

The ratio of η/s of an Interacting Hadrons Gas¹

Anton Wiranata

with V. Koch, X.N. Wang & M. Prakash

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¹Transport Group Meeting

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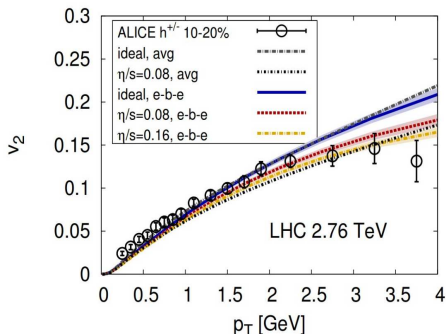
Motivation

- Transport coefficients (shear & bulk viscosities) are important inputs to viscous hydrodynamics simulations of relativistic heavy ion collisions.
- The presence of shear viscosity affects the magnitude of the elliptic and higher order collective flows.
- Comparison of different approximations in the calculation of transport coefficients.
- How jet transport coefficient \hat{q} is related to η/s .
- All the above for bulk viscosity.

The ratio of η/s and elliptic flow

The collective flow with viscous effects reduces the magnitude of elliptic flow relative to ideal (non-viscous) hydrodynamic motion.²

$$\frac{dN}{d\phi} = \frac{N}{2\pi} [1 + 2v_1 \cos(\phi) + 2v_2 \cos(2\phi) + \dots], \quad v_n = \frac{\int d\phi \cos(n\phi) \frac{dN}{d\phi}}{\int d\phi \frac{dN}{d\phi}} \quad (1)$$

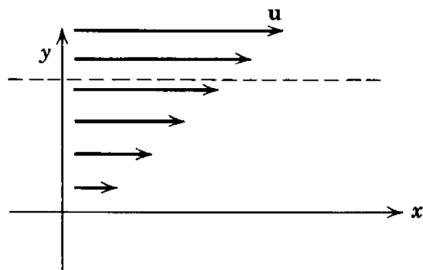


(Color online) Charged hadron v_2 in 10-20 % central collisions for average initial conditions (avg) and event-by-event simulations (e-b-e). The bands indicate the statistical error. Experimental data from ALICE. (Glauber initial condition)

²B. Schanke, PLB 702, 59 (2011)

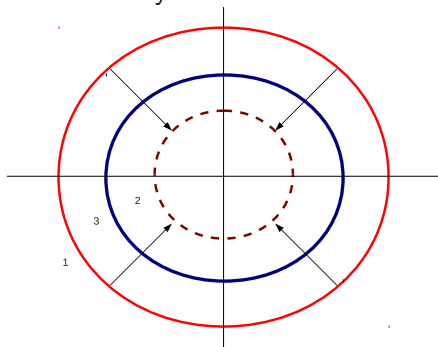
Qualitative aspects of viscosity I

Shear Viscosity



$$F = -\eta_s \frac{dv_x}{dy} \quad (2)$$

Bulk Viscosity



$$F = -\eta_v \nabla \cdot v \quad (3)$$

F. Reif, Fundamental of statistical and Thermal Physics

Qualitative aspects of viscosity II

- Standard unit of viscosity is:
 - ▶ SI unit: Pa.s (Pascal second) = $1 \text{ kg m}^{-1} \text{ s}^{-1}$
 - ▶ cgs unit: P (Poise) = $1 \text{ gr cm}^{-1} \text{ s}^{-1}$
- The viscosity of water is 10^{-3} Pa.s at 20°C ³
- The viscosity of RHIC matter is about 10^{11} Pa.s at 100 MeV ($\approx 10^{12} \text{ }^\circ\text{C}$) ⁴
- Considering only the size of each system:

$$\left(\frac{r_{\text{hadron}}}{r_{\text{molecule}}} \right)^2 \approx 10^{10} \quad (4)$$

Effects of temperature are also important.

³Sears, Zemansky & Young, University Physics, 6th Edition

⁴Wiranata & Prakash, 2009

Temperature dependence of η

- Estimation from kinetic theory:

$$\eta_s \approx \# \frac{\bar{p}}{\bar{\sigma}_T}, \quad \bar{\sigma}_T = \frac{1}{3} \int_0^\infty dg g^7 e^{-g^2} \int_0^\pi d\theta \sin^2 \theta \sigma \left(\frac{2g}{\sqrt{m\beta}, \theta} \right) \quad (5)$$

\bar{p} : Mean momentum, g : Scaled (w.r.t. \bar{p}) relative momentum.

$\bar{\sigma}_t(T)$: T -dependent transport cross section.

- Relativistic expressions are more complicated, but have similar content.

$$\eta \approx \# \frac{\bar{p}}{\bar{\sigma}_t(T)} = \# \frac{\hbar}{\lambda \bar{\sigma}_t(T)} := \frac{\text{action}}{\text{effective physical volume}} \quad (6)$$

- Above, λ is the thermal de-Broglie wavelength.

$$\lambda \ \& \ \eta \propto \begin{cases} \frac{1}{\sqrt{mT}} & \& \ \frac{\sqrt{mT}}{\bar{\sigma}_t(T)} \quad \text{for NR} \\ \frac{1}{T} & \& \ \frac{T}{\bar{\sigma}_t(T)} \quad \text{for UR} \end{cases} \quad (7)$$

Fluidity vs the applicability of hydrodynamics

- The applicability of hydrodynamics requires that the effective mean free path be much smaller compared to the size of the system.
- The concept of fluidity of a substance can only be meaningful if the fluidity is defined exclusively in terms of the properties of the substance itself, i.e.⁵ the mean free path vs inter particle distance (density-density correlation length).

