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Exercise Sheet 11 – Solutions

$\pi^+ \rightarrow \mu^+ + \nu_\mu$

We combine our knowledge about the strong interactions of pions (here charged pions) and PCAC as well as the description of the weak interactions. For these low-energy (here the relevant "electroweak energy scale" is $M_{\text{W}} = (80.2625 \pm 0.0077)\,\text{GeV}$ [T⁺26]) the Fermi-theory like current-current coupling approximation is fully sufficient. The relevant piece of the Lagrangian reads (on the quark level)

$$\mathscr{L} = \frac{G_{\text{F}}}{\sqrt{2}} \left(\overline{u}[\gamma^\lambda(1-\gamma_5)]d'\right)\left(\overline{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu\right) + \text{h.c.} \tag{1}$$

Note that here we write the weak charged quark current in terms of the flavor eigenstates of the down-like quarks (d', s', b') with the CKM-quark-mixing matrix (CKM=Cabibbo, Kobayashi, and Maskawa)

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \hat{V} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \quad \hat{V}^\dagger \hat{V} = \mathbb{1}_3. \tag{2}$$

For the decay of the charged pion, $\pi^+ \rightarrow \mu + \nu_\mu$ we need the invariant matrix element

$$S_{fi} = \left\langle \mu^+(p_1), \nu_\mu(p_2) \middle| \text{S} \middle| \pi^+(p) \right\rangle = -\mathrm{i}(2\pi)^4 \delta^{(4)}(P_f - P_i)\mathscr{M}_{fi}. \tag{3}$$

The relation of the charged-pion fields to the real pion fields in the σ-model discussed in Lect. 9 is $\pi^\pm = (\pi^1 \mp \mathrm{i} p^2)/\sqrt{2}$ the corresponding axialvector isovector currents are

$$J_{\text{A}}^{\pm\mu} = J_{\text{A}}^{1\mu} \mp \mathrm{i} J_{\text{A}}^{2\mu}. \tag{4}$$

The PCAC relation for the real pion fields defines the decay constant $F_\pi \simeq 92\,\text{MeV}$ via

$$\left\langle \Omega \middle| J_{\text{A}}^{a\lambda}(x=0) \middle| \pi^b(p) \right\rangle = \mathrm{i}\delta_{ab} p^\lambda F_\pi. \tag{5}$$

(a) Show that the the relation for the *positively charged* pion is

$$\left\langle \Omega \middle| J_{\text{A}}^{+\mu}(x=0) \middle| \pi^+(p) \right\rangle = \mathrm{i}\delta_{ab} p_\mu \sqrt{2} F_\pi. \tag{6}$$

Solution: The momentum-eigenstate is (with our normalization convention)

$$\left|\pi^+(p)\right\rangle = \sqrt{(2\pi)^3 2E_p} \frac{1}{\sqrt{2}} (\mathbf{a}^1 - \mathrm{i}\mathbf{a}^2)^\dagger \left|\omega\right\rangle = \frac{1}{\sqrt{2}} \left(\left|\pi^1(p)\right\rangle + \mathrm{i} \left|\pi^2(p)\right\rangle \right) \tag{7}$$

With this we get

$$\begin{aligned} \left\langle \Omega \middle| J_{\text{A}}^{+\lambda}(x=0) \middle| \pi^+(p) \right\rangle &= \frac{1}{\sqrt{2}} \left\langle \Omega \middle| [J_{\text{A}}^{1\lambda}(x=0) - \mathrm{i} J_{\text{A}}^{2\lambda}(x=0)] \middle| \pi^1(p) + \mathrm{i}\pi^2(p) \right\rangle \\ &= \frac{1}{\sqrt{2}} \mathrm{i} p^\lambda F_\pi 2 = \mathrm{i} p^\lambda \sqrt{2} F_\pi = \mathrm{i} p^\lambda f_\pi. \end{aligned} \tag{8}$$

Note that we use $F_\pi \simeq 92.2\,\text{MeV}$, while the PDG [T⁺26] uses $f_\pi = \sqrt{2}F_\pi \simeq 130.4\,\text{MeV}$

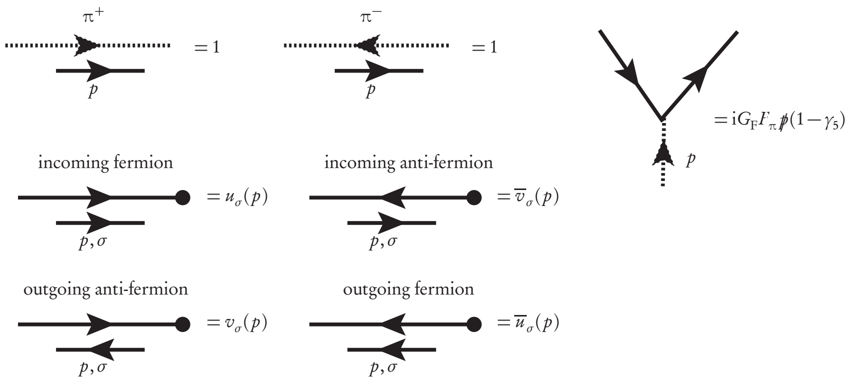
(b) Now use that for the weak current of the pion only the axial piece with $\overline{u}$- and d-fields with the γ_5-matrix in this current contributes (why?) and that this leads to the effective Lagrangian

