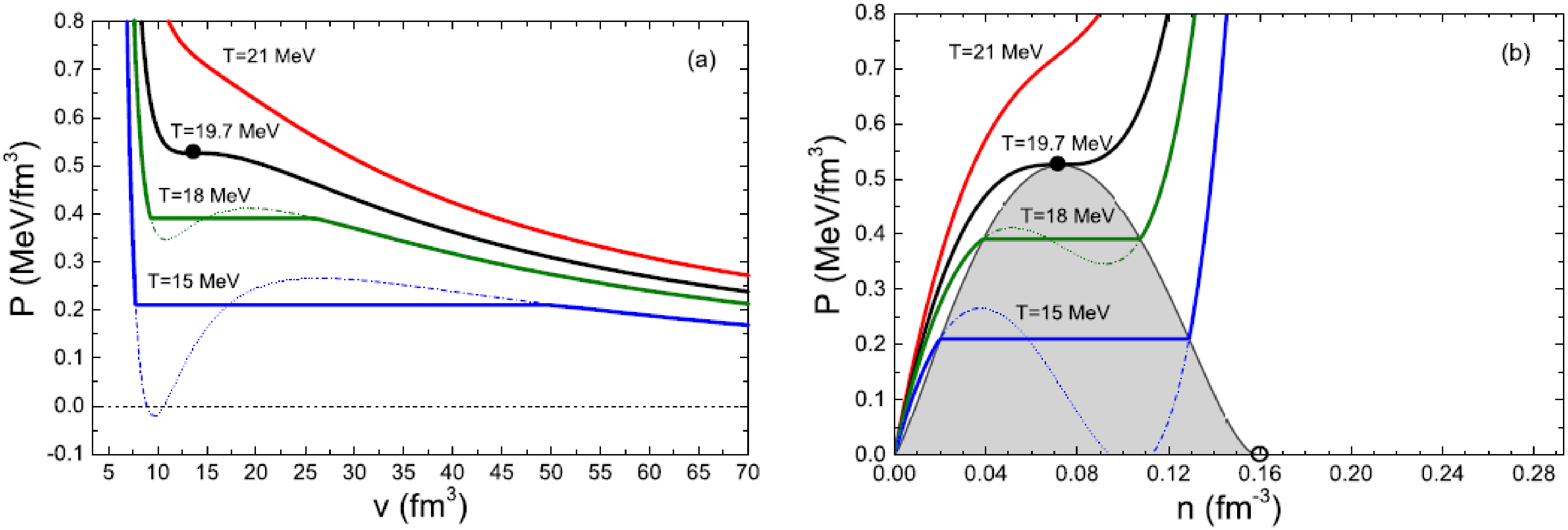
$$p = \frac{nT}{1 - bn} - an^2 \quad \rightarrow \quad p(T,\mu) = p_{\rm id}(T,\mu^*) - abn^2 + 2an$$

$$\mu^* = \mu - bp(T,\mu) - abn^2 + 2an$$

Vovchenko, Anchishkin,
M.I.G.
J.Phys. A (2015)

$$p_{\rm id}(T,\mu) = \frac{g}{6\pi^2}\int_0^\infty k^2 dk \, \frac{k^2}{\sqrt{k^2+m^2}} \left[\exp\left(\frac{\sqrt{k^2+m^2}}{T} - \mu\right) \pm 1\right]^{-1}$$

Nuclear Matter = nucleons with van der Waals EoS



Fermi Statistics, $d = 4, \quad m \cong 938 \text{ MeV} \quad a, b \quad - \quad ?$

$$T = 0, \; p = 0: \quad \varepsilon / n - m \; = \; - \; 16 \text{ MeV}, \quad n = n_0 = \; 0.16 \text{ fm}^{-3}$$

$$a = 329 \text{ MeV fm}^3, \qquad b = 3.42 \text{ fm}^3 \rightarrow r = 0.59 \text{ fm}$$

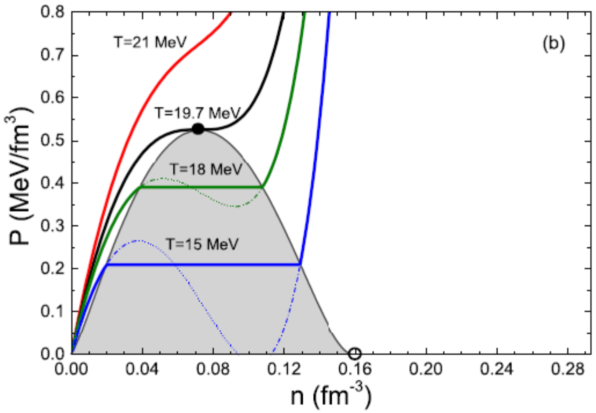
$$T_c \cong 19.7 \text{ MeV},$$

$$n_c \cong 0.07 \text{ fm}^{-3}$$

Vovchenko, Anchishkin, M.I.G., Phys. Rev. C (2015)

III. Critical Point for the Liquid-Gas Transition

Critical Indexes



Order parameter: Non-zero $T < T_c$

and zero at $T > T_c$

$$n_l - n_g \propto (T_c - T)^{\beta} \quad \text{at } T \to T_c - 0, \qquad \beta = \frac{1}{2}$$

liquid gas

Heat Capacity:

$$c_V \propto (T - T_c)^{\alpha} \quad \text{or} \quad (T_c - T)^{\alpha'}$$

$$p = p_c \qquad \alpha = \alpha' = 0$$

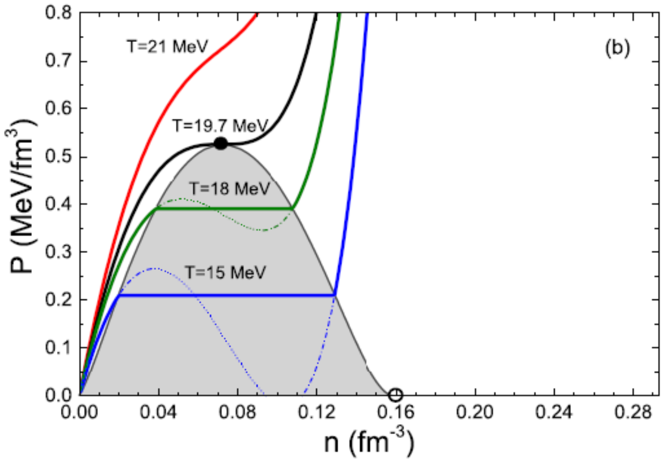
Isothermic Compressibility:

$$\chi_T = \left[n\left(\frac{\partial p}{\partial n}\right) \right]^{-1} \sim \quad (T - T_c)^{-\gamma} \quad \text{or} \quad (T_c - T)^{-\gamma'}$$

$$p = p_c \qquad\qquad \gamma = \gamma' = 1$$

$$p - p_c \sim \mathrm{sgn}(n - n_c)\ |n - n_c|^{\delta}$$

$$T = T_c \qquad\qquad \delta = 3$$



$\alpha' + 2\beta + \gamma' \geq 2$ Rushbooke inequaility Inequalities $\rightarrow$

