Chiral magnetic effect and Berry phase

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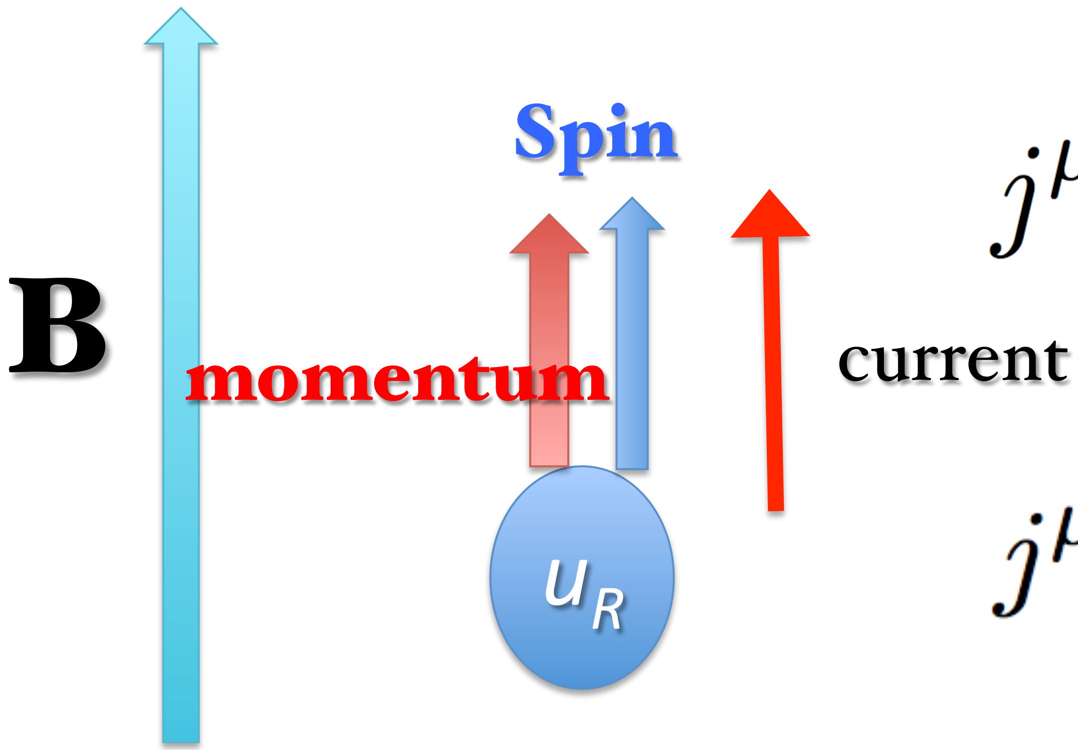
Ref:

- J.H. Gao, Z.T. Liang, SP, Q. Wang, X.N. Wang, PRL 109 (2012) 232301
- J.W. Chen, SP, Q. Wang, X.N. Wang, PRL 110 (2013) 262301

Outline

- Berry phase
- Quantum kinetic theory
- Summary

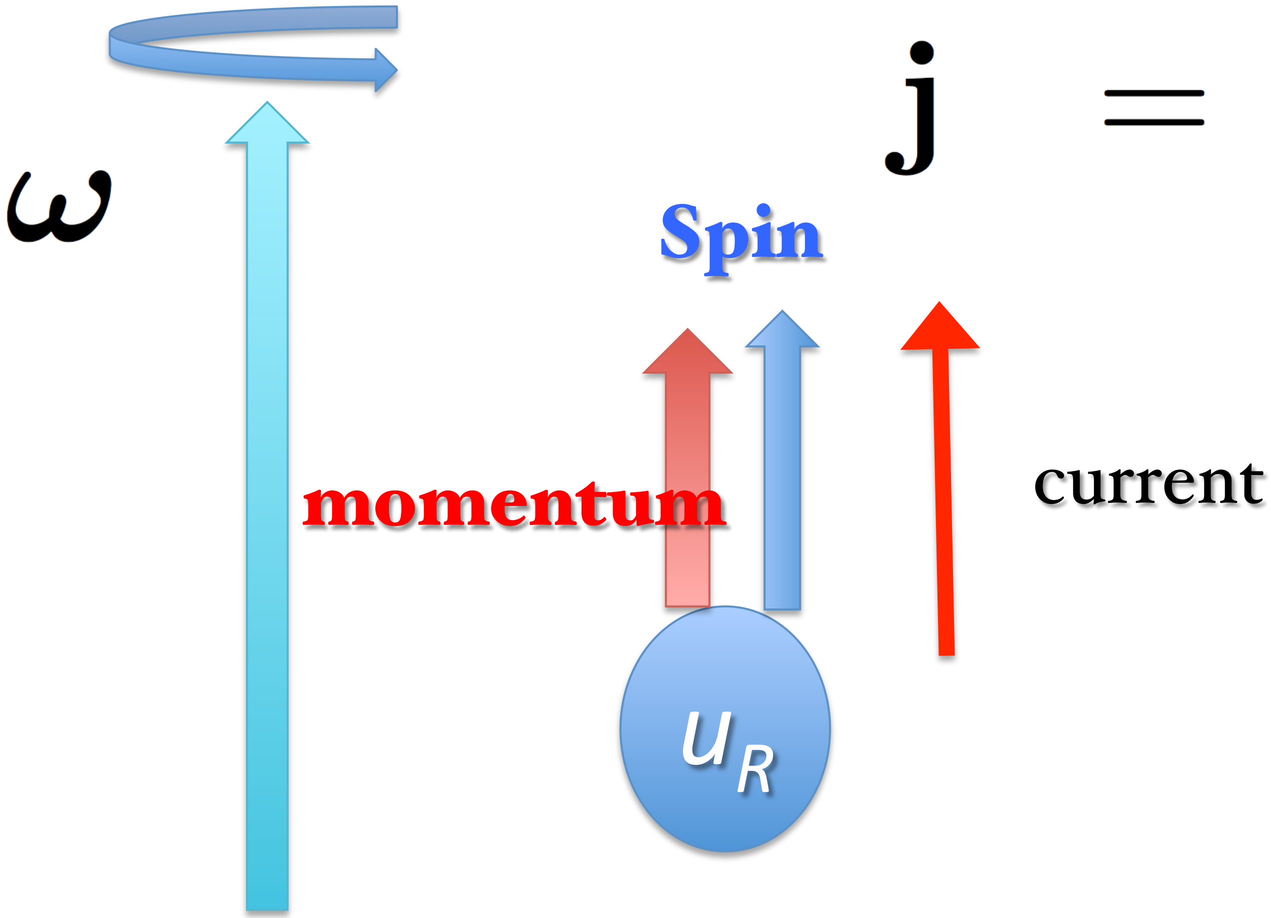
Chiral Magnetic Effect (CME)



$$j^\mu = \xi_B B^\mu,$$

$$j^\mu = \sigma E^\mu,$$

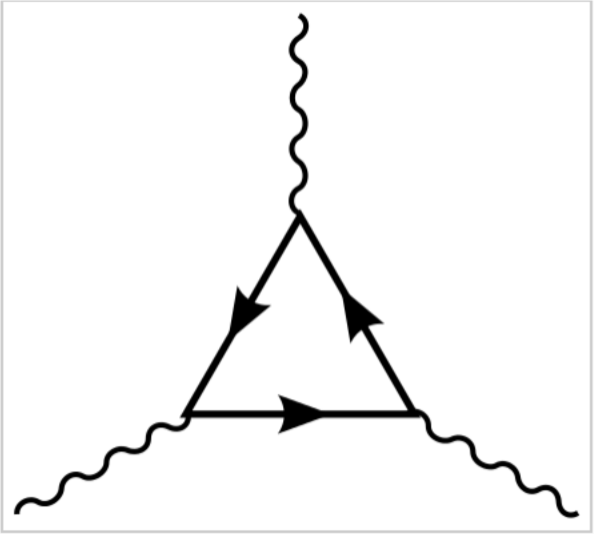
Chiral Vortical Effect (CVE)



$$\mathbf{j} = \xi \boldsymbol{\omega}$$

Chiral anmaly

$$\partial_\mu j_5^\mu = \partial_\mu (j_R^\mu - j_L^\mu) = \frac{Q^2}{2\pi^2}(E \cdot B)$$



