Einführung in die Theoretische Kern- und Elementarteilchenphysik 2

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Exercise Sheet 10

Linear σ model

We start from the QCD Lagrangian with two quark flavors $\psi = (u, d)^{\mathrm{T}}$,

$$\mathscr{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \overline{\psi}(\mathrm{i}\not{\partial} - m)\psi, \tag{1}$$

where we have approximated the mass term by setting $m_{\mathrm{u}} = m_{\mathrm{d}} = m$, where $m = (m_{\mathrm{u}} + m_{\mathrm{d}})/2 \simeq 3.43\,\mathrm{MeV}$[1].

(a) First we determine the transformation properties of the fields, representing the lightest scalar isoscalar and pseudoscalar isovector particles, i.e., the σ-meson and the three pions, by identifying the corresponding (real scalar or pseudoscalar) fields as

$$\Phi_0 = \overline{\psi}\psi, \quad \pi^a = \mathrm{i}\overline{\psi}\gamma_5\tau^a\psi, \tag{2}$$

where the $\tau^a \in \mathbb{C}^{2\times 2}$ are the Pauli matrices acting in flavor space.

Determine, how these fields transform under infinitesimal vector and axial isovector transformations[2], given that the quark fields transform as

$$\mathrm{V}: \quad \delta\psi = -\frac{\mathrm{i}}{2}\delta\alpha^a\tau^a\psi, \quad \delta\overline{\psi} = +\frac{\mathrm{i}}{2}\delta\alpha^a\overline{\psi}\tau^a, \tag{3}$$

$$\mathrm{A}: \quad \delta\psi = -\frac{\mathrm{i}}{2}\delta\beta^a\gamma_5\tau^a\psi, \quad \delta\overline{\psi} = -\frac{\mathrm{i}}{2}\delta\beta^a\overline{\psi}\gamma_5\tau^a. \tag{4}$$

Show that the resulting transformations lead to the o(4) symmetry[3]

$$\delta(\Phi_0^2 + \vec{\pi}^2) = 0. \tag{5}$$

Some helpful formulae

$$\begin{gathered}\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad \gamma_5^2 = \mathbb{1}, \\ \left[\tau^a, \tau^b\right] = \mathrm{i}\epsilon^{abc}\tau^c, \quad \left\{\tau^a, \tau^b\right\} = 2\delta^{ab}.\end{gathered} \tag{6}$$

In our simplified version of two-flavor QCD by setting the u- and d-quark masses equal, of course (1) is invariant under the vector isovector transformations $\mathrm{SU}(2)_{\mathrm{V}}$.

Solution: For the V transformation we find, using (3)

$$\delta\Phi_0 = \delta\overline{\psi}\psi + \overline{\psi}\delta\psi = \frac{\mathrm{i}}{2}\delta\vec{\alpha}\cdot(\overline{\psi}\vec{\tau}\psi - \overline{\psi}\vec{\tau}\psi) = 0, \tag{7}$$

$$\begin{aligned}\delta\pi^a &= \mathrm{i}(\delta\overline{\psi}\tau^a\gamma_5\psi - \overline{\psi}\tau^a\psi) \\ &= +\frac{\delta\alpha^b}{2}\cdot(\overline{\psi}\left[\tau^a, \tau^b\right]\gamma_5\psi \\ &= \mathrm{i}\epsilon^{abc}\delta\alpha^b\pi^c = (\delta\vec{\alpha}\times\vec{\pi})^c.\end{aligned} \tag{8}$$

[1] According to [N⁺24] the light-quark masses are $m_{\mathrm{u}} = (2.16 \pm 0.07)$ MeV, $m_{\mathrm{d}} = (4.70 \pm 0.07)$ MeV, and $m_{\mathrm{s}} = (93.5 \pm 0.8)$ MeV (but here we consider only the u- and d-quarks).

[2] together spanning the $\mathrm{su}(2)_{\mathrm{L}} \oplus \mathrm{sU}(2)_{\mathrm{R}}$ "=Lie algebra of the isovector part of the chiral symmetry, which is only slightly broken by the quark-mass term in (1).