$\alpha' + \beta(1+\delta) \geq 2$ Griffiths inequality $\rightarrow$ Equalities

"Classical" theories:

van der Waals model for liquid-gas Experiment

$\alpha = \alpha' = 0$ $|T - T_c| / T_c \leq 10^{-2}$ $\alpha = \alpha' = 0 \div 0.2$

$\beta = \frac{1}{2}$ $\Delta T / T_c \leq 10^{-4}$ $\beta = 0.33 \div 0.44$

$\gamma = \gamma' = 1$ $c_V \rightarrow \infty$ (eq. time $\sim$ days) $\gamma = \gamma' = 1.2 \div 1.4$

$\delta = 3$ $\chi_T \rightarrow \infty$ (earth gravitation) $\delta = 4.2 \div 4.4$

IV. Long Range Correlations

2-particle correlation function

$$g_2(r) = f_2(\vec{r}_1 - \vec{r}_2) - f_1(\vec{r}_1)\, f_1(\vec{r}_2), \quad r = |\vec{r}_1 - \vec{r}_2|$$

$g_2(r) \to 0$ at $r \to 0$ because of hard-core repulsion

$g_2(r) \equiv 0$ for the ideal classical (Boltzmann) gas

In general,

$$\lim_{V\to\infty} \frac{<N^2> - <N>^2}{<N>} = 1 + n \int_{V=\infty} d^3r\, g_2(r) = T\left(\frac{\partial n}{\partial p}\right)_T \equiv nT\chi_T$$

$g_2(r) \to 0$ at $r \to \infty$

Correlation Length

$$g_2(r) \sim \frac{1}{r}\exp(-r/\xi),$$

ξ correlation length

$$\xi \sim (T - T_c)^{-\nu}, \qquad \xi \sim (T_c - T)^{-\nu'}$$

$$\nu = \frac{1}{2}\gamma, \qquad \nu' = \frac{1}{2}\gamma', \qquad g_2(r)\big|_{T=T_c} \sim r^{-(d-2+\eta)}$$

$$r \to \infty$$

Scaling Hypothesis: β, γ

$$\tau = \frac{T - T_c}{T_c}, \quad \sigma = n_{\mathrm{l}} - n_{\mathrm{g}}$$

$$p - p_c = \sigma \ \psi\left(\tau, \sigma^{1/\beta}\right),$$

$$\psi\left(\lambda\tau, \lambda\sigma^{1/\beta}\right) = \lambda^{\gamma}\psi\left(\tau, \sigma^{1/\beta}\right)$$

Widom (1965)

Kadanoff (1966)

Wilson(1971,1972)

$$\alpha = \alpha'$$

$$\gamma = \gamma'$$

$$\nu = \nu'$$

$\alpha + 2\beta + \gamma = 2$ Rushbooke (in)equaility

$\alpha + \beta(1 + \delta) = 2$ Griffiths (in)equality

$$2 - \alpha = \nu d$$

$$(2 - \eta)\nu = \gamma$$

Magnetic Systems:

Particle magnetic moment $M \leftrightarrow n$

External magnetic field $H \leftrightarrow p$

Equation of state: $M = M(H,T)$

$$\frac{\partial H}{\partial M} = 0\,, \qquad \frac{\partial^2 H}{\partial M^2} = 0$$

$(T_c\ ,\ H_c\ ,\ M_c = 0)$ Critical Point

V. Fluctuations at the Critical point

Scaled Variance

$$\omega[N] \equiv \frac{< N^2 > - < N >^2}{< N >} = \left[\frac{1}{(1-bn)^2} - \frac{2an}{T} \right]^{-1}$$

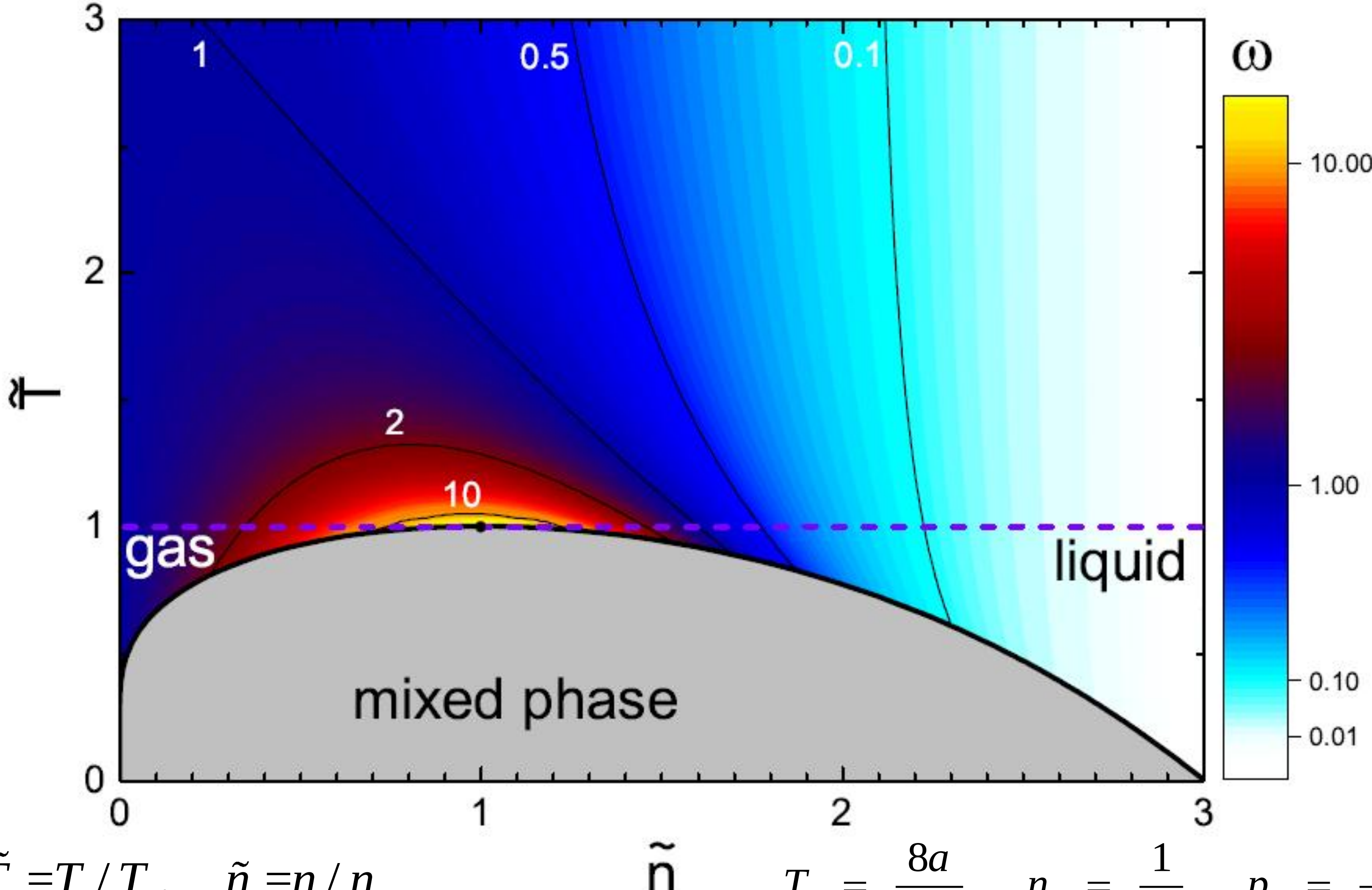
$$a = 0, \quad b = 0 \rightarrow \omega[N] = 1$$