Review the talk @ Palaver

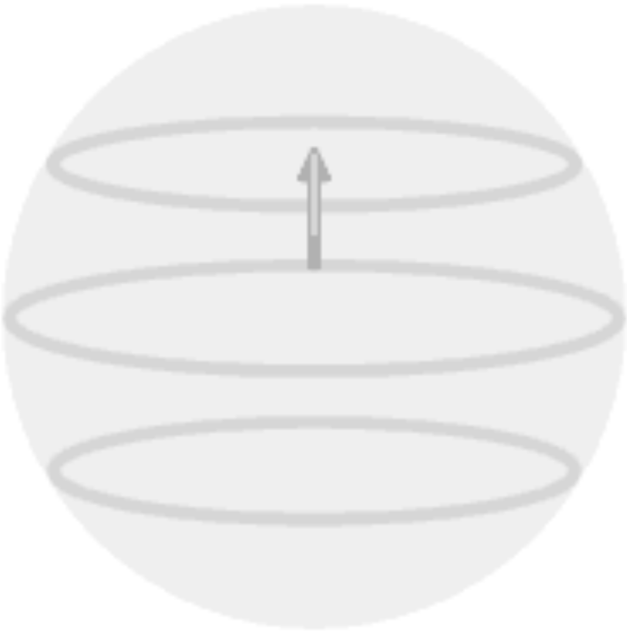
- We have obtained the chiral magnetic and vortical effects, and chiral anomaly by quantum kinetic theory.

- Today, we will give you another interpretation from Berry phase.

Foucault pendulum

Foucault's Pendulum in the Panthéon, Paris

Foucault pendulum

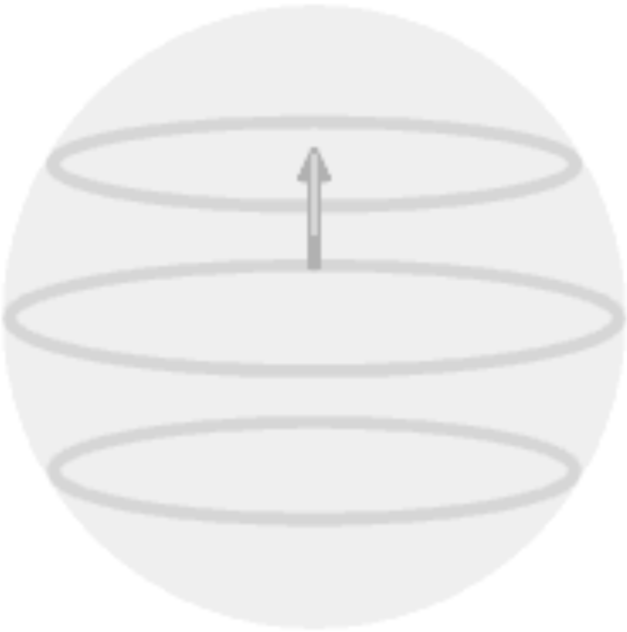


Copy from Wikipedia:

- The animation describes the motion of a Foucault Pendulum at a latitude of 30°N.

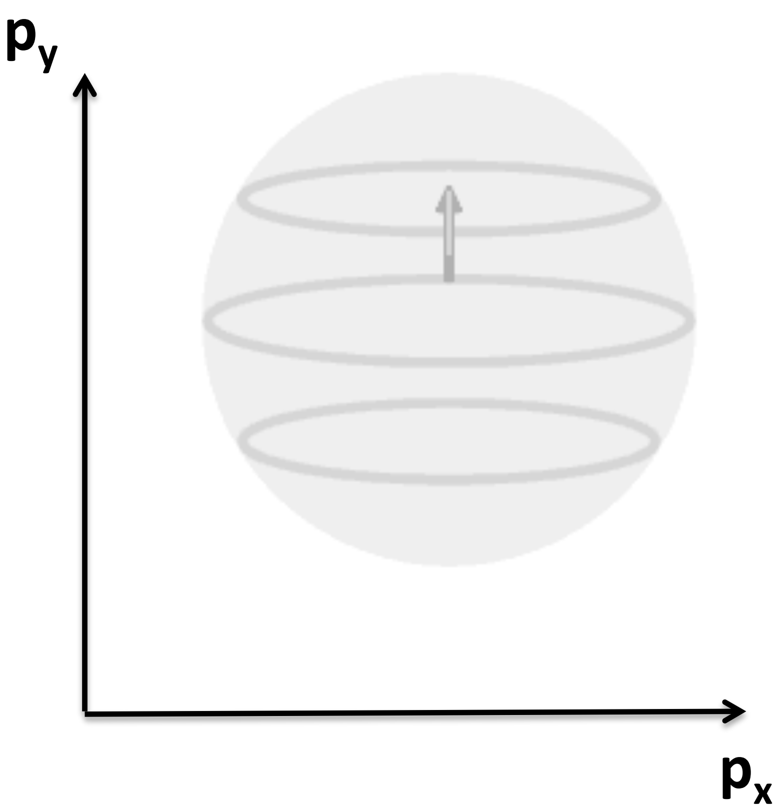
- The plane of oscillation rotates by an angle of −180° during one day, so after two days the plane returns to its original orientation.

Foucault pendulum



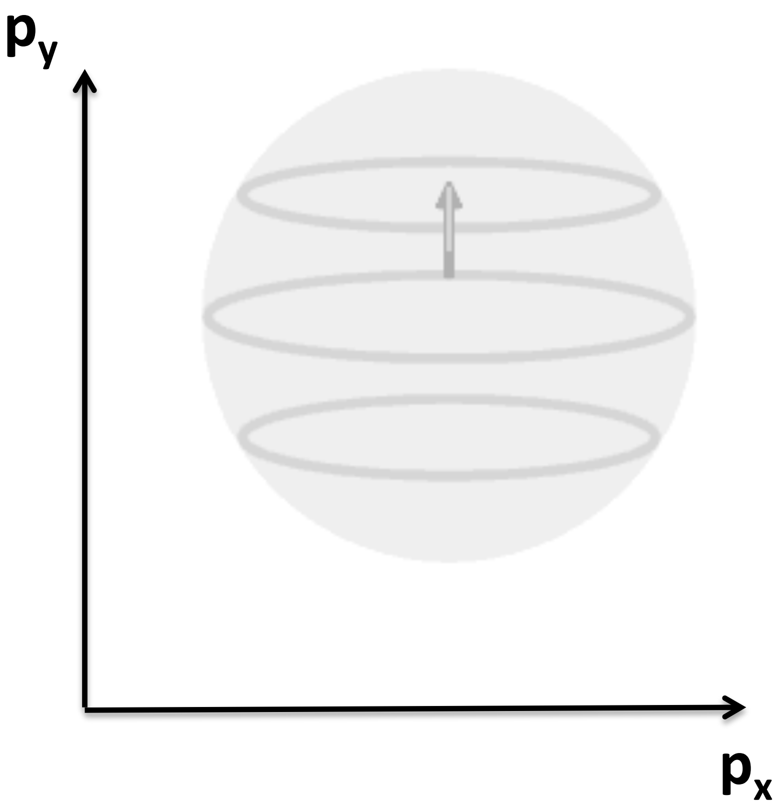
- In our frame (out of the earth), the pendulum goes back to its initial position every day, but with a different phase factor.