[3] Of course this o(4) Lie algebra is a representation, acting on the meson fields, of the original $\mathrm{su}(2)_{\mathrm{L}} \oplus \mathrm{su}(2)_{\mathrm{R}}$ Lie algebra, acting on the quark fields.

For the V transformation we find, using (4)

$$\delta\Phi_0 = \delta\overline{\psi}\psi + \overline{\psi}\delta\psi = -\mathrm{i}\frac{\delta\vec{\beta}}{2}(2\overline{\psi}\vec{\tau}\gamma_5\psi) = -\delta\vec{\beta}\cdot\vec{\pi}, \tag{9}$$

$$\begin{aligned}\delta\pi^a &= \mathrm{i}(\delta\overline{\psi}\tau^a\gamma_5\psi - \overline{\psi}\tau^a\psi)\\ &= \frac{\delta\beta^b}{2}\left(\overline{\psi}\gamma_5\tau^b\gamma_5\tau^a\psi + \overline{\psi}\gamma_5\tau^a\tau^b\gamma_5\psi\right)\\ &\frac{\delta\beta^b}{2}\overline{\psi}\left\{\tau^a,\tau^b\right\}\psi = \delta\beta^a.\end{aligned} \tag{10}$$

Concerning (5) from this we get for the V transformation

$$\delta(\Phi_0^2 + \vec{\pi}^2) = 2\Phi_0\delta\Phi_0 + 2\vec{\pi}\cdot\delta\vec{\pi} = 2\vec{\pi}\cdot(\delta\vec{\alpha}\times\vec{\pi}) = 0 \tag{11}$$

and for the A transformation

$$\delta(\Phi_0^2 + \vec{\pi}^2) = 2\Phi_0\delta\Phi_0 + 2\vec{\pi}\cdot\delta\vec{\pi} = -2\Phi_0\delta\vec{\beta}\cdot\vec{\pi} + 2\vec{\pi}\cdot\vec{\beta} = 0. \tag{12}$$

(b) Next we consider the mesonic part of the linear σ model [GML60], including the chiral-symmetry breaking term. We introduce the four-dimensional SO(4) vector of real scalar fields, $\tilde{\Phi} = (\Phi_0, \vec{\pi})$. Then we write down the Lagrangian (for a renormalizable) field theory, which is approximately symmetric under the SO(4) representation of the chiral symmetry found above:

$$\mathscr{L}_{\mathrm{M}} = \frac{1}{2}(\partial_\mu\tilde{\Phi})\cdot(\partial^\mu\tilde{\Phi} - V(\tilde{\Phi}) \tag{13}$$

with the potential,

$$V = \frac{\lambda}{4}\left(\tilde{\Phi}^2 - v_0^2\right)^2 - \epsilon\Phi_0, \tag{14}$$

where we have used the scalar-product notation $\tilde{\Phi}\cdot\tilde{\Phi} = \tilde{\Phi}^2 = \Phi_0^2 + \vec{\pi}^2$.

Show that the ground state is uniquely given by $(F_\pi, 0, 0, 0)^{\mathrm{T}}$ and determine v_0^2 in terms of F_π and ϵ[4].

Solution: To find the minimum of V we calculate the derivatives wrt. the fields

$$\frac{\partial V}{\partial\Phi_0} = \lambda(\tilde{\Phi}^2 - v_0^2)\Phi_0 - \epsilon, \tag{15}$$

$$\frac{\partial V}{\partial\vec{\pi}} = \lambda(\tilde{\Phi}^2 - v_0^2)\vec{\pi}. \tag{16}$$

Setting both expressions 0 for $\Phi_0 = F_\pi$ to find the minimum, from (15) follows that $\tilde{\Phi}^2 - v_0^2 \neq 0$ and thus from (16) that $\vec{\pi} = 0$. Then (15) becomes with $\Phi_0 = F_\pi$ and $\vec{\pi} = 0$

$$\lambda F_\pi(F_\pi^2 - v_0^2) - \epsilon = 0 \Rightarrow v_0^2 = F_\pi^2 - \frac{\epsilon}{\lambda F_\pi}. \tag{17}$$