⁵J. Liao & V. Koch, 2010

Kinematic Viscosity

From Bjorken expansion (1+1 D)

$$\frac{de}{d\tau} = -\frac{e + P - 4\eta_s/(3\tau)}{\tau} = -\frac{sT}{\tau} \left(1 - \frac{4}{3} \frac{\eta_s}{s} \frac{1}{T\tau} \right), \quad (8)$$

Enthalpy is given as follows:

$$w_R = \epsilon + p = Ts + \mu_R n; \quad \mu_R = \mu_{NR} + m;$$

In general,

$$w = Ts + \mu n + mn; \quad (9)$$

The kinematic viscosity reads as⁶

$$\frac{\eta_s}{s} (\text{Relativistic}) \approx \frac{\eta_s}{\rho} (\text{Non-Relativistic}) \approx \frac{\eta_s}{w} (\text{General}) \quad (10)$$

⁶Landau, Fluid Mechanics

Methods to calculate transport coefficients

In the hadronic system produced in relativistic heavy-ions collisions, there are different approximations/methods to calculate transport coefficients:

- The Chapman-Enskog approximation
- The relaxation time approximation
- The Gree-Kubo method.

The Chapman-Enskog approximation I

“For small deviation from equilibrium, the distribution function can be expressed in terms of hydrodynamic variables and their gradients. Transport coefficients such as shear and bulk viscosities are calculable from kinetic theory.”⁷

$$f(x, p) = f^0 [1 + \phi(x, p)]; \quad \phi(x, p) = \text{deviation function} \quad (11)$$

$$f^0(x, p) = \frac{1}{(2\pi\hbar)^3} \exp \left[\frac{m \mu(x) + p_\alpha u^\alpha(x)}{kT} \right] \quad \text{equilibrium dist. function} \quad (12)$$

where

$\mu(x)$ is the chemical potential,

$u(x)$ is the flow velocity and

T is the temperature.

⁷van Leeuwen

The Chapman-Enskog approximation II

The general form of the deviation function:

$$\begin{aligned}\phi(x, p) &= A \partial_\alpha U^\alpha && \text{(Bulk Viscosity)} \\ &+ B \Delta_{\alpha\beta} p^\beta \Delta^{\alpha\beta} (T \partial_\beta T + C^{-2} D U_\beta) && \text{(Heat Conductivity)} \\ &+ C \langle p_\alpha p_\beta \rangle \langle \partial^\alpha U^\beta \rangle && \text{(Shear Viscosity)}\end{aligned}\quad (13)$$

By using kinetic theory

$$T^{\alpha\beta} = c \int p^\alpha p^\beta f(x, p) \frac{d^3 p}{p^0} \quad (14)$$

and a little bit of math⁸

⁸Details can be found in A.Wiranata, PRC 2012

The Chapman-Enskog approximation III

The working expression

$$\eta_s = \frac{1}{10} (kT)^2 \sum_{m=0}^{r-1} c_m \gamma_m; \quad \sum_{m=0}^{r-1} c_m^{(r)} c_{mn} = \frac{\gamma_n}{\rho kT}; \quad (n = 0, 1, \dots, r-1) \quad (15)$$

r is the order of approximation.

$$[\eta_s]_1 = \frac{1}{10} kT \frac{\gamma_0^2}{c_{00}} \quad \text{1st order approximation} \quad (16)$$

$$[\eta_s]_2 = \frac{1}{10} kT \frac{\gamma_0^2 c_{11} - 2\gamma_0 \gamma_1 c_{01} + \gamma_1^2 c_{00}}{c_{00} c_{11} - c_{01}^2} \quad \text{2nd order} \quad (17)$$

\vdots \vdots

The Chapman-Enskog approximation IV

Shear Viscosity (first order approximation)

$$[\eta_s]_1 = \frac{1}{10} kT \frac{\gamma_0^2}{c_{00}}; \quad \gamma_0 = -10\hat{h}; \quad \hat{h} = \frac{K_3(z)}{K_2(z)}, \quad (18)$$

where $z = m/T$ is the relativity parameter.

$$c_{00} = 16 \left(w_2^{(2)} - \frac{1}{z} w_1^{(2)} + \frac{1}{3z^2} w_0^{(2)} \right); \quad w_i^{(s)} = \text{Rel. Omega Integral} \quad (19)$$

Relativistic Omega Integral:

$$w_i^{(s)} = \frac{2\pi z^3 c}{K_2(z)} \int_0^\infty d\psi \sinh^7 \psi \cosh \psi K_j(2z \cosh \psi) \\ \times \int_0^\pi d\theta \sin \theta \sigma(\psi, \theta) (1 - \cos^2 \theta), \quad (20)$$

where relative momentum $g = mc \sinh \psi$, total momentum $P = 2mc \cosh \psi$, differential cross section $\sigma(\psi, \theta)$ and $j = 5/2 + (-1)^i/2$.