Foucault pendulum in momentum space?



- Image that we have an "earth" in momentum space.
- Considering the Foucault pendulum again in momentum space.
- When it goes back to its initial "position" in momentum space.
- Will it also give us an additional phase factor?

A fermion in momentum space?



- Particles are also waves.
- We replace the pendulum by a fermion.
- After some time, it goes back to its initial state.
- Will it give us an additional phase factor in momentum space?

Action of a single particle

dot: time derivative

$$S = \int dt \left[(\mathbf{p} + e\mathbf{A}) \cdot \dot{\mathbf{x}} - |\mathbf{p}| - A^0 \right]$$

momentum

gauge field

x: coordinate

A natural question:

Why cannot we have a term proportional to $\dot{\mathbf{p}}$?

Action of a single particle

$$S = \int dt \left[(\mathbf{p} + e\mathbf{A}) \cdot \dot{\mathbf{x}} - |\mathbf{p}| - A^0 \right]$$

dot: time derivative

momentum

gauge field

x: coordinate

The gauge field A is coupled to $\dot{\mathbf{x}}$.

Why cannot we have a “gauge field” coupled to $\dot{\mathbf{p}}$?

Two questions:

- Will the fermion have a phase factor in momentum space?

- Why cannot we have a gauge field in momentum space?

Berry phase – gauge fields in momentum space?

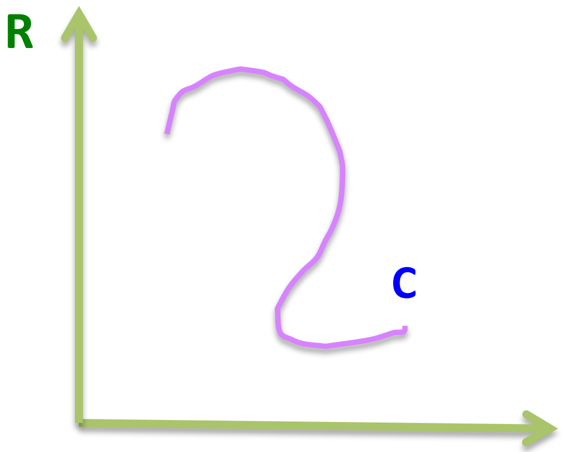
- In quantum mechanics, phase factor is related to gauge field.
- A phase factor in momentum space = an effective gauge field in momentum space.
- The new phase factor is called Berry phase.

Adiabatic processes

- Assuming the system is in an adiabatic evolution, and Hamiltonian is a function of parameters R(t) .

$$H = H(R(t)).$$

- Then the evolution of the system is along a path C in parameter R space.



- At each certain time, the system is at its eigenstate.

$$H(R)|n(R)> = E_n(R)|n(R)> .$$

Additional phase factor

- The wave function is given by,

$$|\psi_n(t)>= e^{i\gamma_n} \exp\left[-i\int_{t_0}^{t} dt' E_n(R(t'))\right] |n(R(t))>,$$

additional phase factor

Normal evolution factor

Eigenstate

which allows an additional phase factor.

Solving phase factor

- Inserting the wavefunction into Schroding Eq,

$$i\partial_t|\psi_n(t)>= H(R)|\psi_n(t)>,$$

we get the phase factor, $e^{i\gamma_n}$

$$\gamma_n = i\int_{t_0}^{t} dt' < n(R)|\partial_{t'}|n(R)> \equiv \int_C d\mathbf{R}\cdot\mathbf{a}_R,$$

$$\mathbf{a}_R = i < n(R)|\frac{\partial}{\partial\mathbf{R}}|n(R)>,$$

Gauge Transform

- Actually, we can still add another phase factor $\exp\left(i\xi(R)\right)$ to the wave function, then,

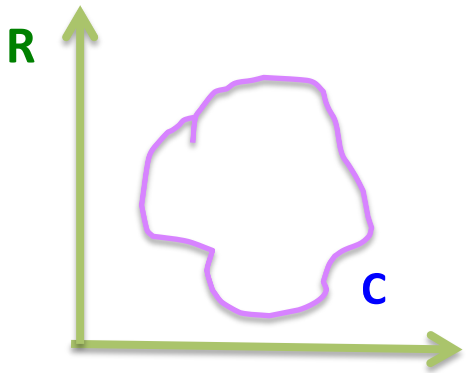
$$\mathbf{a}_R \to \mathbf{a}_R - \frac{\partial}{\partial \mathbf{R}}\xi(R),$$

$$\gamma_n \to \gamma_n - \xi[R(t_0)] + \xi[R(t)].$$

- By choosing suitable values, the total phase factor can be “gauged” away!

A loop: Gauge invariant, Berry phase

- However, *Berry ('84)* has pointed out, if we consider a loop in R-space, i.e. the system goes back to its initial state, then, the phase factor cannot be removed.



Berry phase

$$\gamma = \oint_C d\mathbf{R} \cdot \mathbf{a}_R.$$

a_R : Berry connection

- In 3 dimensional case, using the Stokes's theorem,

$$\gamma = \int d\mathbf{S} \cdot \mathbf{\Omega}_R. \qquad \mathbf{\Omega}_R = \nabla_R \times \mathbf{a}_R,$$

Berry curvature

Berry phase VS gauge theory

Let's set R to be momentum p from now on.

Gauge theory	Berry "things"
local at x space	at p space
gauge field $\overrightarrow{A}$	Berry connection $\overrightarrow{a_p}$
magnetic field $\overrightarrow{B} = \nabla \times \overrightarrow{A}$	Berry curvature $\overrightarrow{\Omega_p} = \nabla_p \times \overrightarrow{a_p}$
Aharonov–Bohm phase $\int_V d\overrightarrow{x} \cdot \overrightarrow{A} = \iint_S d\overrightarrow{S} \cdot \overrightarrow{B}$	Berry phase $\int_V d\overrightarrow{p} \cdot \overrightarrow{a_p} = \iint_S d\overrightarrow{S_p} \cdot \overrightarrow{\Omega_p}$
Dirac monopole (magnetic charge) $\int d^3x \nabla \cdot \overrightarrow{B} = const.$	Berry monople $\int d^3p \nabla_p \cdot \Omega_p = const.$

An example: massless fermions

- Choosing helicity basis, we can get, for right handed fermions,

Berry connection:

(gauge field)

$$\mathbf{a}_p = -\frac{1}{2|\mathbf{p}|}\mathbf{e}_\phi \cot\frac{\theta}{2}.$$

Berry curvature:

(magnetic field)

$$\boldsymbol{\Omega}_p = \frac{\mathbf{p}}{2|\mathbf{p}|^3}.$$

Berry magnetic monopole:

(magnetic monopole)

$$\nabla_p \cdot \boldsymbol{\Omega}_p = 2\pi\delta^3(\mathbf{p}).$$

Path integration of a massless fermion

- Consider a right/left-handed massless fermion,

$$H = i\gamma \cdot \partial \to \gamma \cdot p = \begin{pmatrix} \sigma \cdot p & 0 \\ 0 & -\sigma \cdot p \end{pmatrix}$$

$$< f|\exp\left[-iH(t_f - t_i)\right]|i> \quad = \quad \int DxDp ... < p_{i+1}|\exp[-iH\Delta t]|p_i >$$

Path integration of a massless fermion

$$< f|\exp\left[-iH(t_f - t_i)\right]|i> \;=\; \int DxDp... < p_{i+1}|\exp[-iH\Delta t]|p_i >$$

$$\begin{aligned}
< p_{i+1}|\exp[-iH\Delta t]|p_i > &= e^{-iE_i\Delta t} < p_{i+1}|p_i > \\
&= e^{-iE_i\Delta t} < p_i + \Delta p_i|p_i > \\
&= e^{-iE_i\Delta t}\,[1+ < p_i|\partial_p|p_i > \Delta p_i] \\
&= \exp\left[-iE_i\Delta t - ia_p\Delta p\right] + O(\Delta t^2),
\end{aligned}$$

$$a_p \equiv i < p_i|\partial_p|p_i >$$

Berry connection

Action with Berry connection

- Without external fields, Berry phase is still a phase factor, which cannot modify the evolution of the system.

