Stability analysis in relativistic hydrodynamics

BSc -Talk
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Contents

1. About stability

What is stability?

- We expect any oscillating system to fall back into equilibrium after a perturbation
- A single perturbation can not make a quantity of the system tend to infinity

Is relativistic hydrodynamics stable?

- The equations of **ideal** hydrodynamics are stable

- But: including **dissipative parts** you find

inconsistencies (Hiscock, Lindblom)

2. The equations of a relativistic hydrodynamic system

Governing equations

- Energy and momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

- Number current conservation

$$\partial_\mu N^\mu = 0$$

Ideal and dissipative parts

$$T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - P\Delta^{\mu\nu} + \pi^{\mu\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu}$$

$$N^{\mu} = n u^{\mu} + \nu^{\mu}$$

— ideal part
— dissipative part

The dissipative equations

- In first order theory:

$$\pi^{\mu\nu} = 2\eta P^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$

(unstable?)

$$q^\mu = -\kappa T \Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$ (Eckart frame)

$$\nu^\mu = \frac{\kappa n T}{\varepsilon + P}\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$ (Landau frame)

- In second order theory:

$$\tau_\pi P^{\mu\nu\alpha\beta}\dot{\pi}_{\alpha\beta} + \pi^{\mu\nu} = 2\eta P^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$

$$\tau_q \Delta^{\mu\nu}\dot{q}_\nu + q^\mu = -\kappa T \Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$ (Eckart frame)

$$\tau_\nu \Delta^{\mu\nu}\dot{\nu}_\nu + \nu^\mu = \frac{\kappa n T}{\varepsilon + P}\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$ (Landau frame)

Rest frame configurations

- Landau frame

$$u^{\mu} \equiv eigenvector\ of\ T^{\mu\nu}$$

$$\rightarrow q^{\mu} = 0$$

$$\Rightarrow T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - P\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\Rightarrow q^{\mu} = -\kappa T \Delta^{\mu\nu}\left[\frac{1}{T}\partial_{\nu} T + \dot{u}_{\nu}\right]$$

$$\Rightarrow \tau_q \Delta^{\mu\nu} \dot{q}_{\nu} + q^{\mu} = -\kappa T \Delta^{\mu\nu}\left[\frac{1}{T}\partial_{\nu} T + \dot{u}_{\nu}\right]$$

- Eckart frame

$$u^{\mu} \parallel \mathrm{N}^{\mu}$$

$$\rightarrow \nu^{\mu} = 0$$

$$\Rightarrow N^{\mu} = n u^{\mu}$$

$$\Rightarrow \nu^{\mu} = \frac{\kappa n T}{\varepsilon + P} \Delta^{\mu\nu}\left[\frac{1}{T}\partial_{\nu} T + \dot{u}_{\nu}\right]$$

$$\Rightarrow \tau_{\nu} \Delta^{\mu\nu} \dot{\nu}_{\nu} + \nu^{\mu} = \frac{\kappa n T}{\varepsilon + P} \Delta^{\mu\nu}\left[\frac{1}{T}\partial_{\nu} T + \dot{u}_{\nu}\right]$$

Sets of equations that are analysed

- Landau frame:

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left[\varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}\right]$$
$$\partial_\mu N^\mu = \partial_\mu \left[n u^\mu + \nu^\mu\right]$$

$+$ (first order)

$$\pi^{\mu\nu} = 2\eta P^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$
$$\nu^\mu = \frac{\kappa n T}{\varepsilon + P}\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$

or $+$ (second order)

$$\tau_\pi P^{\mu\nu\alpha\beta}\dot{\pi}_{\alpha\beta} + \pi^{\mu\nu} = 2\eta P^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$
$$\tau_\nu \Delta^{\mu\nu}\dot{\nu}_\nu + \nu^\mu = \frac{\kappa n T}{\varepsilon + P}\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$

- Eckart frame:

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left[\varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu\right]$$
$$\partial_\mu N^\mu = \partial_\mu \left[n u^\mu\right]$$

$+$ (first order)

$$\pi^{\mu\nu} = 2\eta P^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$
$$q^\mu = -\kappa T\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$

or $+$ (second order)

$$\tau_\pi P^{\mu\nu\alpha\beta}\dot{\pi}_{\alpha\beta} + \pi^{\mu\nu} = 2\eta P^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$
$$\tau_\nu \Delta^{\mu\nu}\dot{q}_\nu + q^\mu = -\kappa T\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right]$$

3. How to analyse the stability

Perturbating the equations

Take any quantity of the system and add a perturbation around the hydrostatic equilibrium:

$$Q = Q_0 + \delta Q \cdot e^{i\omega t - ikx}$$

$$\epsilon = \epsilon_0 + \delta\epsilon \cdot e^{i\omega t - ikx},$$

$$P = P_0 + \delta P \cdot e^{i\omega t - ikx},$$

$$n = n_0 + \delta n \cdot e^{i\omega t - ikx},$$

$$u^\mu = u_0^\mu + \delta u^\mu \cdot e^{i\omega t - ikx},$$

$$\pi^{\mu\nu} = \pi_0^{\mu\nu} + \delta\pi^{\mu\nu} \cdot e^{i\omega t - ikx},$$