$$\mathscr{L}_{\text{eff}} = G_{\text{F}} F_\pi (\partial_\lambda \pi^-) V_{ud} \left(\overline{\mu}\gamma^\lambda(1-\gamma_5)\nu_\mu\right) + \text{h.c.} \tag{9}$$

where μ and ν_μ are the Dirac spinor describing muons and muon neutrinos, respectively.

Solution: Since the pion is *pseudo*scalar isovector particle, the overlap of the pion states with the vector isovector vanishes. Now using the PCAC relation (8) "translating" the quark currents to the vector currents, as detailed in the lecture, we immediately get (9).

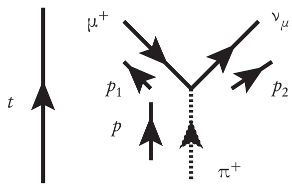
(c) use the resulting Feynman rules,



where the fermions can be $\mu^\pm$ or ν_μ and $\overline{\nu}_\mu$, to evaluate the invariant matrix element $\mathscr{M}_{fi}$ for the charged-pion decay $\pi^+ \to \mu^+ + \nu_\mu$.

Hint: Use that the momenta of the particles are on-shell and $p = p_1 + p_2$ (where p_1 is the four-momentum of the muon and p_2 the four-momentum of the neutrino, which is assumed to be massless.

Solution: The Feynman diagram looks like this



With our orientation of the time axis pointing upwards, we have to read the Feynman diagram from top to bottom. The order of the u/v-spinors has to be read against the direction of the current, indicated by the arrows on the fermion lines. We thus get, using the rules for the external lines and the vertex,

$$\begin{aligned}
\mathscr{M}_{fi} &= \mathrm{i}G_\mathrm{F}F_\pi V_{ud}\overline{u}_{\nu_\mu,\sigma_2}(p_2)\not{p}(1-\gamma_5)v_{\mu,\sigma_1}(p_1) \\
&= \mathrm{i}G_\mathrm{F}F_\pi V_{ud}\overline{u}_{\nu_\mu,\sigma_2}(p_2)(\not{p}_1+\not{p}_2)(1-\gamma_5)v_{\mu,\sigma_1}(p_1) \\
&\overset{(12)}{=} \mathrm{i}G_\mathrm{F}F_\pi V_{ud}\overline{u}_{\nu_\mu,\sigma_2}(p_2)\not{p}_2(1-\gamma_5)v_{\mu,\sigma_1}(p_1) \\
&= \mathrm{i}G_\mathrm{F}F_\pi V_{ud}\overline{u}_{\nu_\mu,\sigma_2}(p_2)(1+\gamma_5)\not{p}_2 v_{\mu,\sigma_1}(p_1) \\
&\overset{(12)}{=} -\mathrm{i}G_\mathrm{F}F_\pi V_{ud}m_\mu\overline{u}_{\nu_\mu,\sigma_2}(p_2)(1+\gamma_5)v_{\mu,\sigma_1}(p_1)
\end{aligned} \tag{10}$$

(d) Evaluate the unpolarized matrix element, i.e., $\overline{|\mathscr{M}_{fi}|^2}$ which is summed over the spins of the outgoing muon and neutrino. Then you can use

$$\not{p}_1 v_{\mu,\sigma}(p_1) = -m_\mu v_{\mu,\sigma}, \quad \overline{u}_{\nu_\mu,\sigma}(p_2)\not{p}_2 = 0. \tag{11}$$

Useful "Diracology": Spin sums

$$\sum_\sigma u_\sigma(p)\overline{u}_\sigma(p) = \not{p} - m, \quad \sum_\sigma v_\sigma(p)\overline{v}_\sigma(p) = \not{p} + m, \tag{12}$$

where m is the mass of the particle/antiparticle described by u/v. Also the following traces over Dirac-matrix expressions are needed:

$$\mathrm{tr}(\not{p}_1\not{p}_2) = 4p_1\cdot p_2, \quad \mathrm{tr}\,\gamma^\mu = \mathrm{tr}\,\gamma_5 = 0 = \mathrm{tr}(\gamma_5\gamma^\mu) = \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0. \tag{13}$$