Vovchenko, Anchishkin, M.I.G.
J.Phys. A (2015)

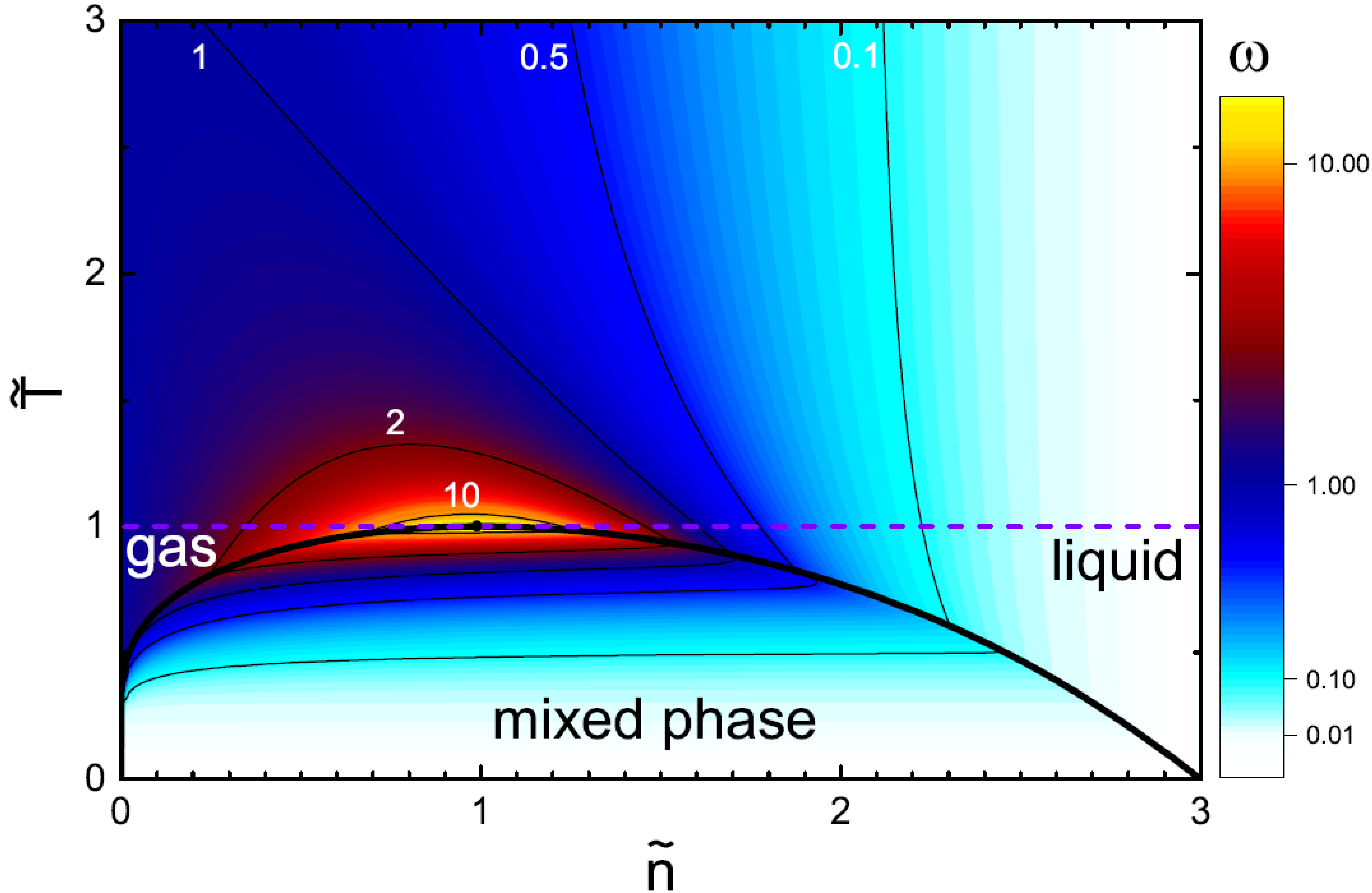
$$a = 0 \rightarrow \omega[N] = (1-bn)^2 < 1$$

$$n \rightarrow 0 \quad \omega[N] = 1$$

$$\omega[N] = \frac{1}{9}\left[\frac{1}{(3-\tilde{n})^2} - \frac{\tilde{n}}{4\tilde{T}}\right]^{-1} \cong \frac{4}{9}\left[\tau + \frac{3}{4}\rho^2 + \tau\rho\right]^{-1}; \quad \begin{array}{l}\tau = \tilde{T} - 1 << 1 \\ \rho = \tilde{n} - 1 << 1\end{array}$$



$$\tilde{T} = T/T_c, \quad \tilde{n} = n/n_c \qquad T_c = \frac{8a}{27b}, \quad n_c = \frac{1}{3b}, \quad p_c = \frac{a}{27b^2}$$



3
2
1
0
1
0.5
0.1
2
10
gas
liquid
mixed phase
0
1
2
3
T̃
ñ
ω
10.00
1.00
0.10
0.01

Skewness and Kurtosis

Central Moments: $<(\Delta N)^2>,\ <(\Delta N)^3>,\ <(\Delta N)^4>,...$

Scaled Variance: $\omega[N] = \dfrac{<(\Delta N)^2>}{<N>}, \qquad \Delta N = N - <N>$