(c) Set now

$$\tilde{\Phi}(\underline{x}) = \begin{pmatrix} F_\pi + \sigma(\underline{x}) \\ \vec{\pi}(\underline{x}) \end{pmatrix} \tag{18}$$

and write the Lagrangian (13) in terms of the fields σ and $\vec{\pi}$. Show in that way that the model parameters are related to the σ and pion masses by

$$m_\sigma^2 = \frac{1}{2}\lambda F_\pi^2 + \frac{\epsilon}{4F_\pi}, \quad m_\pi^2 = \frac{\epsilon}{f\pi}. \tag{19}$$

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[4]That the vacuum expectation value of the Φ_0 field should be the pion-decay constant $F_\pi \simeq 93$ MeV is dictated by phenomenology, e.g., the $\pi^0 \to \gamma + \gamma$ decay rate, discussed in the previous exercise.

Determine the value of the "quark condensate", $-\left\langle \Omega \middle| \overline{\psi}\psi \middle| \Omega \right\rangle$ from the Gell Mann-Oakes-Renner relation (from the "PCAC analysis" given in the lecture)[5]

$$f_\pi^2 m_\pi^2 = -m \left\langle \Omega \middle| \overline{\psi}\psi \middle| \Omega \right\rangle. \tag{20}$$

Solution: Plugging (18) into the potential and using (17) for v_0 we get the potential expressed with the new fields (usually called the potential "after spontaneous symmetry breaking") and ordered in powers of the fields:

$$\begin{aligned} V =& \frac{\epsilon^2}{4\lambda^2 F_\pi^2} - \epsilon F_\pi & (21) \\ &+ \underbrace{\left(\lambda F_\pi^2 + \frac{\epsilon}{2F_\pi}\right)}_{m_\sigma^2/2} \sigma^2 + \underbrace{\frac{\epsilon}{2F_\pi}}_{m_\pi^2} \vec{\pi}^2 & (22) \\ &+ \lambda F_\pi \sigma(\sigma^2 + \vec{\pi}^2) & (23) \\ &+ \frac{\lambda}{4}(\sigma^2 + \vec{\pi})^2. & (24) \end{aligned}$$

The field-independent term (26) is of course physically irrelevant and can be abandoned. In the 2nd line we have he quadratic terms in the fields, i.e., the mass terms, from which we find

$$m_\sigma^2 = 2\lambda F_\pi^2 + \frac{\epsilon}{F_\pi} = 2\lambda F_\pi^2 + m_\pi, \quad m_\pi = \frac{\epsilon}{f_\pi}. \tag{25}$$

Eqs. (31) and (32) are cubic and quartic meson interactions with a specific pattern, dictated by the underlying chiral symmetry, of which in this form one does not see much anymore.

(d) Now consider the nucleon part of the Lagrangian of the Gell-Mann Levy model with only the nucleons, p and n, (with parity +1), using the hadronic isodoublet $\Psi = (p, n)^{\mathrm{T}}$ with the Dirac spinors p and n being the proton and nucleon fields,

$$\mathscr{L}_{\text{nucl}} = \overline{\Psi} \mathrm{i} \not{\partial} \Psi - g_{\pi NN} \left[\Phi_0 \overline{\Psi}\Psi + \mathrm{i}\vec{\pi} \cdot \overline{\Psi}\gamma_5 \vec{\tau}\psi \right]. \tag{26}$$

Why is it chirally symmetric and how get the nucleons their mass? How do the vector and axialvector isovector currents look like? What is the nucleon mass in this minimal linear-σ model?