Relaxation time approximation

Relaxation time approximation

The effect of collisions is always to bring the perturbed distribution function close to the equilibrium distribution function. In other words, the effect of collisions is to restore local the equilibrium distribution function exponentially with the relaxation time $\tau(p)$ which is of order the time required between collisions.

$$C[f] \cong \frac{f(x, p) - f_0^{eq}(x, p)}{\tau(p)} = D_t f(x, p) \quad (21)$$

Shear viscosity (for $a + b \rightarrow c + d$) is given by ⁹

$$\eta_s = \frac{1}{15T} \int_0^\infty \frac{d^3 p_a}{(2\pi)^3} \frac{|p_a|^4}{E_a^2} \frac{1}{w_a(E_a)} f_a^{eq} \quad (22)$$

The collision frequency, $w_a(E_a)$ is given by ¹⁰

$$w_a(E_a) = \int_0^\infty \frac{d^3 p_b}{(2\pi)^3} \frac{\sqrt{s(s - 4m^2)}}{2E_a 2E_b} f_b^{eq} \sigma_T, \quad (23)$$

where σ_T is the total cross section.

⁹F. Reif, Fund. of Statistical, J. Kapusta (2010) & S. Gavin(1985)

¹⁰W. Greiner & M. D. R. ... 2012

The Green-Kubo method I

The Green-Kubo method I

The Green-Kubo method enables us to extract transport coefficients in terms of the correlation functions of fluctuations in the stress-energy tensor for the system¹¹. Shear viscosity is given by

$$\eta_s = \frac{1}{T} \int d^3r \int dt \langle T^{xy}(\vec{0}, 0) T^{xy}(\vec{r}, t) \rangle, \quad (24)$$

where T is the temperature, T^{xy} is the the shear component of the stress energy tensor. The averaging symbol refers to the Gibbs grand canonical average.

The stress-energy tensor is given by

$$T^{\mu\nu}(\vec{r}, t) = \int d^3p \frac{p^\mu p^\nu}{p^0} f(\vec{r}, \vec{p}, t), \quad (25)$$

where $f(\vec{r}, \vec{p}, t)$ is the distribution function of the system.

¹¹S.G. Brush, Kinetic Theory, 1972

The Green-Kubo methods II

In the case of point particles (monte-carlo simulations & lattice calculations), the distribution function and stress energy tensor of the system can be written as follows¹²:

$$f(\vec{r}, \vec{p}) = \frac{1}{V} \sum_{j=1}^{N_{part}} \delta^{(3)}(\vec{p} - \vec{p}_j), \quad T^{xy} = \frac{1}{V} \sum_{j=1}^{N_{part}} \frac{p^x(j)p^y(j)}{p^0(j)} \quad (26)$$

and then the shear viscosity reads as

$$\eta_s = \frac{T}{V} \int_0^\infty dt \langle T^{xy}(0) T^{xy}(t) \rangle \quad (27)$$

Requires knowledge of all components of the stress energy-momentum tensor for all times. This entails dynamical simulations, often termed as molecular dynamical calculations with specified force laws and equation of motions.

¹²Demir & Bass, PRL 2009

Interactions

Interactions enter the calculation of shear viscosity through the scattering cross section $\sigma(s, \theta)$.

Shear Viscosity:

$$\eta \approx \# \frac{\bar{p}}{\bar{\sigma}_t(T)}, \quad \text{differential cross section}$$

$$\eta_s = \frac{1}{15T} \int_0^\infty \frac{d^3 p_a}{(2\pi)^3} \frac{|p_a|^4}{E_a^2} \frac{1}{w_a(E_a)} f_a^{eq}, \quad \text{total cross section}$$

$$\eta_s = \frac{T}{V} \int_0^\infty dt \langle T^{xy}(0) T^{xy}(t) \rangle, \quad \text{total cross section}$$

Cross-Sections from:

- 1 Phase shifts parameterization (M.Prakash, et.al, 1993).
- 2 Breit-Wigner parameterization (Bass, et.al, 1999).
- 3 K-Matrix formalism (This work).

$\pi - \pi$ phase shifts parameterization

Diff. Cross-section ($\pi - \pi \rightarrow M_R \rightarrow \pi - \pi$):

$$\sigma = \frac{4}{q_{cm}^2} \left(\frac{1}{9} \sin^2 \delta_0^0 + \frac{5}{9} \sin^2 \delta_2^0 + \frac{1}{3} \sin^2 \delta_1^1 \right)$$

Phase-shifts:

$$\delta_0^0(\epsilon) = \frac{\pi}{2} + \arctan \left(\frac{\epsilon - m_\sigma}{\Gamma_\sigma/2} \right)$$

$$\delta_1^1(\epsilon) = \frac{\pi}{2} + \arctan \left(\frac{\epsilon - m_\rho}{\Gamma_\rho/2} \right)$$

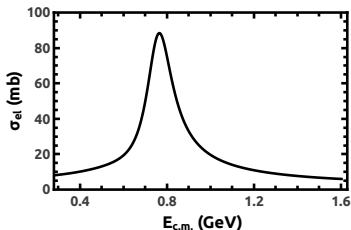
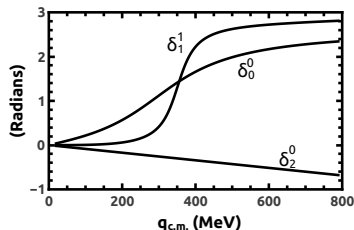
$$\delta_2^0(\epsilon) = -\frac{0.12q}{m_\pi}$$

Decay widths:

$$\Gamma_\rho = 0.095 q \left(\frac{q/m_\pi}{1 + (q/m_\rho)^2} \right)^2$$

$$\Gamma_\sigma = 2.06 q$$

Parameterization from Bertsch et.al, PRD 37 (1988) 1202.