$$q^\mu = q_0^\mu + \delta q^\mu \cdot e^{i\omega t - ikx},$$

$$\nu^\mu = \nu_0^\mu + \delta\nu^\mu \cdot e^{i\omega t - ikx}.$$

Deriving the perturbated equations

Example: $$\partial_\mu T^{\mu\nu} = \partial_\mu [\epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}] = 0$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} T^{00} + \frac{\partial}{\partial x} T^{x0} \\
&= \frac{\partial}{\partial t}\left\{\epsilon u^0 u^0 - P\Delta^{00} + \pi^{00}\right\} + \frac{\partial}{\partial x}\left\{\epsilon u^x u^0 - P\Delta^{x0} + \pi^{x0}\right\} \\
&= \frac{\partial}{\partial t}\left\{(\epsilon_0 + \delta\epsilon \cdot e^{i\omega t - ikx}) \cdot (u_0^0)^2 - (P_0 + \delta P \cdot e^{i\omega t - ikx})(1 - (u_0^0)^2) + \delta\pi^{00} \cdot e^{i\omega t - ikx}\right\} \\
&+ \frac{\partial}{\partial x}\left\{(\epsilon_0 + \delta\epsilon \cdot e^{i\omega t - ikx})\delta u^x \cdot e^{i\omega t - ikx} + (P_0 + \delta P \cdot e^{i\omega t - ikx})\delta u^x \cdot e^{i\omega t - ikx} + \delta\pi^{x0} \cdot e^{i\omega t - ikx}\right\} \\
&= \frac{\partial}{\partial t}[\delta\epsilon \cdot e^{i\omega t - ikx}] + \frac{\partial}{\partial x}[(\epsilon_0 + P_0)\delta u^x \cdot e^{i\omega t - ikx}] \\
&= i\omega\delta\epsilon \cdot e^{i\omega t - ikx} - ik(\epsilon_0 + P_0)\delta u^x \cdot e^{i\omega t - ikx},
\end{aligned}$$

$$\Leftrightarrow 0 = i\omega\delta\epsilon - ik(\epsilon_0 + P_0)\delta u^x$$

Formatting the linearized equations

$$A \cdot X = 0$$

Example:

$$A = \begin{pmatrix} i\omega & -ik(\epsilon_0 + P_0) & & & & & & & \\ -ikc_s^2 & i\omega(\epsilon_0 + P_0) & -ik & & & & & & \\ & -ik\frac{4}{3}\eta & 1 & & & & & & \\ & & & i\omega(\epsilon_0 + P_0) & -ik & & & & \\ & & & -ik\eta & 1 & & & & \\ & & & & & i\omega(\epsilon_0 + P_0) & -ik & & \\ & & & & & -ik\eta & 1 & & \\ & ik\frac{2}{3}\eta & & & & & & 1 & \\ & & & & & & & & 1 \end{pmatrix}$$

$$X = (\delta\epsilon, \delta u^x, \delta\pi^{xx}, \delta u^y, \delta\pi^{xy}, \delta u^z, \delta\pi^{xz}, \delta\pi^{yy}, \delta\pi^{yz})^T$$

Getting the dispersion relation

$$\underline{\omega(k)}$$

Calculate the determinant of A and set it equal to 0:

$$\text{Det } A = \text{Det}\begin{bmatrix} i\omega & -ik(\epsilon_0+P_0) & {} \\ -ikc_s^2 & i\omega(\epsilon_0+P_0) & -ik \\ {} & -ik\frac{4}{3}\eta & 1 \end{bmatrix} \cdot \text{Det}\begin{bmatrix} i\omega(\epsilon_0+P_0) & -ik \\ -ik\eta & 1 \end{bmatrix}^2 \cdot \text{Det}\begin{bmatrix} 1 & {} \\ {} & 1 \end{bmatrix} = 0$$