- In the present of electromagnetic fields, the action of a single massless fermion becomes,

dot: time derivative

gauge field

$$S = \int dt \left[(\mathbf{p} + Q\mathbf{A}) \cdot \dot{\mathbf{x}} - |\mathbf{p}| - A^0 - \mathbf{a}_p \cdot \dot{\mathbf{p}} \right]$$

momentum

x: coordinate

Stephanov&Yin('13), Yamamoto&Son('13),JW Chen, JY Pang, SP, Q Wang ('14)

Equation of motion

$$\sqrt{\gamma}\dot{\mathbf{x}} = \frac{\mathbf{p}}{|\mathbf{p}|} + Q\mathbf{E}\times\mathbf{\Omega}_p + Q\mathbf{B}\left(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\mathbf{\Omega}_p\right),$$

$$\sqrt{\gamma}\dot{\mathbf{p}} = Q\mathbf{E} + Q\frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{B} + Q^2\mathbf{\Omega}_p\left(\mathbf{E}\cdot\mathbf{B}\right),$$

$$\sqrt{\gamma} = 1 + Q\mathbf{B}\cdot\mathbf{\Omega}_p.$$

Berry curvature

The invariant volume of phase space becomes

$$dV = \sqrt{\gamma}dxdp$$

Liouville's theorem

- From df/dt=0, we can get a kinetic theory,

$$\partial_t f + \dot{\mathbf{x}} \cdot \nabla_x f + \dot{\mathbf{p}} \cdot \nabla_p f = 0,$$

with f the distribution function for right/left handed fermions,

$$\begin{aligned} \sqrt{\gamma}\dot{\mathbf{x}} &= \frac{\mathbf{p}}{|\mathbf{p}|} + Q\mathbf{E} \times \mathbf{\Omega}_p + Q\mathbf{B}\left(\frac{\mathbf{p}}{|\mathbf{p}|} \cdot \mathbf{\Omega}_p\right), \\ \sqrt{\gamma}\dot{\mathbf{p}} &= Q\mathbf{E} + Q\frac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{B} + Q^2\mathbf{\Omega}_p\left(\mathbf{E} \cdot \mathbf{B}\right), \end{aligned}$$

Anomalous current

- Integrating over d^3p, we get the divergence of a chiral current

$$\partial_t n_5 + \nabla \cdot \mathbf{j}_5 = \frac{Q^2}{2\pi^2}(\mathbf{E}\cdot\mathbf{B})\,.$$ chiral anomaly

and, chiral magnetic effects,

$$\mathbf{j} = \frac{Q}{2\pi^2}\mu_5\mathbf{B}. \qquad \mathbf{j}_5 = \frac{Q}{2\pi^2}\mu\mathbf{B}.$$

Stephanov&Yin(PRL13), Yamamoto&Son(PRL,PRD13)

What can we learn?

- Berry phase gives an effective gauge field in momentum space.
- Berry curvature (magnetic field of such gauge field) will couple to external fields and modify the equation of motion for single fermions.
- The kinetic theory will also be modified and give the chiral magnetic effects and chiral anomaly.

You might say,

“mmm… sounds interesting, but useless …”

or,

“We are working on a relativistic theory. Who cares a XXX phase factor in quantum mechanics!”

Or, “Can you get it from Dirac equations or other relativistic theory?”

Motivation of our work

- We want to understand (or obtain) the contributions of Berry phase from a quantum kinetic theory.

Wigner function for fermions

- Wigner function: a quantum distribution function, ensemble average, normal ordering

Vasak, Gyulassy and Elze ('86,'87,'89)

$$W(x,p) =<: \int \frac{d^4y}{(2\pi)^4} e^{-ipy} \overline{\psi}(x+\frac{1}{2}y) \otimes \mathcal{P}U(x,y) \psi(x-\frac{1}{2}y) :>$$

Gauge link

$\overline{\psi}(x+\frac{1}{2}y)$ **X** $\psi(x-\frac{1}{2}y)$

Macroscopic quantities

Charge current

$$j^\mu(x) \equiv <: \overline{\psi}(x)\gamma^\mu\psi(x) :>= \int d^4p \mathrm{Tr}\ (\gamma^\mu W),$$

Axial (chiral) current

$$j_5^\mu(x) \equiv <: \overline{\psi}(x)\gamma^5\gamma^\mu\psi(x) :>= \int d^4p \mathrm{Tr}\ (\gamma^5\gamma^\mu W),$$

Master equation from Dirac Eq.

- Massless, constant external electromagnetic fields $F_{ext}^{\mu\nu}$, turn off all internal interactions

$$[\gamma^\mu p_\mu + \frac{1}{2} i \ \gamma^\mu (\partial^x_\mu - Q F^{ext}_{\mu\nu} \partial^p_\mu)] W = 0,$$

- First order differential equation, solve it order by order

Solve the Master equation

- Gradient expansion to Wigner function W and its master equation,
  - expand all quantities at the power of derivatives

    $O(\partial_x^1), O(\partial_x^2),$
  - external fields are weak $F^{\mu\nu} \sim \partial_x^\mu A^\nu \sim O(\partial^1),$

Leading order

- 0^{th} order, non-interacting ideal gas
  - classical Fermi-Dirac distribution

- input
  - finite temperature T,
  - chemical potential $\mu = \mu_R + \mu_L$,
  - chiral chemical potential $\mu_5 = \mu_R - \mu_L$

1st order, Chiral anomaly

- Remarkable, we obtain the chiral anomaly by Wigner function!