Skewness: $S\sigma = \dfrac{<(\Delta N)^3>}{<(\Delta N)^2>},$

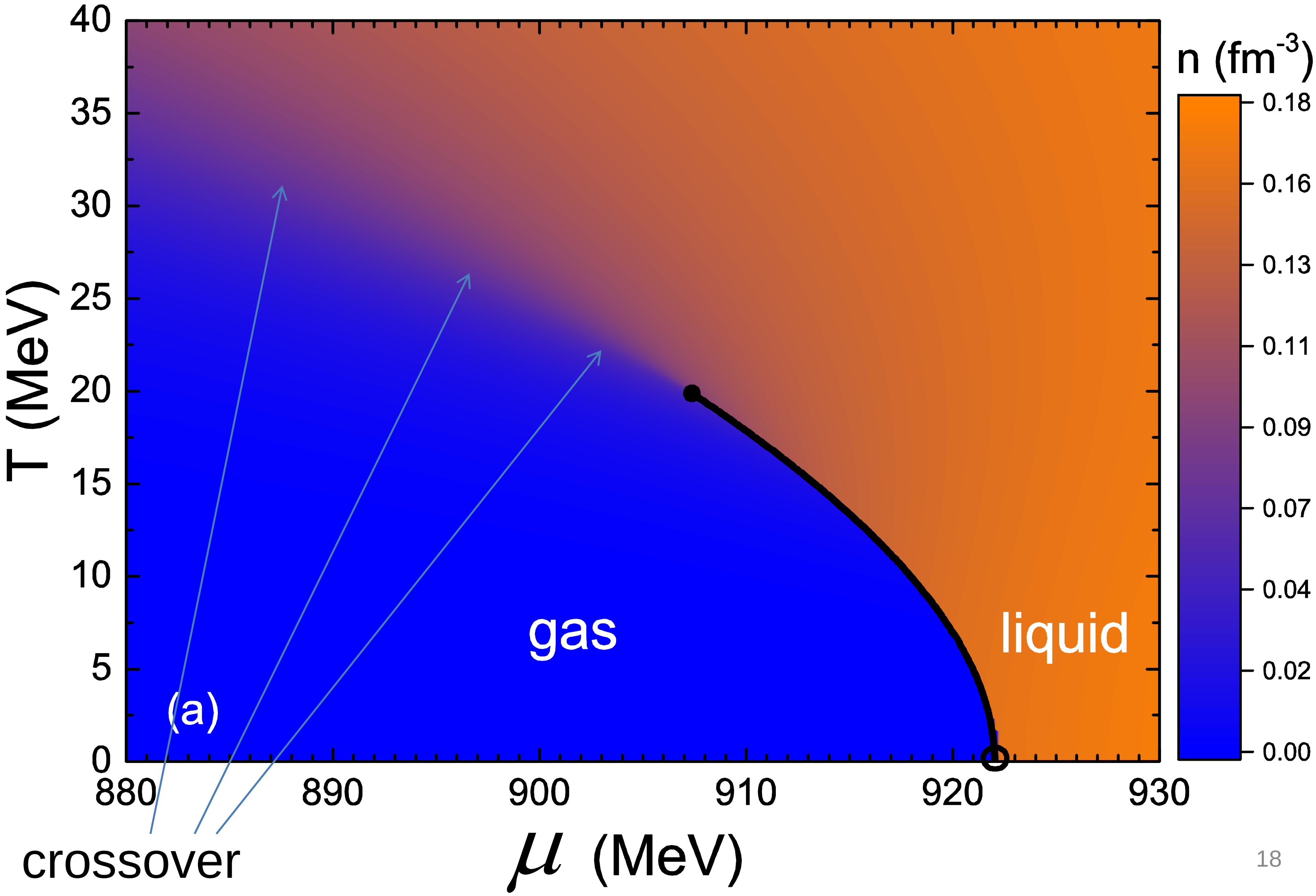
Kurtosis: $\kappa\sigma^2 = \dfrac{<(\Delta N)^4 - 3<(\Delta N)^2>^2}{<(\Delta N)^2>}.$

Cummulants: $\mathrm{k}_n = \dfrac{\partial^n (p/T^4)}{\partial(\mu/T)^n}, \quad n = 1, 2, ...$

$\omega[N] = \dfrac{\mathrm{k}_2}{\mathrm{k}_1}, \quad S\sigma = \dfrac{\mathrm{k}_3}{\mathrm{k}_2}, \quad \kappa\sigma^2 = \dfrac{\mathrm{k}_4}{\mathrm{k}_2}.$

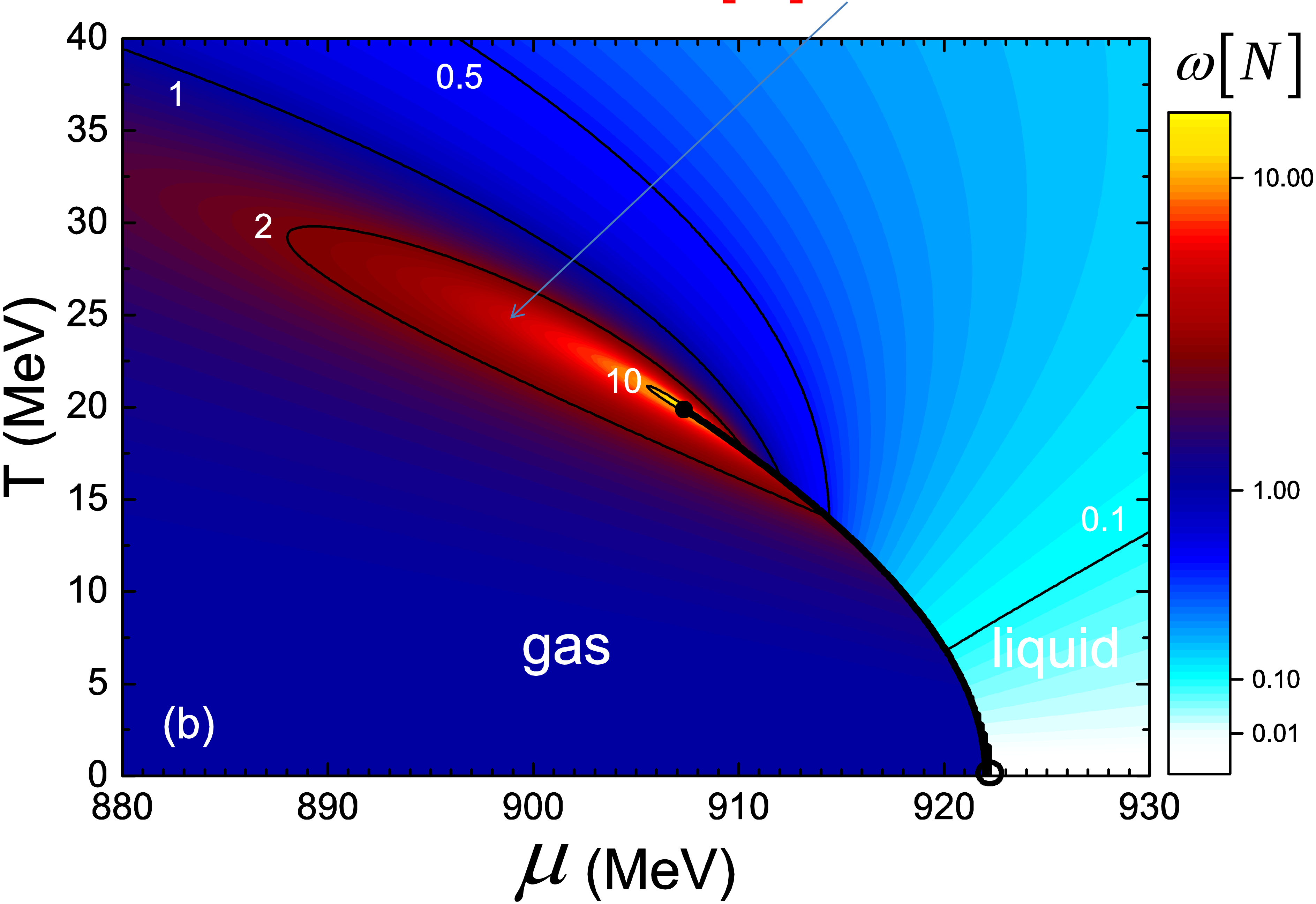
Vovchenko, Anchishkin,
M.I.G., Poberezhnjuk,
Phys. Rev. C (2015)