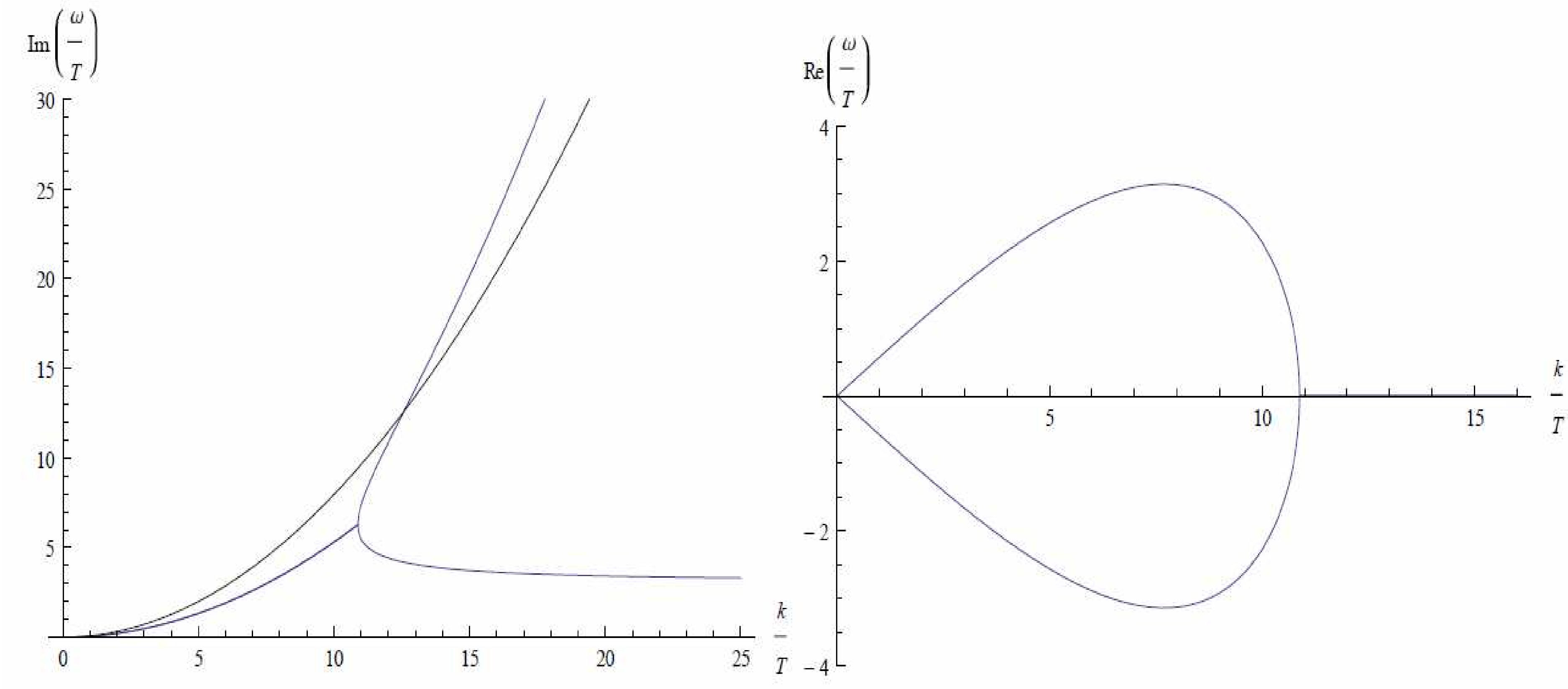
$$\downarrow$$

$$\text{Det}\begin{bmatrix} i\omega & -ik(\epsilon_0+P_0) & {} \\ -ikc_s^2 & i\omega(\epsilon_0+P_0) & -ik \\ {} & -ik\frac{4}{3}\eta & 1 \end{bmatrix} = 0$$

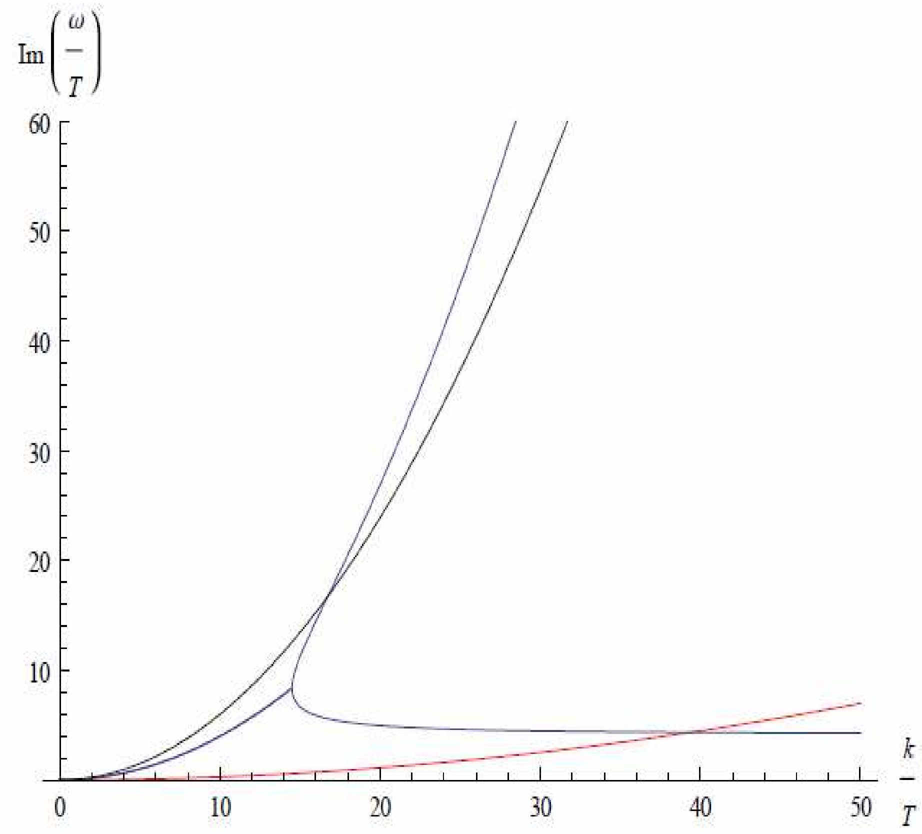
$$\text{Det}\begin{bmatrix} i\omega(\epsilon_0+P_0) & -ik \\ -ik\eta & 1 \end{bmatrix} = 0 \quad \longrightarrow \quad \omega(k) = \frac{ik^2\eta}{\epsilon_0+P_0}$$