Energy momentum conservation

$$\partial_\mu T^{\mu\nu} = QF^{\nu\rho} j_\rho,$$

$$\partial_\mu j^\mu \ = \ 0,$$

Triangle anomaly

$$\partial_\mu j_5^\mu \ = \boxed{-\frac{Q^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}}$$

$$\propto E \cdot B$$

Chiral magnetic and vortical effect

$$j^\mu = \xi_B B^\mu + \xi \omega^\mu,$$
$$j_5^\mu = \xi_{5B} B^\mu + \xi_5 \omega^\mu,$$

Consistent with other approaches!

$$\xi = \frac{1}{\pi^2}\mu\mu_5,$$
$$\xi_B = \frac{Q}{2\pi^2}\mu_5,$$
$$\xi_5 = \frac{1}{6}T^2 + \frac{1}{2\pi^2}\left(\mu^2+\mu_5^2\right),$$
$$\xi_{B5} = \frac{Q}{2\pi^2}\mu.$$

Q: charge

T: temperature

Chemical potentials

$$\mu = \mu_R + \mu_L,$$
$$\mu_5 = \mu_R - \mu_L,$$

4D kinetic equation for massless fermions

- By using the solutions of Wigner function, we rewrite our master equations in a proper way,

$$\left[\frac{dx^\mu}{d\tau}\partial^x_\mu + \frac{dp^\mu}{d\tau}\partial^p_\mu\right] f_{R/L} = 0,$$

f: distribution function

Effective velocity

$$\frac{dx^\sigma}{d\tau} = p^\sigma \pm Q\left[(u\cdot b)B^\sigma - (b\cdot B)u^\sigma + \epsilon^{\sigma\alpha\beta\gamma}u_\alpha b_\beta E_\gamma\right] \pm \left[\frac{1}{2}\omega^\sigma + \omega^\sigma(p\cdot u)(b\cdot u) - u^\sigma(p\cdot\omega)(b\cdot u)\right],$$

Effective force

$$\frac{dp^\sigma}{d\tau} = -Qp_\rho F^{\rho\sigma} \mp Q^2(E\cdot B)b^\sigma \pm Q\frac{1}{2}(\omega\cdot E)u^\sigma \mp Q(p\cdot\omega)b_\eta F^{\sigma\eta}.$$

3D chiral kinetic equations

- Integrating over dp^0,

$$\frac{dt}{d\tau}\partial_t f_{R/L} + \frac{d\mathbf{x}}{d\tau}\cdot\nabla_{\mathbf{x}} f_{R/L} + \frac{d\mathbf{p}}{d\tau}\cdot\nabla_{\mathbf{p}} f_{R/L} = 0,$$

$$\frac{dt}{d\tau} = 1 \pm Q\boldsymbol{\Omega}\cdot\mathbf{B} \pm 4|\mathbf{p}|(\boldsymbol{\Omega}\cdot\boldsymbol{\omega}),$$

$$\frac{d\mathbf{x}}{d\tau} = \hat{\mathbf{p}} \pm Q(\hat{\mathbf{p}}\cdot\boldsymbol{\Omega})\mathbf{B} \pm Q(\mathbf{E}\times\boldsymbol{\Omega})$$

$$\frac{d\mathbf{p}}{d\tau} = Q(\mathbf{E} + \hat{\mathbf{p}}\times\mathbf{B}) \pm Q^2(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}$$
$$\mp Q|\mathbf{p}|(\mathbf{E}\cdot\boldsymbol{\omega})\boldsymbol{\Omega} \pm 3Q(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\hat{\mathbf{p}},$$

(1)Berry phase is embedding in quantum kinetic theory.

(2)Consistent with other approaches.

(3) Vortical dependence is new!

Embedded Berry phase

- The chiral current is,

$$\partial_\sigma j^\sigma_{R/L} = \mp Q^2 (E \cdot B) \int d^4 p \partial^p_\sigma [b^\sigma \delta(p^2)] f_{R/L}$$

$$b^\mu = -\frac{p^\mu}{p^2}, \qquad \int dp_0 \delta(p^2) b^\sigma = \left(0, \frac{1}{2}\mathbf{\Omega}\right)$$

4D Berry monopole

- Taking analytic continuation, we find

$$\int d^4p \partial_\mu \left[b^\mu \delta(p^2) \right] = 2\pi^2 .$$

since, b^μ embedded the Berry curvature (Berry magnetic field), so the divergence of this quantity is an effective charge in momentum space. We called it Berry monopole.

Chiral anomaly

$$\partial_\sigma j^\sigma_{R/L} = \mp Q^2 (E \cdot B) \int d^4p \partial^p_\sigma [b^\sigma \delta(p^2)] f_{R/L}$$

$$= \mp \frac{Q^2}{4\pi^2} (E \cdot B).$$

Then, the chiral anomaly is given by the Berry monopole.

Summary

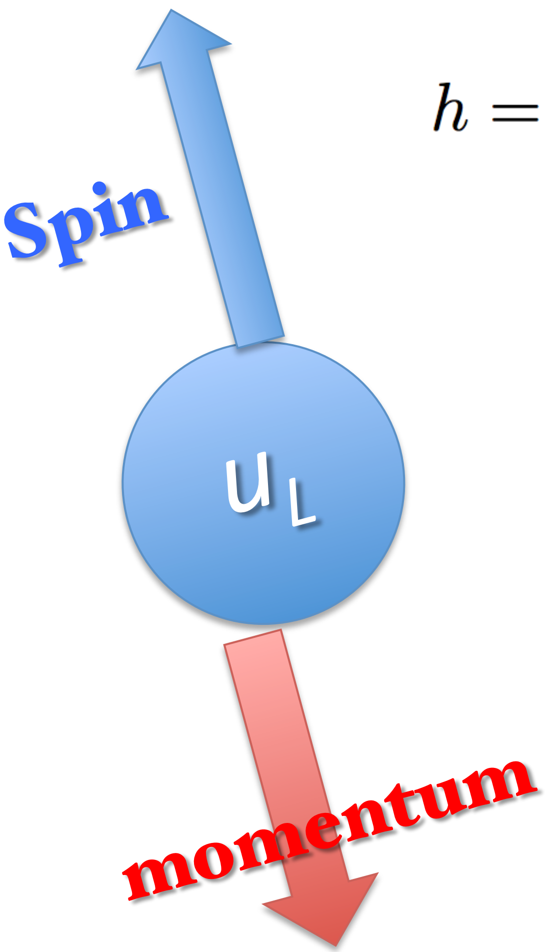
- We have derived a 4D chiral kinetic theory from Wigner functions.
- Reducing to 3D, it is consistent with the approaches including Berry phase.
- We also find the chiral anomaly is given by a 4D Berry monopole.

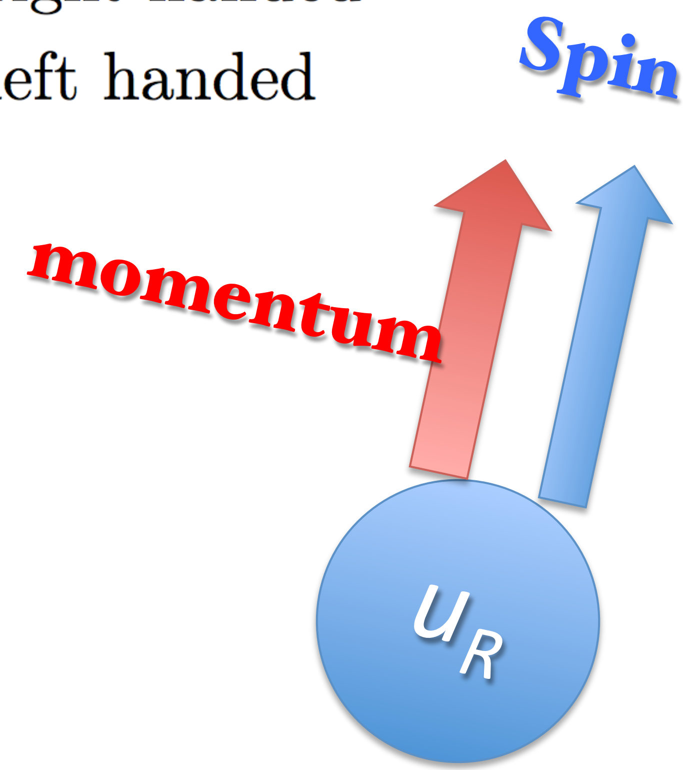
Thank you!