Particle Number Density $n(T,\mu)$ $\mu_c \cong 908\ \mathrm{MeV}$



Scaled Variance

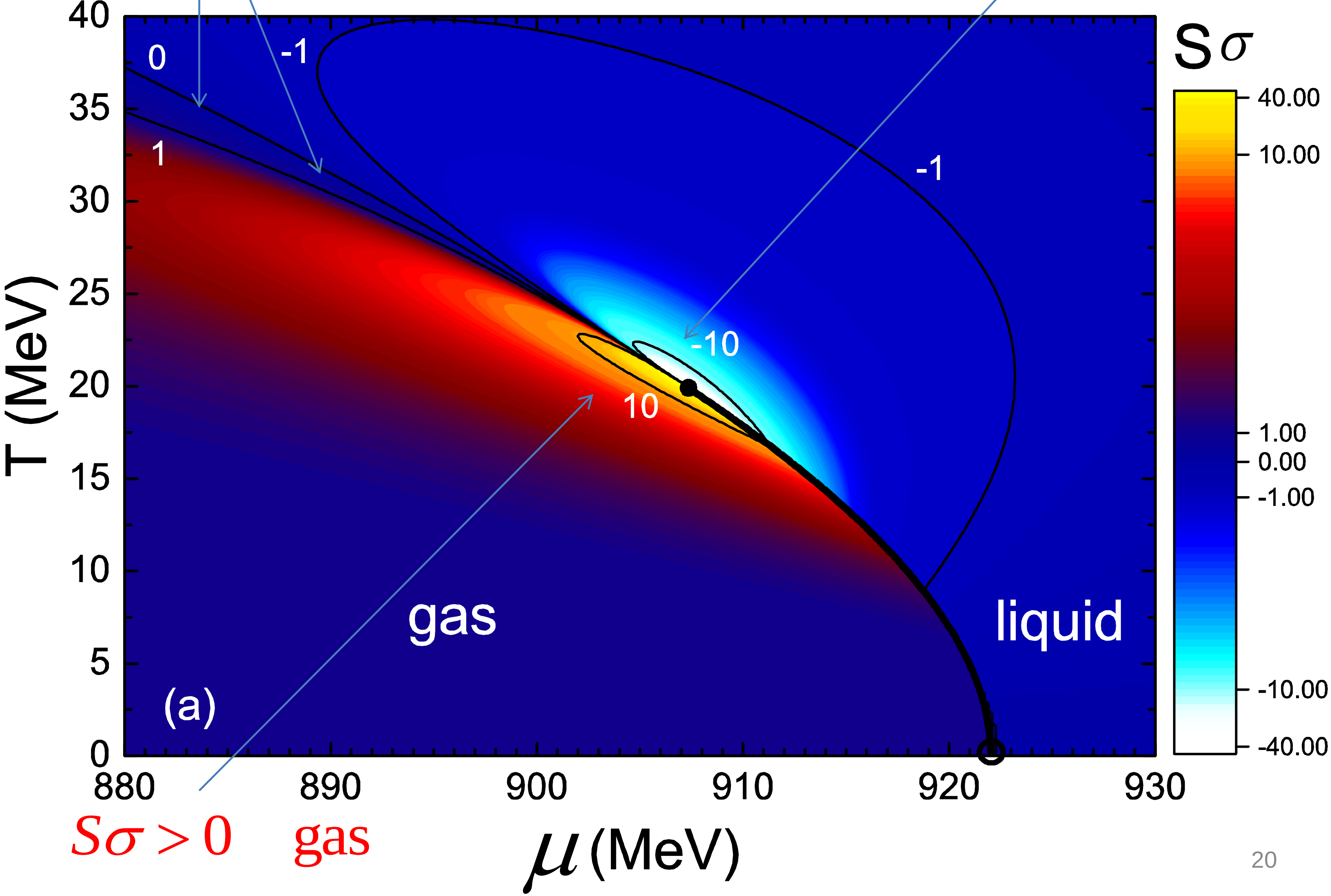
$\omega[N] >> 1$ along crossover



Skewness

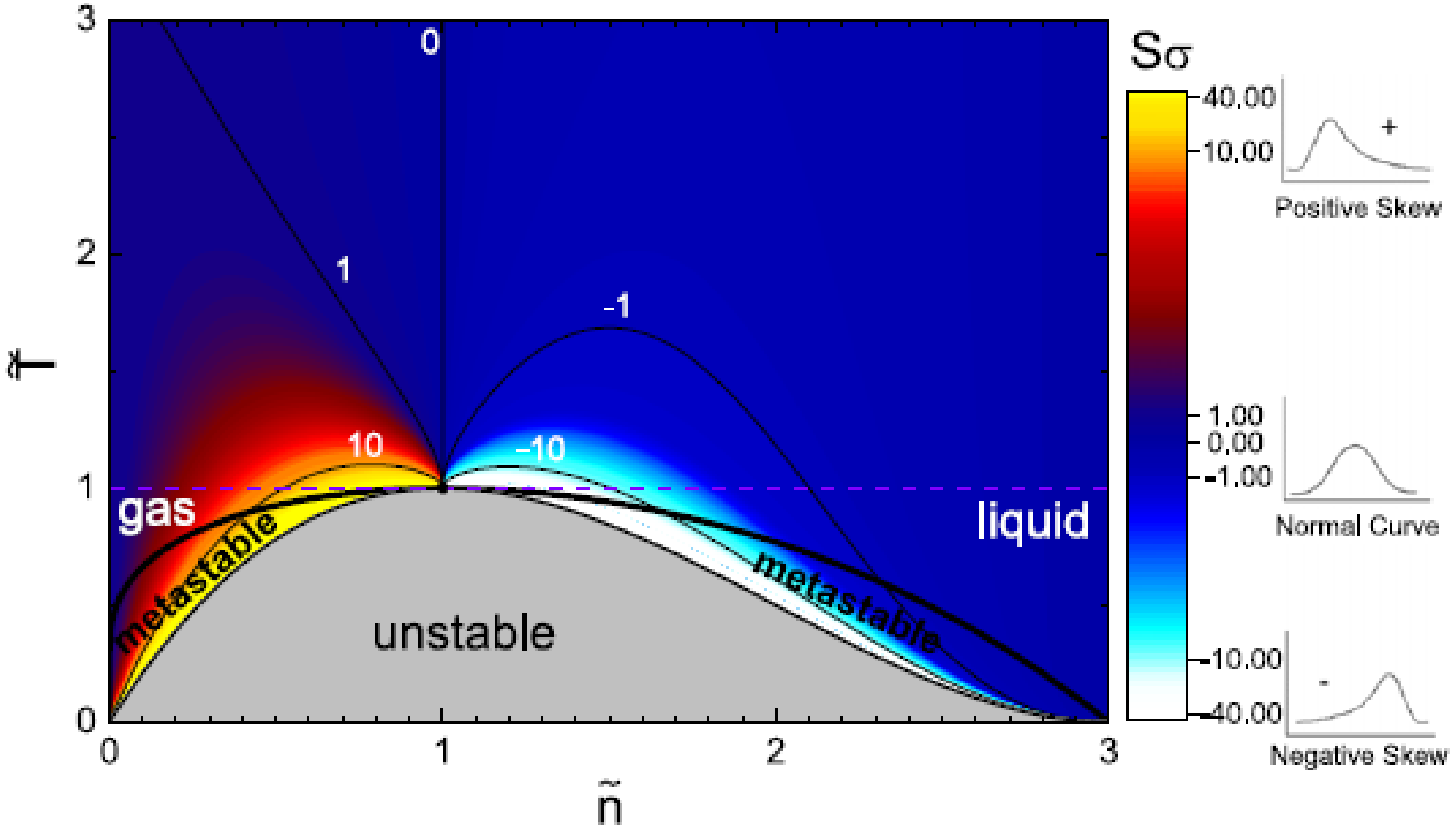
$S\sigma = 0$ crossover

$S\sigma < 0$ liquid



$S\sigma > 0$ gas

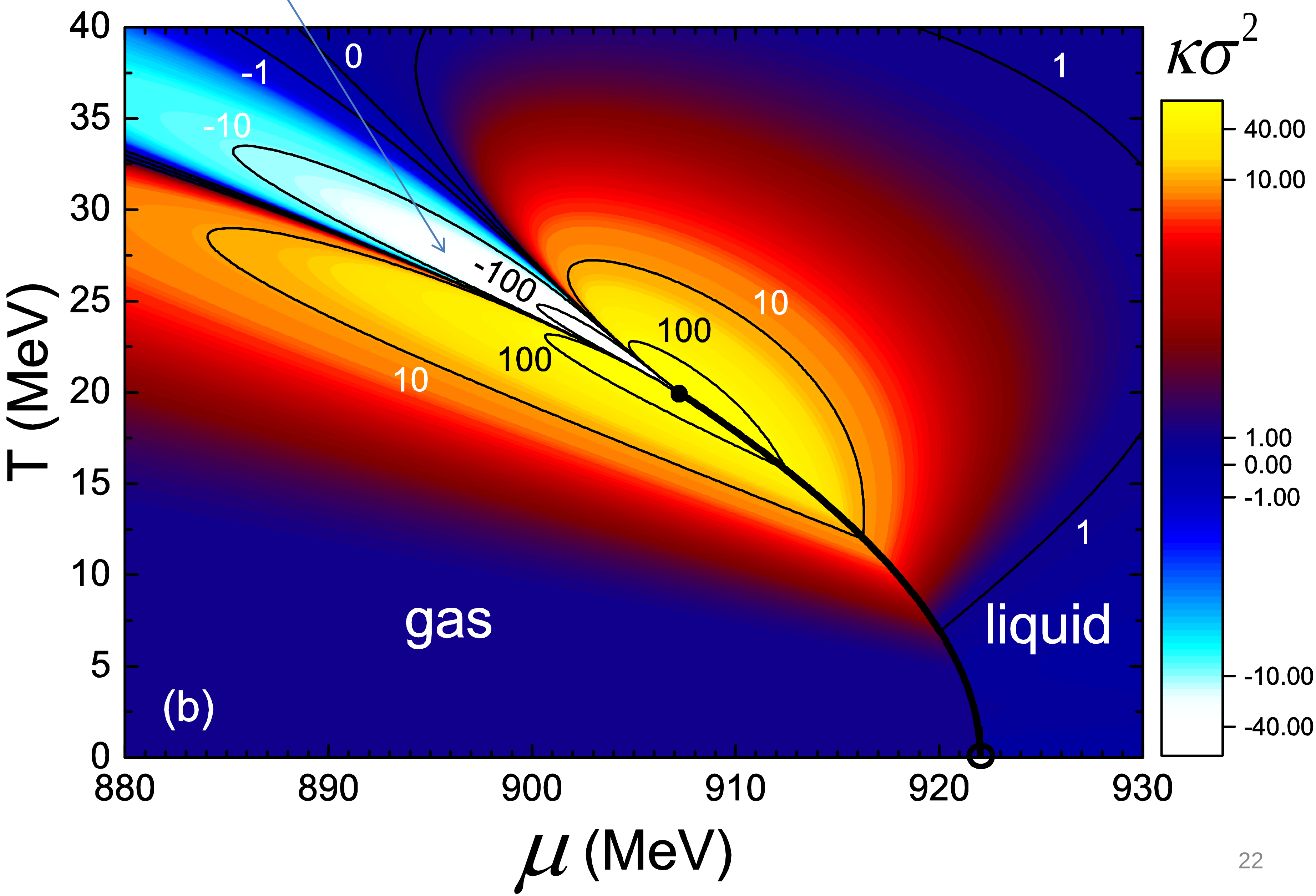
Skewness



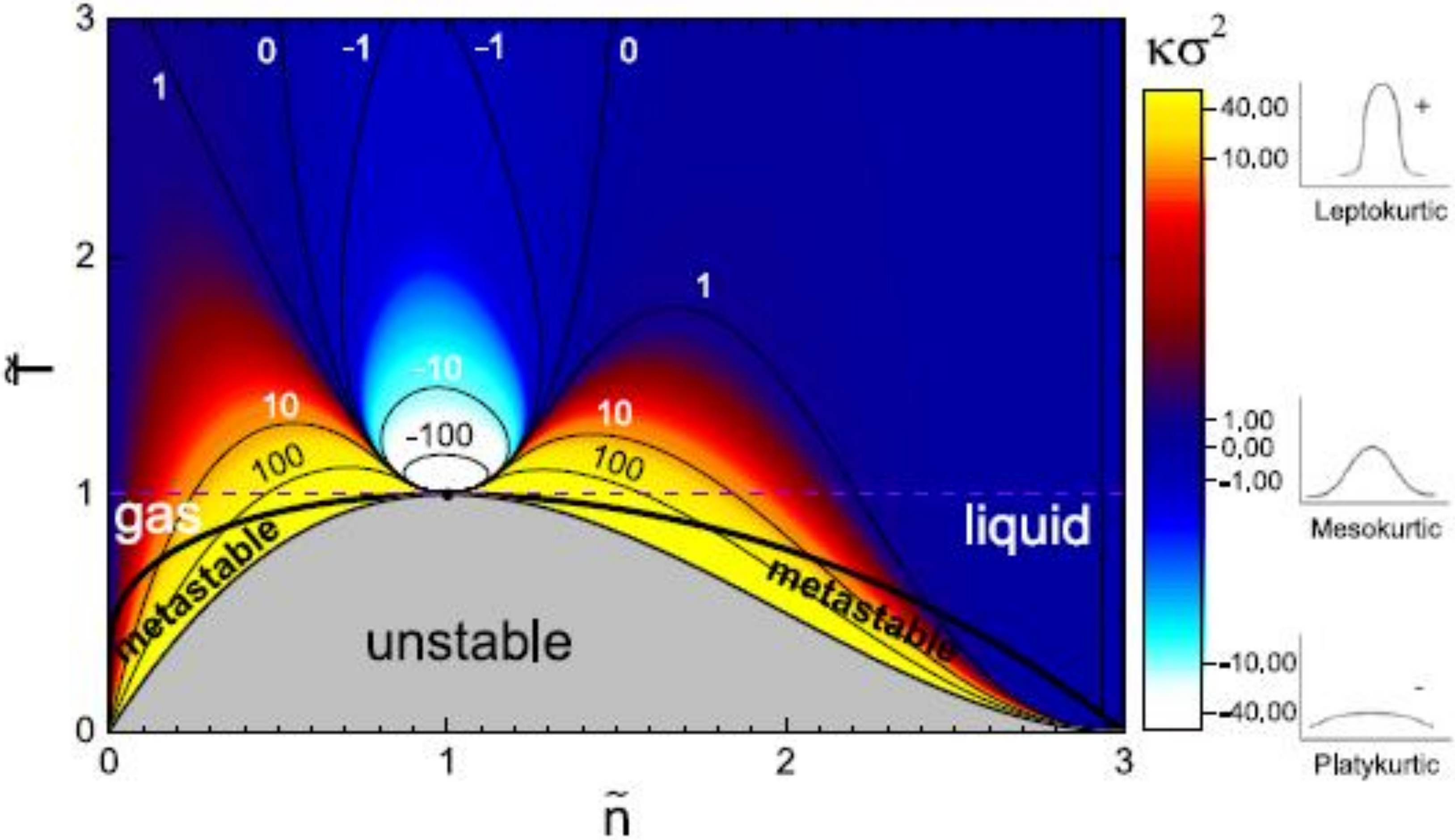
3
2
1
0
$\tilde{T}$
0
1
-1
10
-10
gas
liquid
metastable