Breit-Wigner parameterization

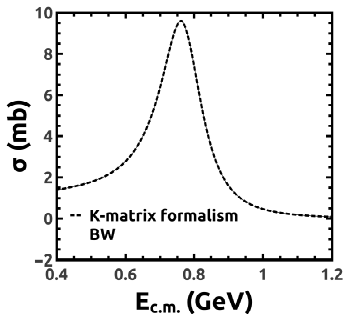
Tot. Cross-section (mesons-baryons):

$$\sigma_t = \sum_{\text{Res}} \langle j_B, m_B, j_M, m_M || j_R, m_R \rangle \frac{2S_R + 1}{(2S_B + 1)(2S_m + 1)} \\ \times \frac{\pi}{p_{\text{cms}}^2} \frac{\Gamma_{R \rightarrow M, B} \Gamma_{\text{tot}}}{(M_R - \sqrt{s})^2 + \frac{\Gamma_{\text{tot}}^2}{4}},$$

Decay widths:

$$\Gamma_{i,j}(\sqrt{s}) = \Gamma_R^{i,j} \frac{M_R}{\sqrt{s}} \left(\frac{p_{i,j}(\sqrt{s})}{p_{i,j}(M_R)} \right)^{2l+1} \\ \times \frac{1.2}{1 + 0.2 \left(\frac{p_{i,j}(\sqrt{s})}{p_{i,j}(M_R)} \right)^{2l}},$$

where M_R is the resonance mass, $\Gamma_R^{i,j}$ is the decay width at the pole and l is the angular momentum of the exit channel.



Parameterization from Bass et.al, 1999

K-Matrix cross section I (Why another cross section ??)

This formalism was introduced by Wigner and Eisenbud for the study of resonances in nuclear reactions in order to maintain unitarity of the T-matrix¹³

The scattering operator, S , can be decomposed as

$$S = I + 2iT; \quad I \text{ (Identity operator); } \quad T \text{ - matrix (Interaction terms)} \quad (28)$$

The K operator is introduced as follows:

$$K^{-1} = T^{-1} + iI, \quad K = K^\dagger. \quad (29)$$

The K-matrix is symmetric and real.

We can write the T matrix into its real and imaginary parts as follows:

$$\begin{aligned} \text{Re}T &= (I + K^2)^{-1}K = K(I + K^2)^{-1} \\ \text{Im}T &= (I + K^2)^{-1}K^2 = K^2(I + K^2)^{-1}, \quad T = e^{i\delta} \sin \delta \end{aligned} \quad (30)$$

Details of calculations for hadronic resonances can be found in A. Wiranata et.al
(In progress)

¹³Wigner & Eisenbud, 1946 & 1947

K-Matrix cross section II

In the case of resonances, the K -matrix can be written as follows:

$$K_{ij} = \sum_{\alpha} \frac{m_{\alpha} \Gamma_{\alpha i}}{m_{\alpha}^2 - m^2}, \quad \alpha = \text{Resonances}$$

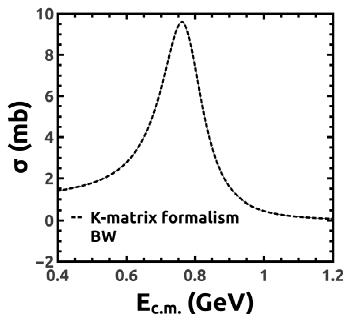
where the partial decay width is given by:

$$\Gamma_{\alpha i}(m) = \Gamma_0 \left(\frac{M_R}{m} \right) \left(\frac{q(m)}{q(m_R)} \right) (B_{\alpha i}^l(q, q_{\alpha}))^2$$

and the B -factor is

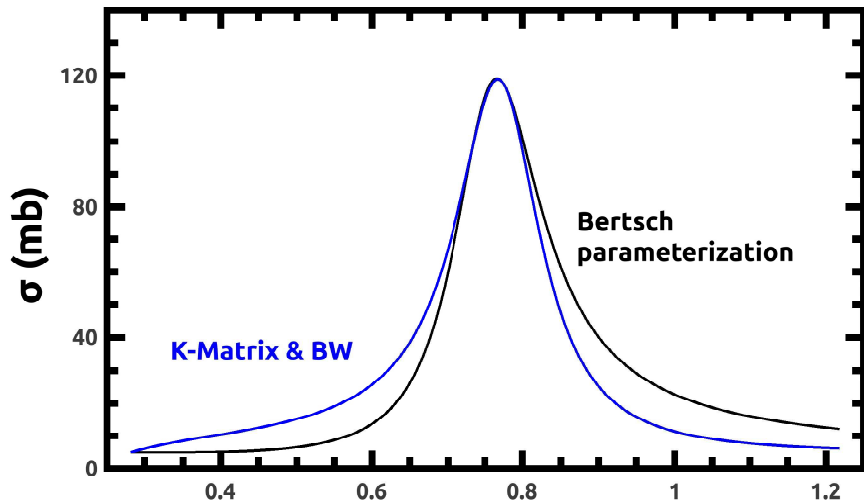
$$B_{\alpha i}^l(q, q_{\alpha}) = \frac{F_l(q)}{F_l(q_{\alpha})} \quad (31)$$

details of f -factor can be found in A. Wiranata (in progress).