Backup

Chirality of massless fermions

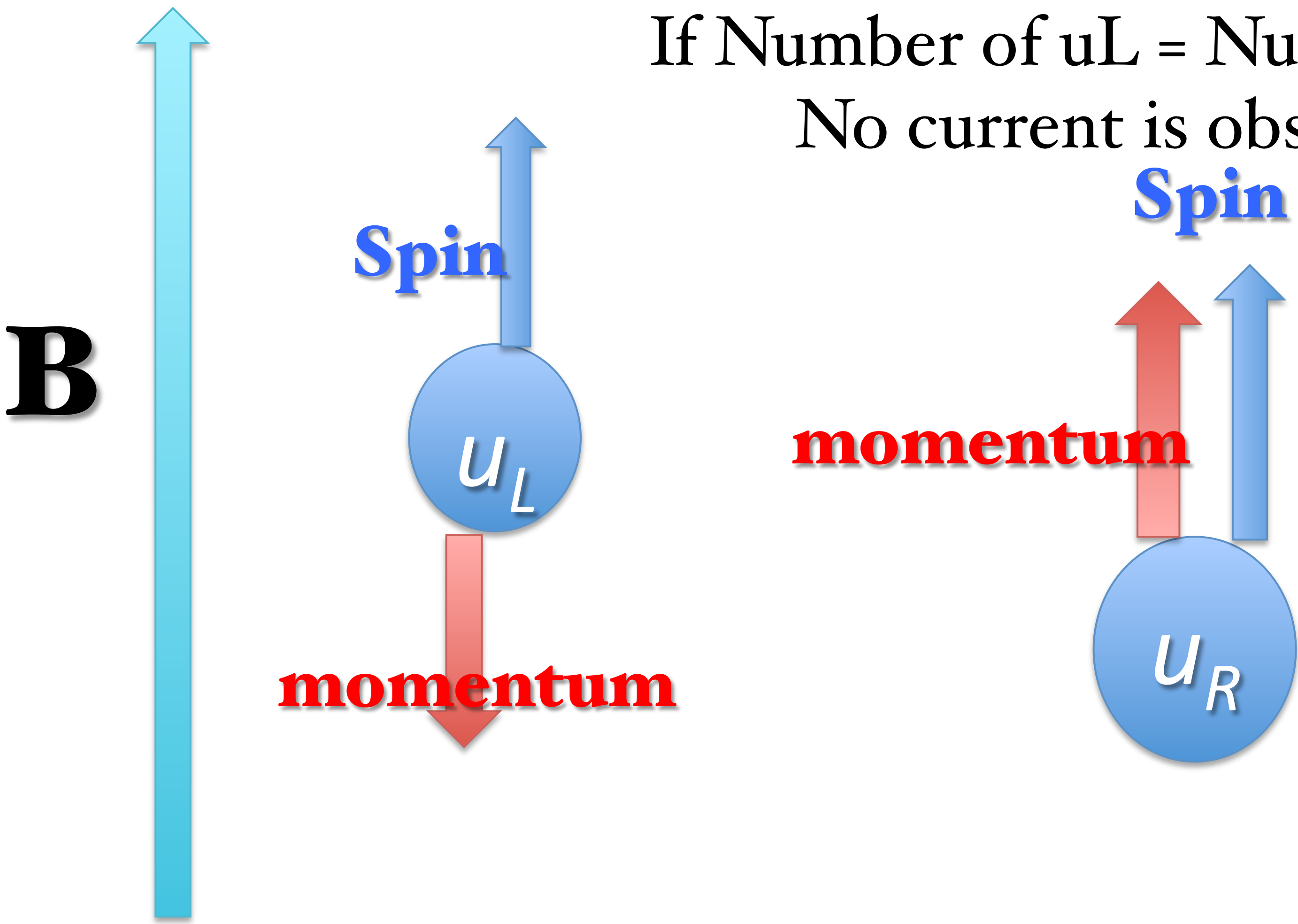
$$h = \frac{\sigma \cdot p}{|\mathbf{p}|} = \begin{cases} +1, & \text{right handed} \\ -1, & \text{left handed} \end{cases}$$





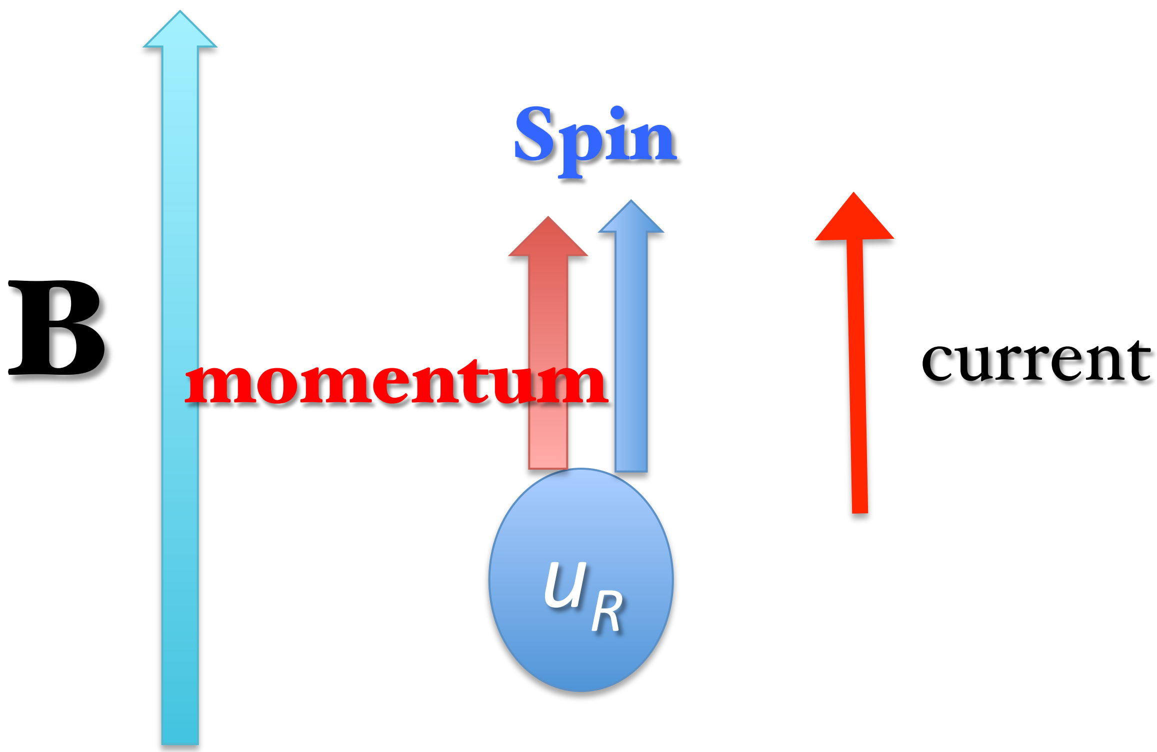
Chirality

If Number of uL = Number of uR
No current is observed.