metastable
unstable
0
1
2
3
$\tilde{n}$
Sσ
40,00
10,00
1,00
0,00
-1,00
-10,00
-40,00
+
Positive Skew
Normal Curve
-
Negative Skew

Kurtosis

$\kappa\sigma^2 << 0$ crossover



Kurtosis



0
-1
-1
0
1
1
-10
-100
10
100
10
100
gas
liquid
metastable
metastable
unstable
$\tilde{T}$
$\tilde{n}$
$\kappa\sigma^2$
40,00
10,00
1,00
0,00
-1,00
-10,00
-40,00
Leptokurtic
Mesokurtic
Platykurtic

VI. Strongly Intensive Measures of Fluctuations

$$\Delta\left[A, B\right] = \frac{1}{C_{\Delta}} \left[< B > \omega[A] - < A > \omega[B] \right]$$

$$\Sigma\left[A, B\right] = \frac{1}{C_{\Sigma}} [< B > \omega[A] + < A > \omega[B]$$

$$- 2\left(< AB > - < A > < B > \right)]$$

$$< C_{\Delta} >, \; < C_{\Sigma} > \sim < V >$$

These combinations of second moments $< A^2 >, \; < B^2 >, \; < AB >$
are independent of $< V >$ and $\omega\left[V\right]$

M.I.G., Gazdzicki, Phys. Rev. C (2011)