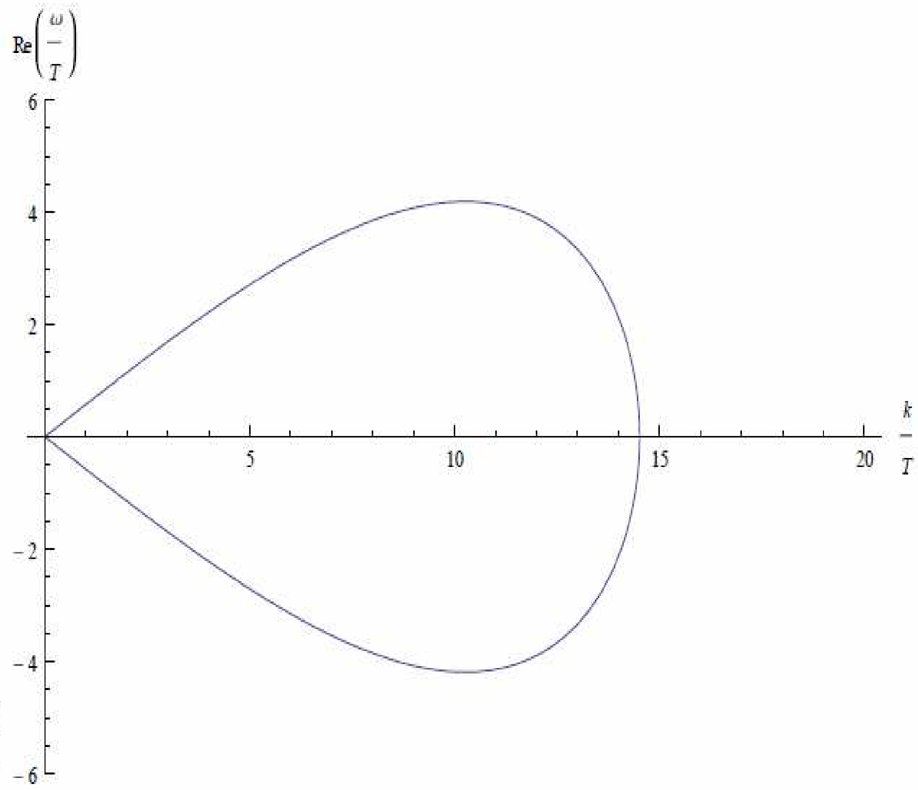
4. Results

First order Landau frame with $N^{\mu} = 0$:



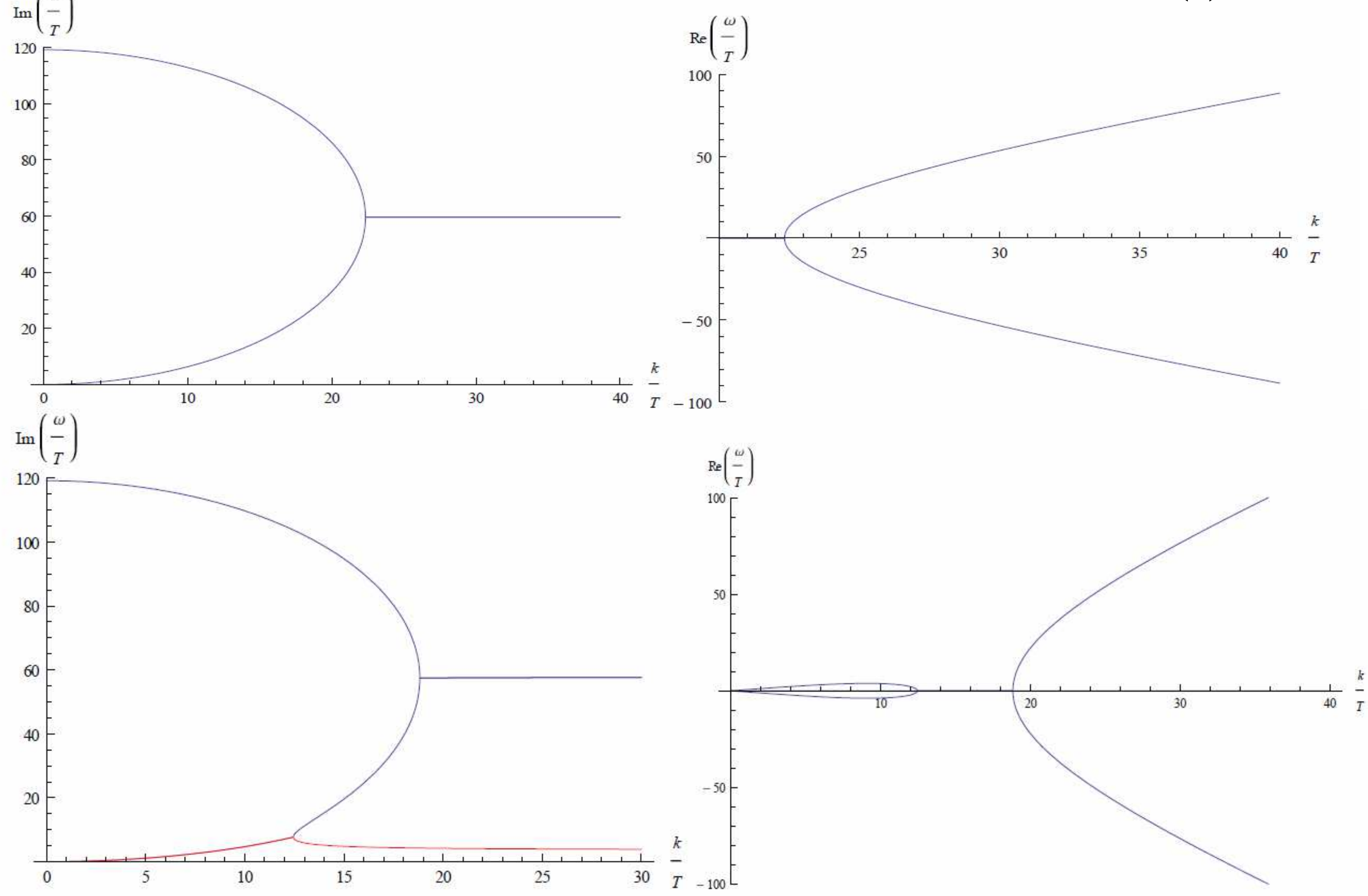
First order Landau Frame:





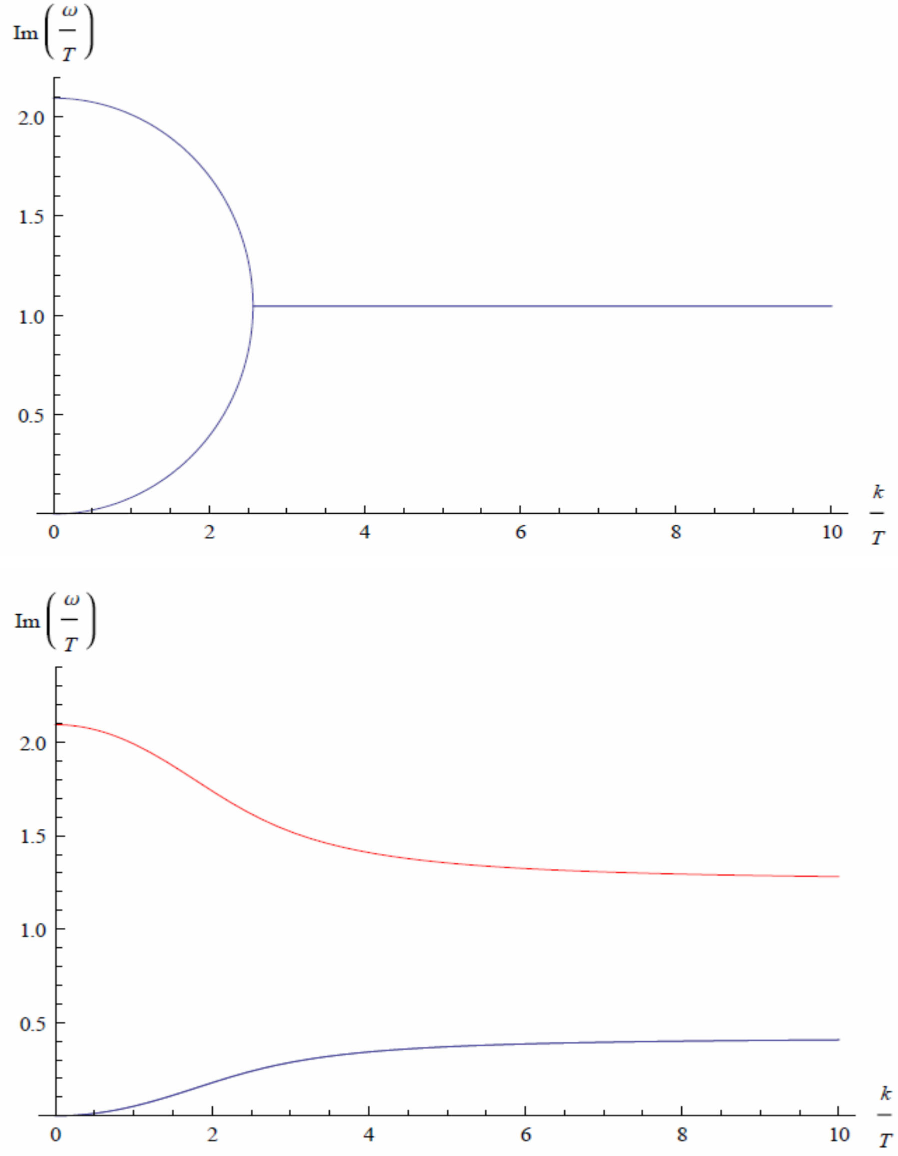
First oder Eckart frame:

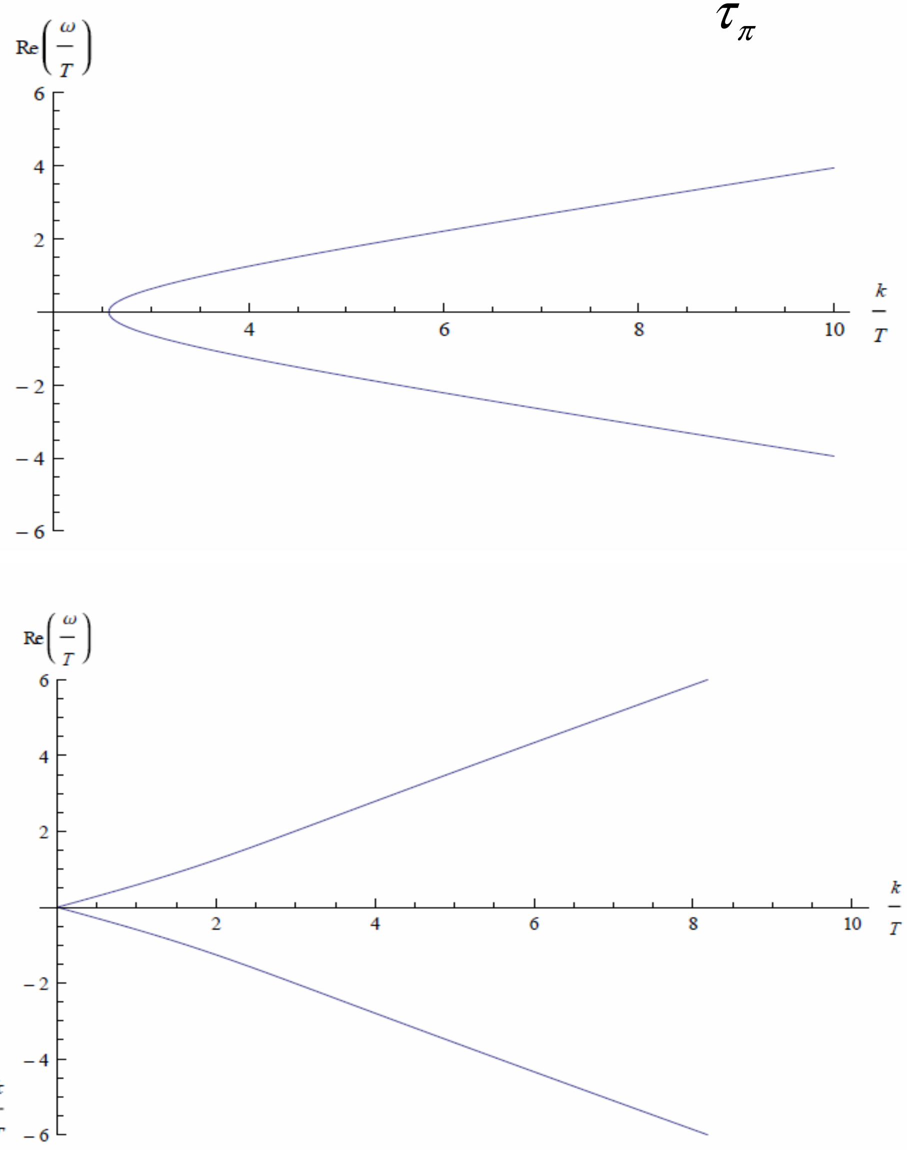
Additional solution: $\omega(k) = 0$



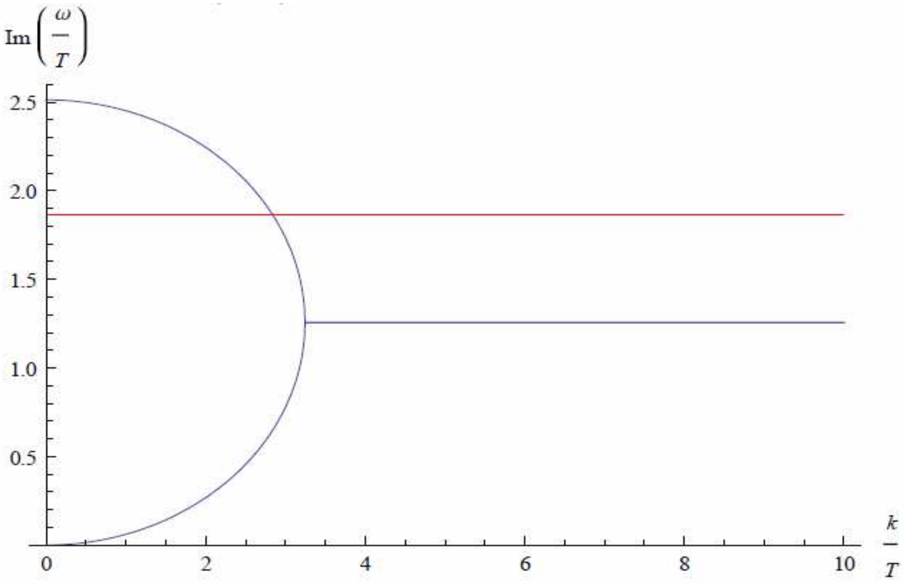
Second order Landau frame with $N^{\mu} = 0$:

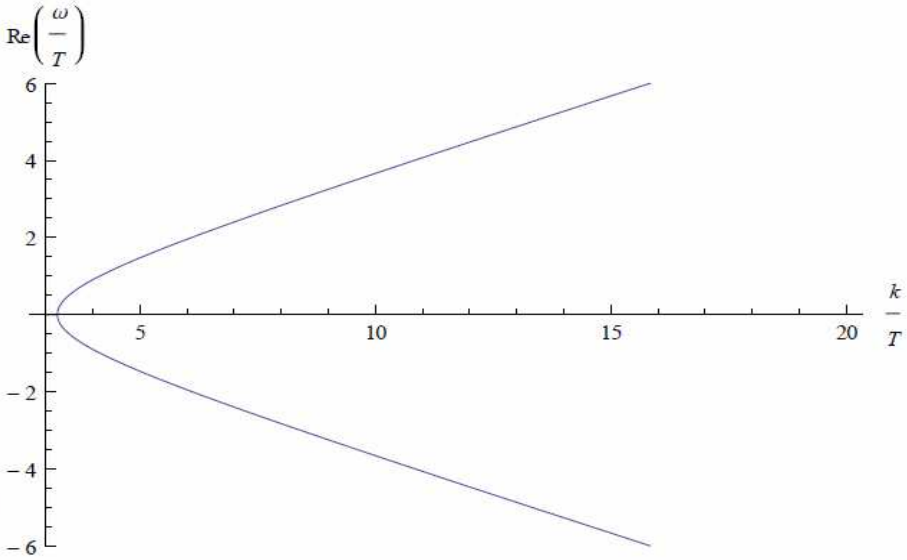
Additional solution: $\omega(k) = \dfrac{i}{\tau_{\pi}}$