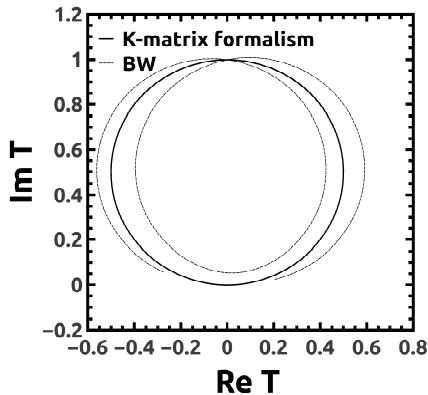
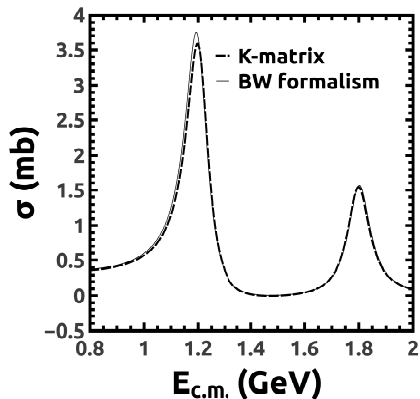
Single resonances

$$\pi - \pi \rightarrow \rho \rightarrow \pi - \pi$$



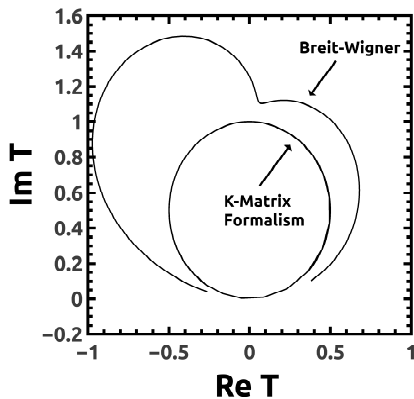
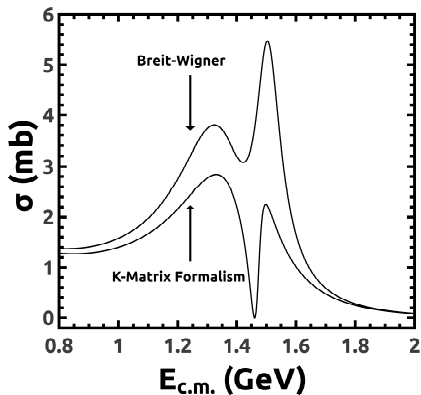
Two and more resonances

$$\pi - \pi \rightarrow M_{R1}, M_{R2} \rightarrow \pi - \pi$$



Two and more resonances

$$\pi - \pi \rightarrow M_{R1}, M_{R2} \rightarrow \pi - \pi$$



Take home points

- ① Experimental phase shifts for all resonances not available.
- ② Breit-Wigner/UrQMD parameterizations only “good” for a single resonance and not overlapping resonances.
- ③ K-matrix can include higher resonances and maintain unitarity at the same time even for overlapping resonances.

Methods to calculate equation of state

- 1 Non-interacting hadrons gas: Pressure, energy density and entropy density of the system are sums of individual pressure, energy density and entropy density of particles.
- 2 Cluster-expansion:¹⁴ Pressure, energy density and entropy density of the system can be calculated from virial expansion.¹⁵

3

$$P = \frac{1}{\beta} \frac{1}{2\pi^3} \int_{2m_\pi}^{\infty} d\epsilon \epsilon^2 K_1(\epsilon\beta) \sum_{l,I} g_l^I \delta_l^I(\epsilon), \quad (32)$$

$$\mathcal{E} = \frac{1}{4\pi^3} \int_{2m_\pi}^{\infty} d\epsilon \epsilon^3 [K_2(\epsilon\beta) + K_0(\epsilon\beta)] \sum_{l,I} g_l^I \delta_l^I(\epsilon), \quad (33)$$

$$s = \beta \frac{1}{2\pi^3} \int_{2m_\pi}^{\infty} d\epsilon \epsilon^3 K_2(\epsilon\beta) \sum_{l,I} g_l^I \delta_l^I(\epsilon). \quad (34)$$

¹⁴Kahn & Uhlenbeck, 1938; Batx & Uhlenbeck

¹⁵Dashen, Ma, & Bernstein & R. Venugopalan & M. Prakash, 1992

Results

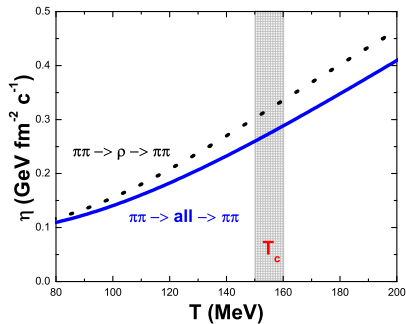
The ratio of η_s/s an interacting pion gas

Table: List of resonances in the case of $\pi - \pi$ interaction.