Normalization: For the Independent Particle Model: $\Delta[A, B] = 1$

IB-GCE ; Mixed Event Model $\Sigma[A, B] = 1$

$$C_{\Delta} = C_{\Sigma} = \omega[p_T] < N > \qquad [A = P_T,\ B = N]$$

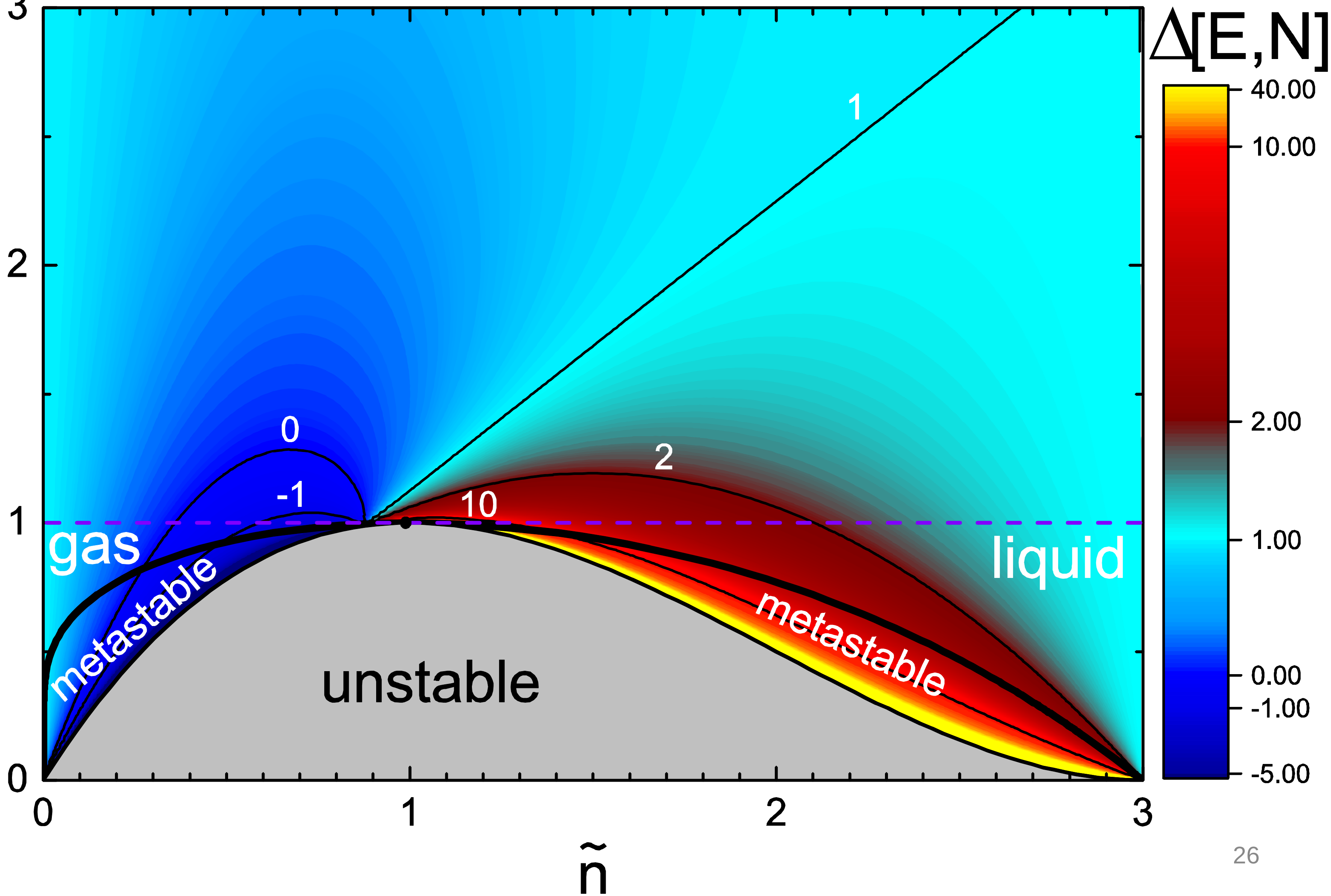
$$C_{\Delta} = < N_1 > - < N_2 >$$

$$C_{\Sigma} = < N_1 > + < N_2 >$$

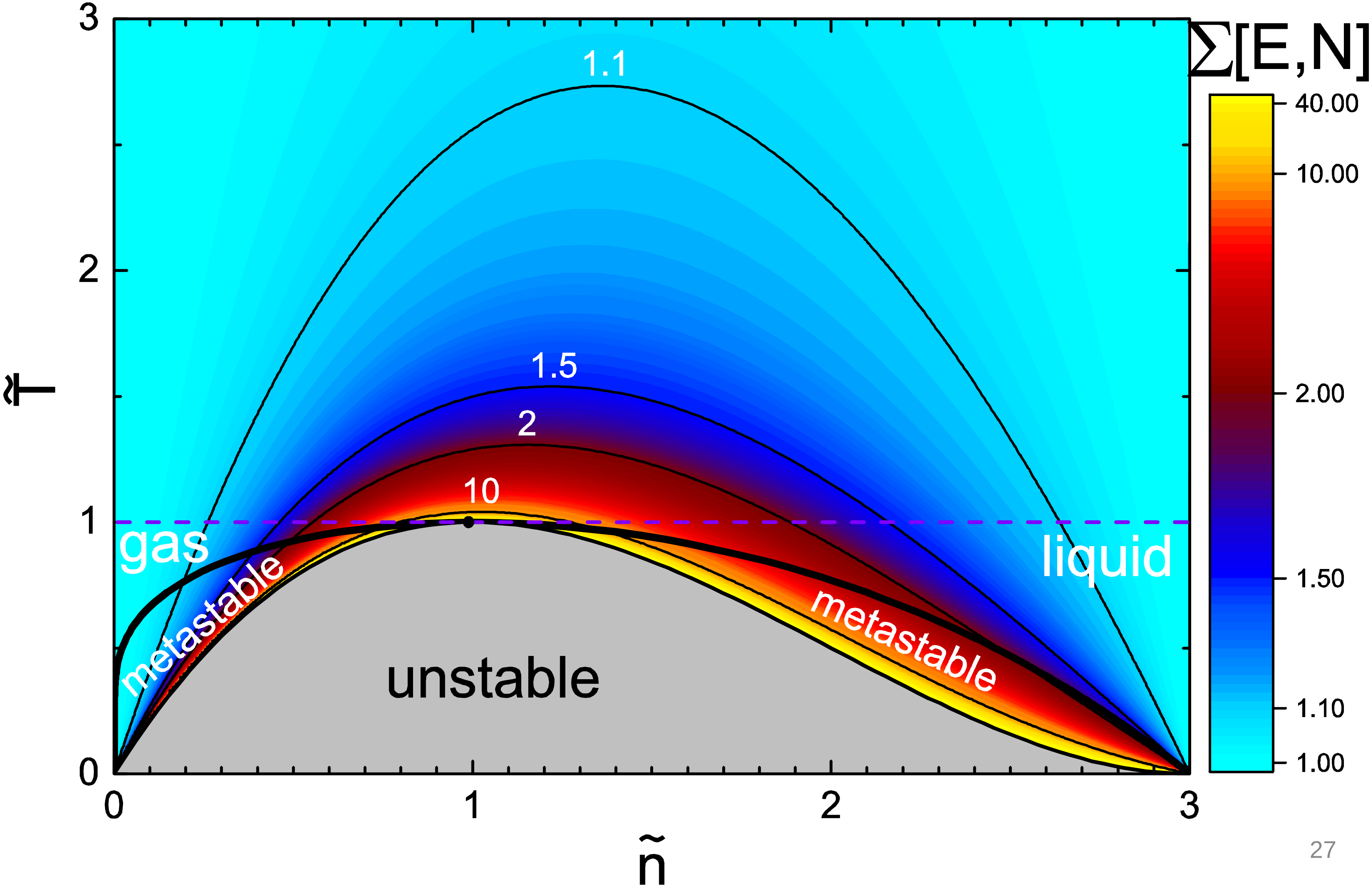
$$[A = N_1,\ B = N_2]$$

Gazdzicki, M.I.G., Mackowiak-Pawlowska, Phys. Rev. C (2013)

$$\Delta[E,N]=1-\frac{2an(T-an)}{T^2}\omega[N]$$



$$\Sigma[E,N] = 1 + \frac{2a^2n^2}{3T^2}\omega[N]$$



Summary

1. Van der Waals Equation of State for Nuclear Matter
   Provides an analytical example of the systems with 1st order liquid-gas phase transition and critical point.

2. Particle Number Fluctuations:
   Scaled Variance increases at a vicinity of the critical point
   (goes to infinity at the CP)
   For Skewness and Kurtosis the CP is a point of essential singularity:
   i.e., the limiting singular values of skewness and kurtosis depend on the path of approach to the CP

3. Strongly Intensive Measures $\Delta[A,B]$ and $\Sigma[A,B]$

$\Delta[A,B]=1$ and $\Sigma[A,B]=1$ if $a=0$, i.e., in the excluded volume model

$\Sigma[A,B]>1$, $\Delta[A,B]>0$ and $\Delta[A,B]<0$

$\Sigma[A,B]\to\infty$ and $\Delta[A,B]\to\infty$ at the CP