Hint: To determine the currents, it is most convenient to vary the action with infinitesimal symmetry transformations for *local* transformations, i.e., using space-time-dependent infinitesimal group parameters, $\delta\vec{\alpha}(\underline{x})$ and $\delta\vec{\beta}(\underline{x})$, and noting that under these transformations only $\mathscr{L}_{\text{kin}}$ and $\mathscr{L}_{\text{SB}}$ with

$$\mathscr{L}_{\text{kin}} = \frac{1}{2}(\partial_\mu \tilde{\Phi}) \cdot (\partial_\mu \tilde{\Phi}) + \mathrm{i}\overline{\Psi} \not{\partial} \Psi, \quad \mathscr{L}_{\text{SB}} = \epsilon \Phi_0 \tag{27}$$

are *not* invariant under the transformations and that then the variation of $\mathscr{L}$ under the vector and axialvector isovector transformation reads

$$\delta_{\mathrm{V}} \mathscr{L} = (\partial_\mu \delta\vec{\alpha}) \cdot \vec{j}_{\mathrm{V}}^\mu, \tag{28}$$

$$\delta_{\mathrm{A}} \mathscr{L} = (\partial_\mu \delta\vec{\beta}) \cdot \vec{j}_{\mathrm{A}}^\mu + \delta\mathscr{L}_{\text{SB}}. \tag{29}$$

Note: The result shows that there is a discrepancy with the empirically better prediction of the Goldberger-Treiman relation from the PCAC argument and QCD given in the lecture,

$$g_{\pi NN} = \frac{g_{\mathrm{A}} m_N}{F_\pi}, \quad m_N = \frac{m_{\mathrm{p}} + m_{\mathrm{n}}}{2} \simeq 939\,\text{MeV}, \quad g_{\mathrm{A}} \simeq 1.27. \tag{30}$$

For a possible way to cure this with (non-renormalizable) derivative meson-nucleon couplings see [DM00]. A very recent paper on this topic within a chiral doubler (mirror-assignment) model, where the discrepancy between (30) without derivative couplings is even worse than in the here discussed Gell Mann-Levy model is [KLS26].

[5] The (average) pion mass $m_\pi = (m_{\pi^\pm} + m_{\pi^0})/2 \simeq 137\,\text{MeV} \simeq 140\,\text{MeV}$.

Solution: We follow the hint and use local V and A in the hadronic distribution. Since the nucleons have isospin $1/2$ the Ψ fields transform in the same way as the quarks, and the meson fields transform as given by (14)[6]. The Lagrangian total Lagrangian is invariant under both *global* V and A transformations except $\mathscr{L}_{\text{SB}}$ which explicitly breaks the A symmetry. Making the transformations local, i.e., making $\delta\vec{\alpha}$ and $\delta\vec{\beta}$ dependent on the space-time variables $\underline{x}$ in addition all terms in the Lagrangian with derivatives become non-symmetric, because of contributions where terms with $\partial_\mu \delta\vec{\alpha}$ and $\partial_\mu \delta\vec{\beta}$ occur. That is why we need to consider only the two pieces of the Lagrangian given in (27).

After some algebra one gets for the V transformations (for which $\delta\mathscr{L}_{\text{SB}} = 0$, as it should be in our idealized world, where the $\text{SU}(2)_\text{V}$ symmetry is assumed to be exact,

$$\delta\mathscr{L} = (\partial_\mu \delta\vec{\alpha}) \cdot \underbrace{\left[\overline{\Psi}\gamma^\mu \frac{\vec{\tau}}{2}\psi + \vec{\pi} \times (\partial_\mu \vec{\pi})\right]}_{\vec{J}_\text{V}^\mu}, \tag{31}$$

where $\vec{J}_\text{V}^\mu$ is the vector isovector current of the hadronic model.