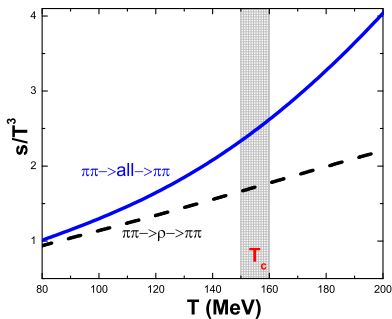
Particle	Mass (Mev)	Width (MeV)	$\pi - \pi$
ρ	774	150	1
ω	782	8	0.02
f_0	980	100	0.7
f_{02}	1370	200	0.1
f_2	1270	185	0.5
f_2'	1525	76	0.01
ρ_2	1465	310	0.5
ρ_3	1700	235	0.1

The ratio of η_s/s of an interacting pion gas

Shear viscosity of pion gas



Entropy density of pion gas

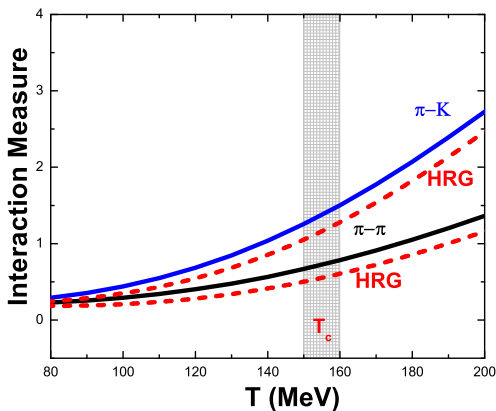


Interaction measure (comparison with hadron resonance gas)

Interaction measure is $I = (\epsilon - 3P)/T^4$.

Table: List of resonances in the case of $\pi - K$ interaction.

$\pi - \pi$
$\rho, \omega, f_0, f_{02}, f_2, f_2', \rho_2, \rho_3$
$\pi - K$
$K^*, K_0^*, K_2, K_2^*, K_3^*$
$K - K$
$a_0, \phi, f_0^*, a_2, \phi_2, \phi_3$



Resonances in the $\pi - K - \eta - N$ mixture

$\pi - \pi$

$\rho, \omega, f_0, f_0', f_2, f_2', \rho_2, \rho_3$
8 resonances

$\pi - K$

$K^*, K_0^*, K_2, K_2^*, K_3^*$
5 resonances

$K - K$

$a_0, \phi, f_0^*, a_2, \phi_2, \phi_3$
6 resonances

$\pi - \eta$

a_0 & a_2 - 2 resonances

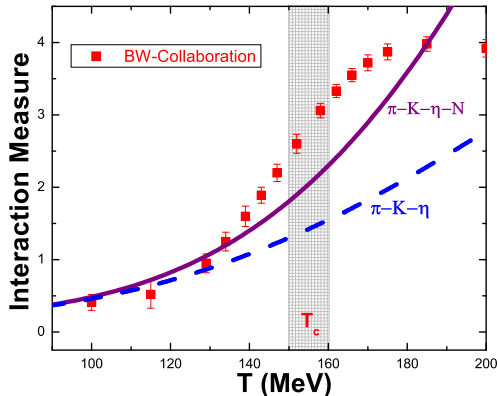
$\pi - N$

$\Delta(1232 - 1950)$ - 10 resonances
 $N^*(1440 - 1990)$ - 11 resonances

$K - N$

$\Lambda^*(1520 - 1890)$ - 9 resonances
 $\Sigma^*(1440 - 1940)$ - 6 resonances

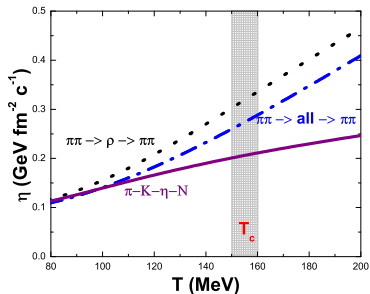
Interaction Measure (comparison with lattice results)



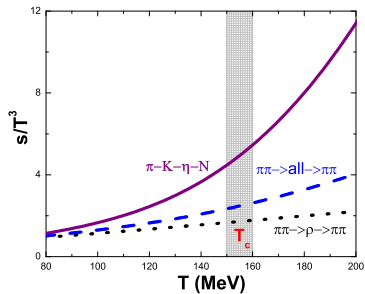
- Interaction measure is $I = (\epsilon - 3P)/T^4$.
- 3 quarks ($n_f = 2 + 1$) and $m_\pi = 135$ MeV, BMW collaboration 2012.
- Lattice spacing is $N_t = 8$, BMW collaboration 2012.