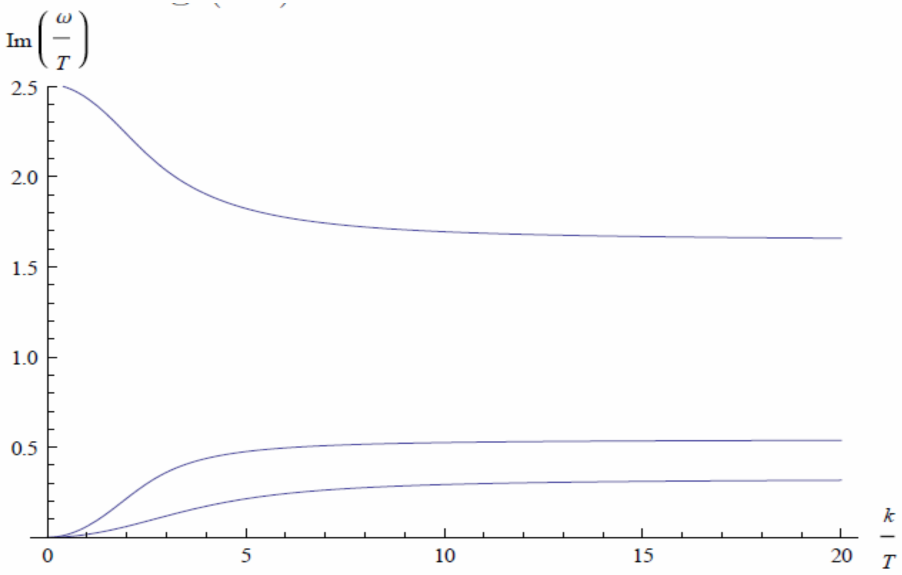


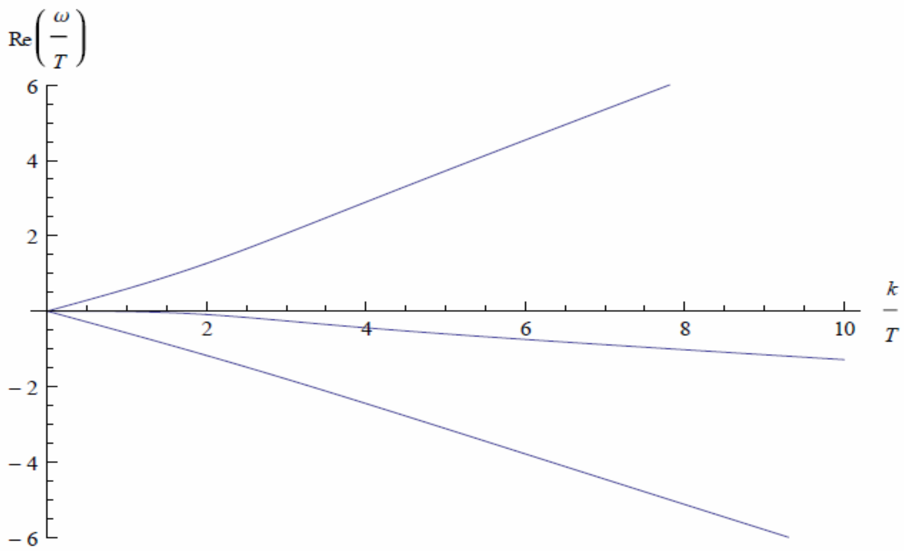


Second order Landau frame (1):