Solution: To get $|\mathcal{M}_{fi}|^2$ we make use of the "Diracology rule" that the conjugate complex of a Dirac-spinor expression is given by the Dirac adjoint. Using $\overline{\gamma}_5 = -\gamma_5$ we thus find, summing also over the spins of the anti-muon and the neutrino in the final state

$$\begin{aligned}
\overline{|\mathcal{M}_{fi}|^2} &= G_{\mathrm{F}}^2 F_\pi^2 |V_{ud}|^2 m_\mu^2 \sum_{\sigma_1\sigma_2} \overline{u}_{\nu_\mu,\sigma_2}(p_2)(1+\gamma_5)v_{\mu,\sigma_1}(p_1)\overline{v}_{\mu,\sigma_1}(p_1)(1-\gamma_5)u_{\nu_\mu,\sigma_2}(p_2) \\
&\stackrel{(12)}{=} G_{\mathrm{F}}^2 F_\pi^2 |V_{ud}|^2 m_\mu^2 \operatorname{tr}\left[\not{p}_2(1+\gamma_5)(\not{p}_1+m_\mu)(1-\gamma_5)\right] \\
&= G_{\mathrm{F}}^2 F_\pi^2 |V_{ud}|^2 m_\mu^2 \operatorname{tr}\left[\not{p}_2(\not{p}_1+m_\mu)(1-\gamma_5)^2\right] \\
&= 2G_{\mathrm{F}}^2 F_\pi^2 |V_{ud}|^2 m_\mu^2 \operatorname{tr}\left[\not{p}_2(\not{p}_1+m_\mu)(1-\gamma_5)\right] \\
&\stackrel{(13)}{=} 8G_{\mathrm{F}}^2 F_\pi^2 |V_{ud}|^2 m_\mu^2 p_1 \cdot p_2.
\end{aligned} \tag{14}$$

Now we use that

$$p^2 = m_\pi^2 = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m_\mu^2 + 2p_1 \cdot p_2 \Rightarrow p_1 \cdot p_2 = \frac{1}{2}(m_\pi^2 - m_\mu^2), \tag{15}$$

which finally gives

$$\overline{|\mathcal{M}_{fi}|^2} = 4G_{\mathrm{F}}^2 F_\pi^2 |V_{ud}|^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \tag{16}$$

(e) Finally use the formula for the differential decay width (see Sect. 49.4.2 of [T+26])

$$\mathrm{d}\Gamma = \frac{1}{64\pi^2} \frac{m_\pi^2 - m_\mu^2}{m_\pi^3} \overline{|\mathcal{M}_{fi}|^2} \mathrm{d}\Omega \tag{17}$$

to derive the total decay width and the lifetime of the charged pion (the branching ratio to this decay mode is very close to 100%). For comparison: the empirical value for the lifetime is $\tau_{\pi^+} = 2.6033 \cdot 10^{-8}$ s. The small difference to the above derived theoretical result is due to higher-order corrections.

Solution: Using (16) in (17) and noting the $\mathcal{M}_{fi}$ does not depend on any angles of the final-state vectors integration over the solid angle simply gives a factor 4π. Thus the total decay width is

$$\Gamma = \frac{1}{4\pi} G_{\mathrm{F}}^2 F_\pi^2 m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2. \tag{18}$$

Plugging in the values of the constants, given below, we finally find

$$\Gamma = 2.479 \cdot 10^{-17}\,\mathrm{GeV}, \quad \tau = \frac{\hbar c}{\Gamma c} \simeq \frac{0.197\,\mathrm{GeV\,fm}}{\Gamma c} = 2.65 \cdot 10^{-8}\,\mathrm{s}. \tag{19}$$

(f) Calculate the branching ratio $\Gamma_{\pi^+ \rightarrow e^+ + \nu_e} / \Gamma_{\pi^+ \rightarrow e^+ + \nu_e}$.

Solution: The only difference in the same calculation for the $\pi^+ \rightarrow e^+ + \nu_e$-decay is that in the final formula we have to substitute m_e instead of m_μ, leading to

$$\frac{\Gamma_{\pi^+ \rightarrow e^+ + \nu_e}}{\Gamma_{\pi^+ \rightarrow \mu^+ + \nu_\mu}} = \frac{m_e^2}{m_\mu^2} \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \simeq 1.283 \cdot 10^{-4} \tag{20}$$

vs. the empirical value of $(1.230 \pm 0.004) \cdot 10^{-4}$.

Values of the constants: $G_{\mathrm{F}} = 1.16637908 \cdot 10^{-5}\mathrm{GeV}^{-2}$, $F_\pi = 92.2$ MeV, $|V_{ud}| = 0.9740$, $m_{\pi^+} = 139.5704$ MeV, $m_\mu = 105.6584$ MeV, $m_e = 0.511$ MeV.

References

[T+26] F. Takahashi et al. (Particle Data Group), Review of Particle Physics, Int. J. Mod. Phys. A **41**, 2630011 (2026), doi:10.1142/S0217751X26300115, `https://pdg.lbl.gov`.