Multi-component mixtures

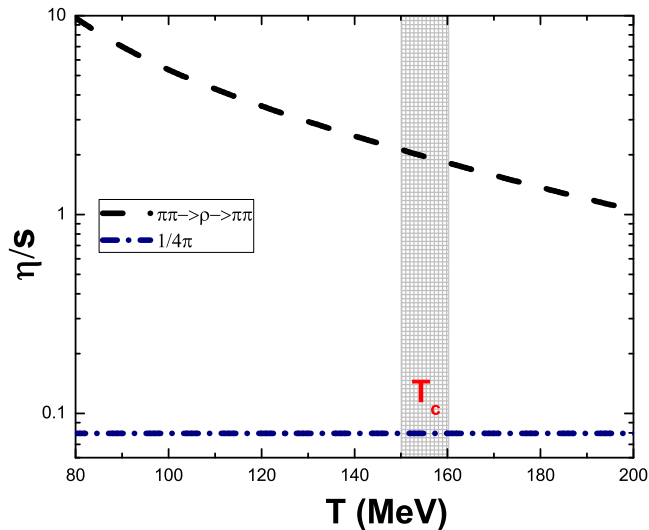
The coefficient of shear viscosity



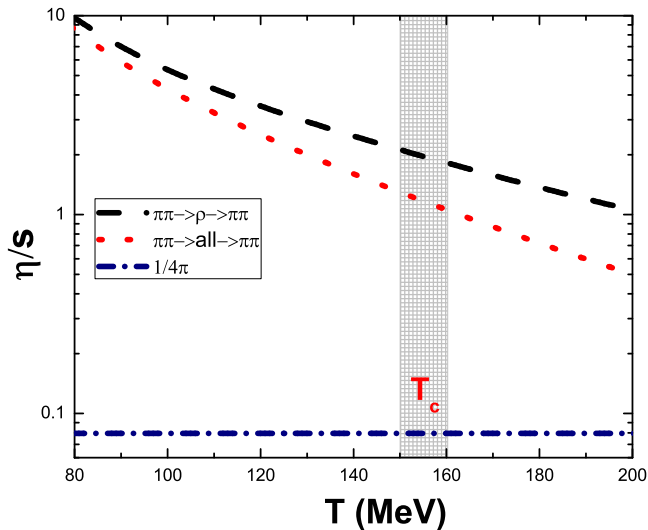
The coefficient of entropy density



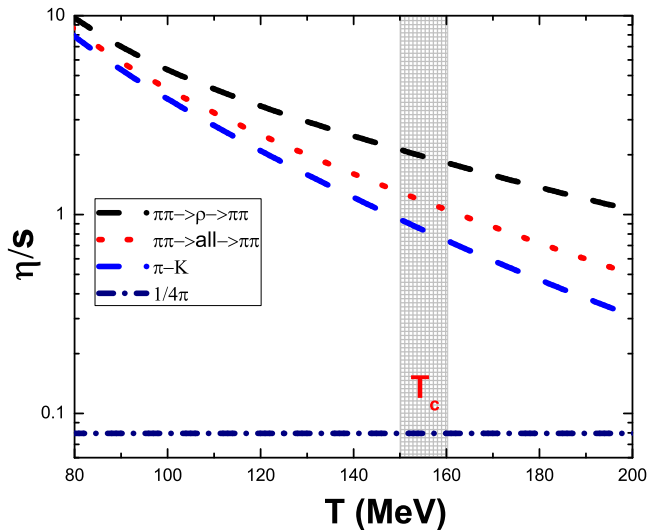
The ratio of η_s/s Mixture



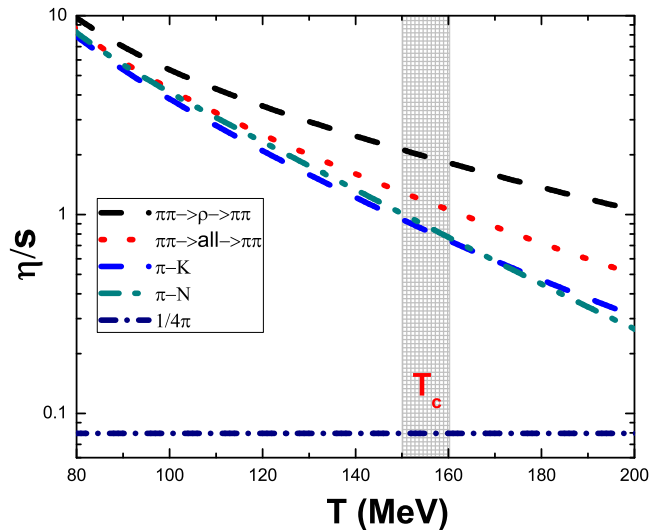
The ratio of η_s/s Mixture



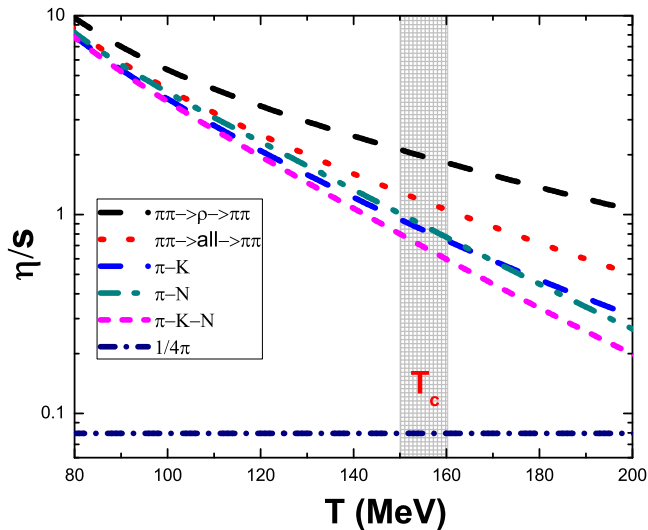
The ratio of η_s/s Mixture



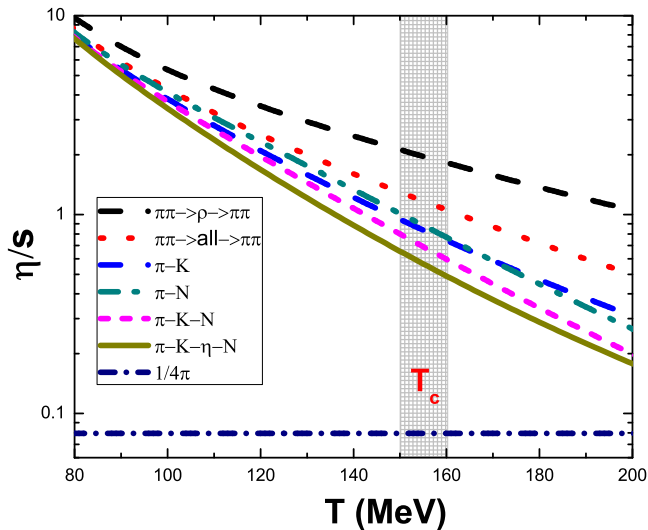
The ratio of η_s/s Mixture



The ratio of η_s/s Mixture



The ratio of η_s/s Mixture



Summary & Outlook

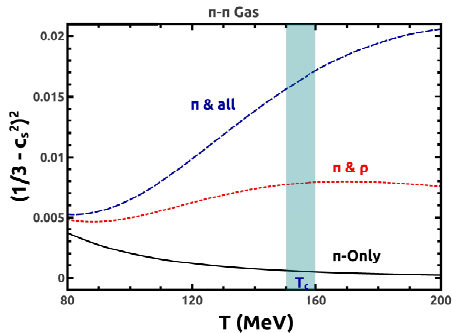
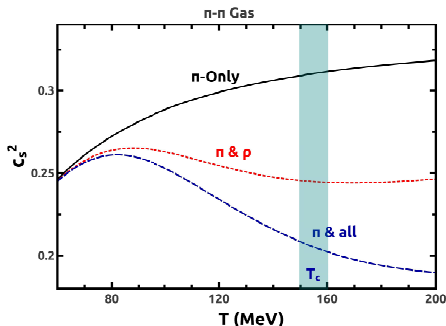
1 Summary:

- ▶ It is important to include massive resonances in the shear viscosity and entropy density calculation, particularly close to T_c . The magnitude of the shear viscosity is controlled by low-mass components of the mixture. Entropy density receive contribution also from the high mass components.
- ▶ For nearby resonance, the K -matrix preserve unitarity.

2 Outlook:

- ▶ Calculations for μ_b, μ_s not equal to zero case to address low energy scan at RHIC and future experiments.