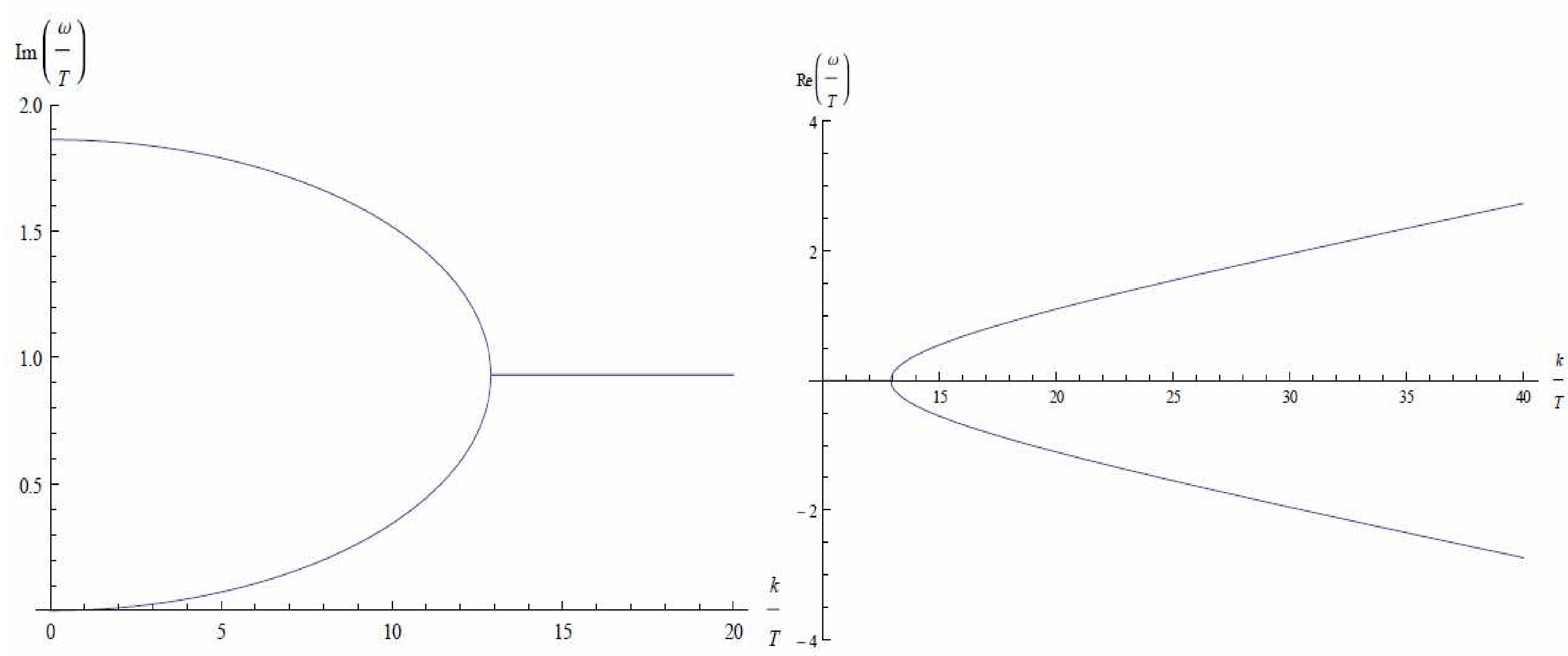




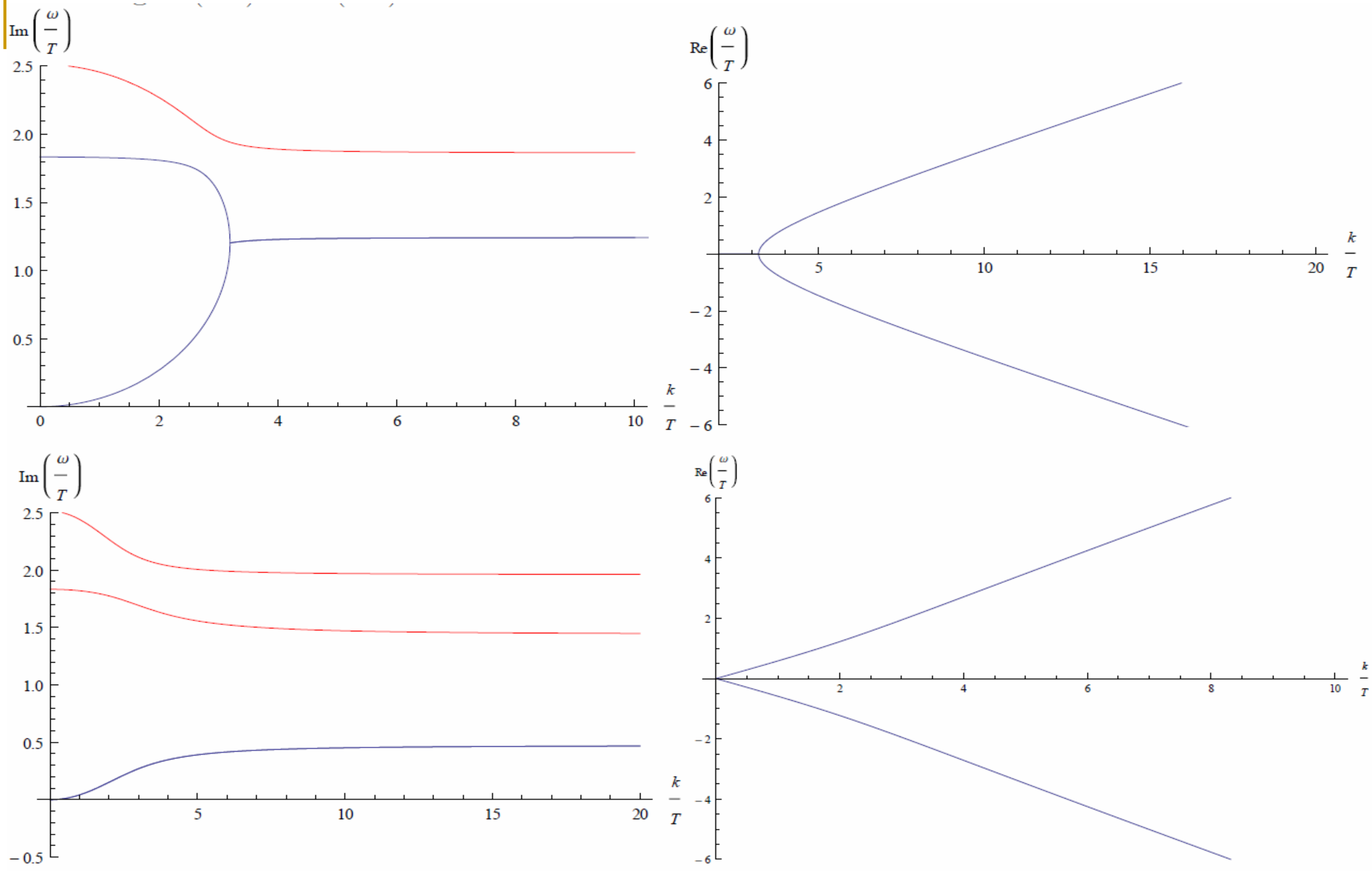




Second order Landau frame (2):



Second order Eckart frame:



5. Conclusions

- First oder: instabilites occur in a general rest frame configuration (W. Hiscock, L. Lindblom).

  But: If you choose a distinguished rest frame configuration you possibly find stable equations.

  Still: Causality issues appear. And: Away from the rest frame the first order equations generate instabilities (W. Hiscock, L. Lindblom).
- Second order: No instabilities found; stability is independent from frame configurations (even away from the rest frame – Shi Pu, T. Koide, D. Rischke).