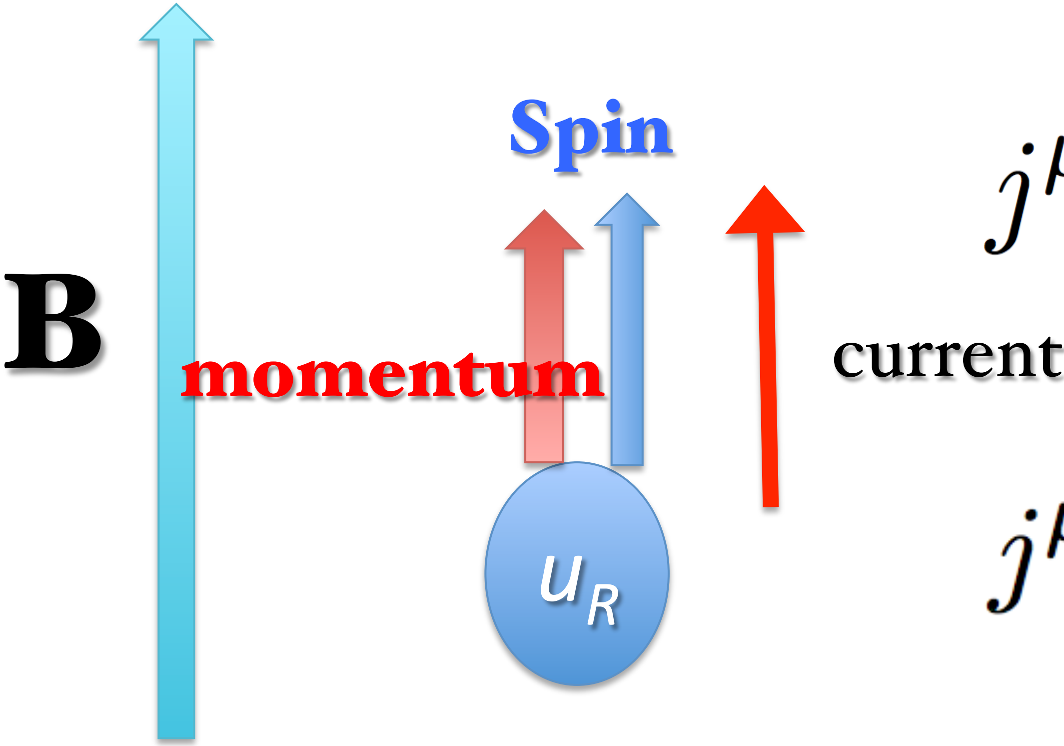


Chiral Magnetic Effect

If Number of uL ≠ Number of uR
A electric current will be observed.



Chiral Magnetic Effect (CME)

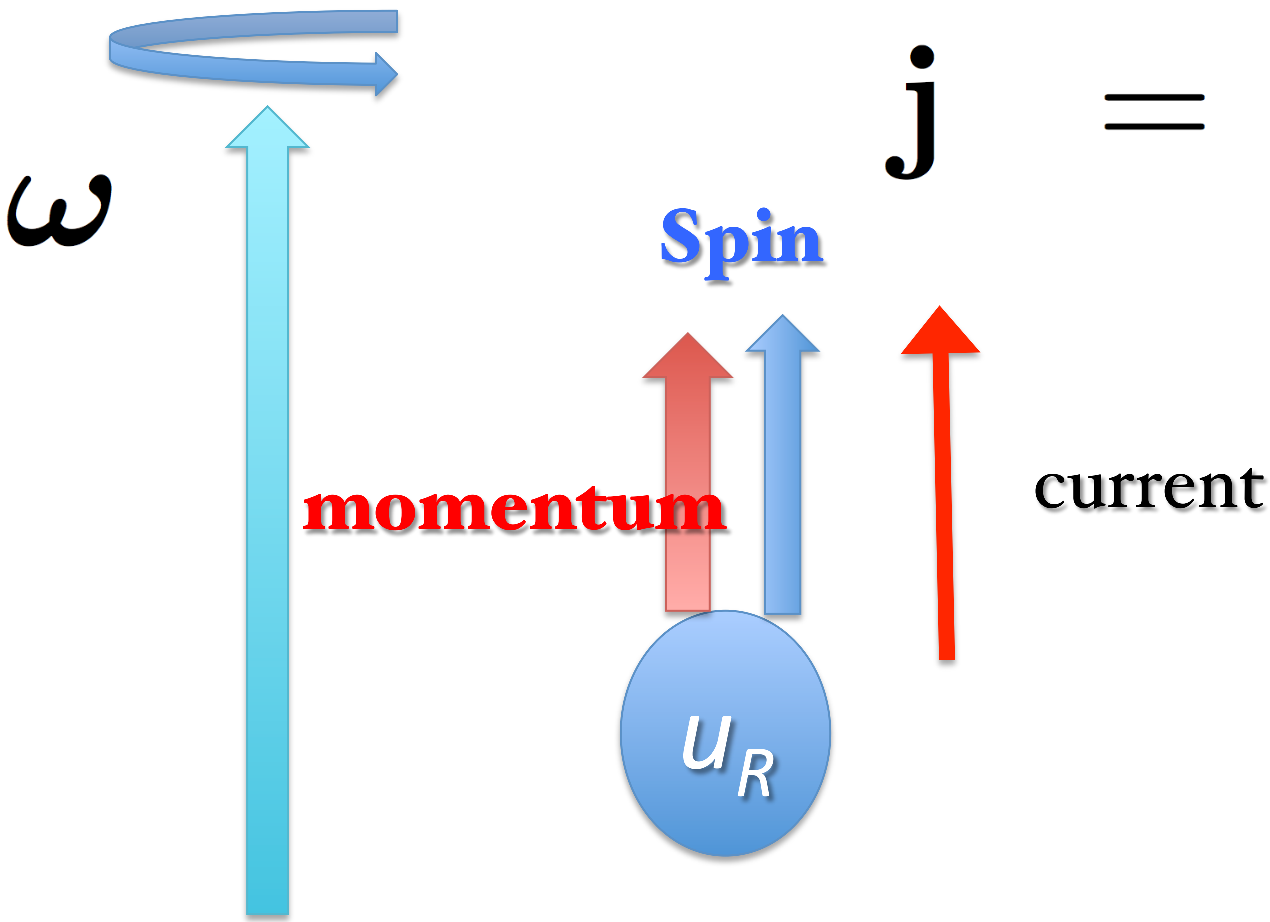


$$j^\mu = \xi_B B^\mu,$$

$$j^\mu = \sigma E^\mu,$$

Chiral Vortical Effect (CVE)

$$\mathbf{j} = \xi \boldsymbol{\omega}$$



Chiral Magnetic and Vortical Effect

Charge current

Magnetic field

Vorticity

$$j^\mu = \xi_B B^\mu + \xi \omega^\mu,$$

$$j_5^\mu = \xi_{5B} B^\mu + \xi_5 \omega^\mu,$$

Axial current

New Transport coefficients

$$\begin{aligned} j^\mu &= \xi_B B^\mu + \xi \omega^\mu, \\ j_5^\mu &= \xi_{5B} B^\mu + \xi_5 \omega^\mu, \end{aligned}$$

- Strong coupling, AdS/CFT duality,

(Erdmenger('09), Banerjee('11), Torabian('11), ...)

- Weakly coupling, Kubo formula

(Fukushima('08), Kharzeev('11), Landsteiner('11), Hou('12), ...)

Anomalous fluid dynamics

- Those new terms are forbidden by the entropy principle of a normal fluid.

- Son and Suro´wka ('09) pointed out these terms are crucial to cancel the production of negative entropy in an anomalous fluid.