- ▶ The ratio of bulk viscosity to entropy density.
- ▶ Hadronic energy loss.

Bulk Viscosity (Preliminary results)

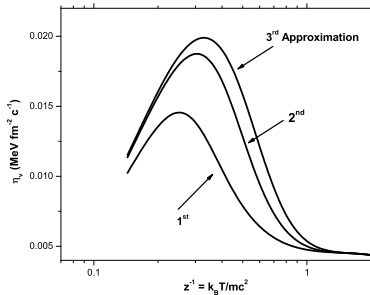
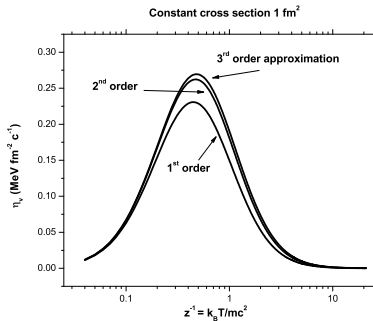


$$\eta_v = kT \frac{a^2 (1/3 - v_s^2)^2 + 2ab (1/3 - v_s^2) + b^2}{2\omega_0^{(2)}}, \quad (35)$$

where $z = m/T$, $a = -3/2(z\hat{h} + 1)$, $b = -1/2(z\hat{h} - 4)$ and $\hat{h} = K_3(z)/K_2(z)$.

Bulk Viscosity

$\pi - \pi \rightarrow$ Resonances $\rightarrow \pi - \pi$



Bulk viscosity & speed of sound

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Nearly massless particle or very high temperature $z \ll 1$.

as $z \rightarrow 0$; $z\hat{h} \rightarrow 4$, $a \rightarrow -\frac{15}{2}$ and $b \rightarrow 0$

$$\eta_v = \frac{225}{8} \frac{kT}{w_0^{(2)}} \left(\frac{1}{3} - v_s^2 \right)^2 \quad v_s^2 = 1/3 \rightarrow \eta_s = 0 \quad (37)$$

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Massive particle or very low temperature $z \gg 1$.

as $a \rightarrow -\frac{3}{2}z$ and $b \rightarrow -1/2z$

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