This implies that the variation of the action under the local V transformation is

$$\delta S = \int_{\mathbb{R}^4} \mathrm{d}^4\underline{x} (\partial_\mu \delta\vec{\alpha}) \cdot \vec{J}_\text{V}^\mu = -\int_{\mathbb{R}^4} \mathrm{d}^4\underline{x}\, \delta\vec{\alpha} \cdot \partial_\mu \vec{J}_\text{V}^\mu. \tag{32}$$

Since $\delta S = 0$ for the solutions of the equations of motion (given by the Euler-Lagrange equations) and since (32) hold for arbitrary $\delta\vec{\alpha}(\underline{x})$ this implies that the vector isovector current is conserved in our model, which is of course Noether's theorem for the assumed symmetry under $\text{SU}(2)_\text{V}$ transformations. As we have seen in Lect. 8 these transformations also leave the fermionic path-integral measure invariant, i.e., that there is no anomaly, also the generating functional for Green's functions, connected Green's functions and proper vertex diagrams (the latter functional being the quantum action) are invariant under these transformations.

For the A transformations we get

$$\delta\mathscr{L} = (\partial_\mu \delta\vec{\beta}) \cdot \underbrace{\left[\overline{\Psi}\gamma^\mu \gamma_5 \vec{\tau}\Psi + \Phi^0 \partial_\mu \vec{\pi} - \vec{\pi}\partial_\mu \Phi^0\right]}_{\vec{J}_\text{A}^\mu} - \delta\vec{\beta} \cdot (\epsilon\vec{\pi}). \tag{33}$$

From the same argument as with the variation of the action under V transformations we can conclude for A transformations that

$$\partial_\mu \vec{J}_\text{A}^\mu = -\epsilon\vec{\pi} \neq 0 = -F_\pi m_\pi^2 \vec{\pi}. \tag{34}$$

This is also in accordance with Noether's theorem since the symmetry under $\text{SU}(2)_\text{A}$ transformations is explicitly broken by construction, and indeed, in the chiral limit, $\epsilon = 0$, also the axial isovector current is conserved.

Identifying the vacuum expectation values of the symmetry-breaking terms in the hadronic and the QCD Lagrangian we can conclude again the Gell-Mann-Oakes-Renner relation, which we have derived in the lecture directly from the PCAC argument,

$$\epsilon \langle \Omega | \sigma | \Omega \rangle = \epsilon F_\pi = F_\pi^2 m_\pi^2 = -m \left\langle \Omega \middle| \overline{\psi}\psi \middle| \Omega \right\rangle. \tag{35}$$

Plugging in the given values for F_π, m_π, and m leads to

$$\left\langle \Omega \middle| \overline{\psi}\psi \middle| \Omega \right\rangle = -\frac{F_\pi^2 m_\pi^2}{m} \simeq -(360\,\text{MeV})^3. \tag{36}$$

Usually one quotes the value for the condensate of a single quark flavor. Since $\left\langle \Omega \middle| \overline{\psi}\psi \middle| \Omega \right\rangle = \left\langle \Omega \middle| \overline{u}u + \overline{d}d \middle| \Omega \right\rangle = 2 = \langle \Omega | \overline{u}u | \Omega \rangle$, where we have used the assumed exact isospin symmetry again, we finally get

$$\langle \Omega | \overline{u}u | \Omega \rangle = \langle \Omega | \overline{u}u | \Omega \rangle \simeq -(285\,\text{MeV})^3. \tag{37}$$

[6] Of course we work with the "old fields" $\tilde{\Phi}$ instead with σ and $\vec{\pi}$ first, because they have the convenient transformation properties given in (14).

References

[DM00] V. Dmitrasinovic and F. Myhrer, Pion nucleon scattering and the nucleon sigma term in an extended linear sigma model, Phys. Rev. C **61**, 025205 (2000), https://doi.org/10.1103/PhysRevC.61.025205.

[GML60] M. Gell-Mann and M. Levy, The axial vector current in beta decay, Nuovo Cim. **16**, 705 (1960), https://dx.doi.org/10.1007/BF02859738.

[KLS26] C. Kummer, S. Leupold and L. von Smekal, Kinetic mixing and axial charges in the parity-doublet model, Phys. Rev. D **113**, 116010 (2026), https://doi.org/10.1103/qc6c-wg6h.

[N+24] S. Navas et al. (Particle Data Group), Review of particle physics, Phys. Rev. D **110**, 030001 (2024), https://doi.org/10.1103/PhysRevD.110.030001.