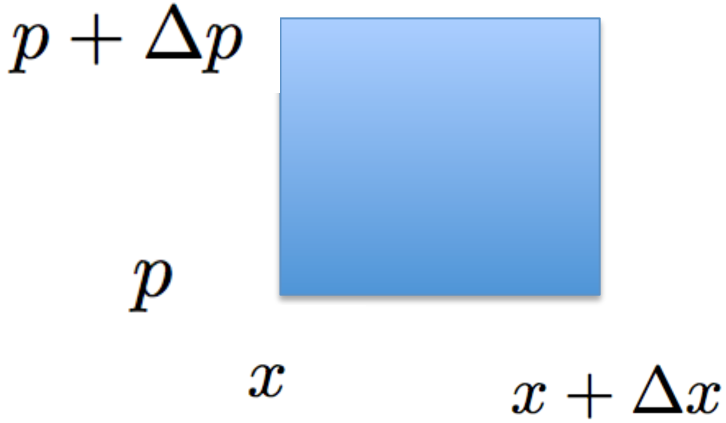
$$\partial_\mu T^{\mu\nu} = QF^{\nu\rho} j_\rho,$$

$$\partial_\mu j^\mu = 0, \qquad \partial_\mu j_5^\mu = -\frac{Q^2}{2\pi^2} E_\rho B^\rho,$$

Kinetic theory

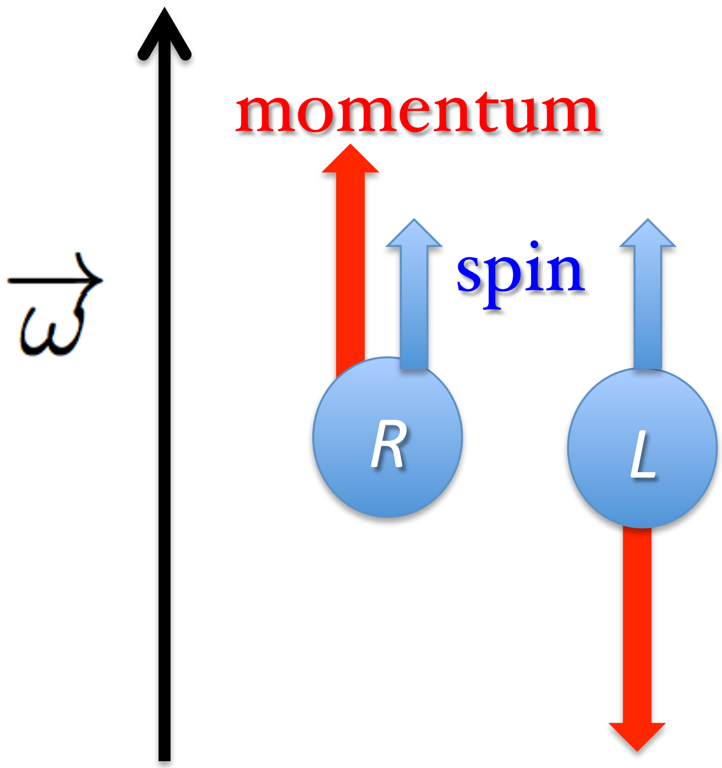
- Kinetic theory: a microscopic dynamic theory for many-body system, to compute transport coefficients.

- distribution function, e.g. Fermi-Dirac distribution f(x,p)



Local Polarization Effect

Spin local polarization effect



Axial current

$$j_5^\mu \equiv j_R^\mu - j_L^\mu = \xi_5 \omega^\mu,$$

$$\xi_5 = \frac{1}{6} T^2 + \frac{1}{2\pi^2} (\mu^2 + \mu_5^2),$$

Can be observed in both high/low energy collisions

3-dimensional Chiral kinetic equation

Integral over p0

$$\frac{dt}{d\tau}\partial_t f_{R/L} + \frac{d\mathbf{x}}{d\tau}\cdot\nabla_{\mathbf{x}} f_{R/L} + \frac{d\mathbf{p}}{d\tau}\cdot\nabla_{\mathbf{p}} f_{R/L} = 0,$$

$$\frac{dt}{d\tau} = 1 \pm Q\mathbf{\Omega}\cdot\mathbf{B} \pm 4|\mathbf{p}|(\mathbf{\Omega}\cdot\boldsymbol{\omega}),$$

velocity

$$\frac{d\mathbf{x}}{d\tau} = \hat{\mathbf{p}} \pm Q(\hat{\mathbf{p}}\cdot\mathbf{\Omega})\mathbf{B} \pm Q(\mathbf{E}\times\mathbf{\Omega}) \pm \frac{1}{|\mathbf{p}|}\boldsymbol{\omega},$$

force

$$\frac{d\mathbf{p}}{d\tau} = Q(\mathbf{E}+\hat{\mathbf{p}}\times\mathbf{B}) \pm Q^2(\mathbf{E}\cdot\mathbf{B})\mathbf{\Omega} \mp Q|\mathbf{p}|(\mathbf{E}\cdot\boldsymbol{\omega})\mathbf{\Omega} \pm 3Q(\mathbf{\Omega}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\hat{\mathbf{p}},$$

Talk @ Transport Meeting June 24

- What are those new terms in Boltzmann Eq. ?

  - They are related to a topologic phase factor, called Berry phase.

- Are these related to chiral anomaly?

  - Chiral anomaly is given by an effective monopole in momentum space.

- Are you serious?

  - All of these can be obtained by path integral of a single massless fermion.

Summary

- We obtain the chiral magnetic and vortical effect, chiral anomaly by Wigner function.
- We derive the chiral kinetic equation (modified Boltzmann equation).

Thank you!

3-dimensional Chiral kinetic equation

Integral over p0

$$\frac{dt}{d\tau}\partial_t f_{R/L} + \frac{d\mathbf{x}}{d\tau}\cdot\nabla_{\mathbf{x}} f_{R/L} + \frac{d\mathbf{p}}{d\tau}\cdot\nabla_{\mathbf{p}} f_{R/L} = 0,$$

$$\frac{dt}{d\tau} = 1 \pm Q\mathbf{\Omega}\cdot\mathbf{B}$$

$$\frac{d\mathbf{x}}{d\tau} = \hat{\mathbf{p}} \pm Q(\hat{\mathbf{p}}\cdot\mathbf{\Omega})\mathbf{B} \pm Q(\mathbf{E}\times\mathbf{\Omega})$$

$$\frac{d\mathbf{p}}{d\tau} = Q(\mathbf{E}+\hat{\mathbf{p}}\times\mathbf{B}) \pm Q^2(\mathbf{E}\cdot\mathbf{B})\mathbf{\Omega}$$

if set ω=0,
it is as the same as the Hamiltonian approaches.

3-dimensional Chiral kinetic equation

Integral over p0

$$\frac{dt}{d\tau}\partial_t f_{R/L} + \frac{d\mathbf{x}}{d\tau}\cdot\nabla_{\mathbf{x}} f_{R/L} + \frac{d\mathbf{p}}{d\tau}\cdot\nabla_{\mathbf{p}} f_{R/L} = 0,$$

$$\frac{dt}{d\tau} = 1 \pm Q\boldsymbol{\Omega}\cdot\mathbf{B} \boxed{\pm 4|\mathbf{p}|(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})},$$

$$\frac{d\mathbf{x}}{d\tau} = \hat{\mathbf{p}} \pm Q(\hat{\mathbf{p}}\cdot\boldsymbol{\Omega})\mathbf{B} \pm Q(\mathbf{E}\times\boldsymbol{\Omega}) \boxed{\pm \frac{1}{|\mathbf{p}|}\boldsymbol{\omega}},$$

$$\frac{d\mathbf{p}}{d\tau} = Q(\mathbf{E} + \hat{\mathbf{p}}\times\mathbf{B}) \pm Q^2(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}$$
$$\boxed{\mp Q|\mathbf{p}|(\mathbf{E}\cdot\boldsymbol{\omega})\boldsymbol{\Omega} \pm 3Q(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\hat{\mathbf{p}},}$$

ω dependence is new!