

## Exercise Sheet 9

### 1. Chiral anomalies in the standard model

#### QCD-like non-Abelian gauge theory

We consider a generalization of the chiral anomaly (Adler-Bell-Jackiw anomaly) in QED to more general cases. We start with a general non-Abelian gauge theory suited to analyse the possible anomalies of the (approximate) chiral symmetries of QCD. We consider a slightly more general case and consider “quarks” with  $N_f$  flavors. For the QCD in the standard model we have 6 flavors, ordered in 3 families, each family containing an up- and a downlike quark, (u,d), (c,s), and (t,b) (called up, down, charm, strange, top, and bottom), of which only u, d, s have small enough masses such that the mass terms can be treated as a perturbation of the massless case, giving rise to the said approximate chiral symmetries. In this case there is no danger that the gauged symmetry group,  $SU(3)_{\text{col}}$ , itself is anomalously broken, because the QCD Lagrangian does not contain  $\gamma_5$  and thus the symmetry transformations are all unitary, and the path-integral measure of the quark fields is invariant under the transformation.

We consider only the relevant part of the Lagrangian, containing the kinetic terms as well as the coupling of the quarks to the gauge field (with the corresponding particles being the gluons). We add the possibility that the gluons couple to a linear combination of the flavors. Writing  $\psi = (u, d, c, s, t, b)^T$  with  $u, \dots, b$  Dirac spinor fields. In addition each quark field carries a color index, i.e.,  $u \equiv u_c$  with  $c \in \{r, g, b\}$ . The generators of the gauge group, which describes the local color symmetry are  $\hat{T}_c^a = \hat{\lambda}^a/2$  with  $a \in \{1, 2, \dots, 8\}$  and  $\hat{\lambda}^a$  being the Gell-Mann matrices. Further  $\hat{T}_f$  is a matrix acting in flavor space, mixing the different quark flavors. In Nature, of course,  $\hat{T}_f = \mathbb{1}$  since the strong interaction is “flavor blind”. Our Lagrangian thus reads

$$\mathcal{L} = \bar{\psi}(i\mathcal{D} - \hat{M})\psi, \quad (1)$$

where  $\hat{M} = \text{diag}(m_u, m_d, m_c, m_s, m_t, m_b)$  is the mass matrix of the quarks, and the gauge-covariant derivative reads

$$D_\mu = \partial_\mu + ig_s \hat{T}_c^a \hat{T}_f^a G_\mu^a \quad (2)$$

with  $G_\mu^a$  ( $a \in \{1, 2, \dots, 8\}$ ) being the gluon fields.

We consider global chiral symmetries of the kind

$$\psi' = \psi - i\delta \vec{\alpha} \vec{t} \gamma_5 \psi, \quad (3)$$

where the  $\vec{t}$  act in flavor space (in QCD in the light quark sector it's either an  $su(2)$ -Lie algebra acting on the (u,d)-isospin doublet or an  $su(3)$ -Lie algebra acting on (u,d,s) (Gell-Mann's “eightfold way”), or just  $\mathbb{1}$  for the axial iso-scalar current. The corresponding currents, which are conserved in the classical theory in the massless limit  $\hat{M} = 0$  are

$$\vec{j}_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \vec{t} \psi. \quad (4)$$

Then the same analysis as the one shown in the lecture for the case of the  $U(1)_{\text{em}}$  gauge theory, QED, can be used also in our non-Abelian case. The space-time and Dirac trace from the Fujikawa regulator is the same as in the QED calculation. With the various matrices in color and flavor space we simply get another trace over these matrices, which factor in the product over the traces over color and flavor matrix products, respectively. The anomaly finally reads

$$\partial_\mu \langle \Omega | \vec{j}_5^\mu(\underline{x}) | \Omega \rangle_{\text{an}} = -\frac{g_s^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}_{\text{col}} \hat{T}_c^a \hat{T}_c^b \text{tr}_{\text{flav}} (\vec{t} \hat{T}_f^2) G_{\mu\nu}^a G_{\rho\sigma}^b \quad (5)$$

with the gluon field-strength tensor

$$G_{\mu\nu}^a = (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_s f^{abc} A_\mu^b A_\nu^c), \quad (6)$$

where the  $f^{abc}$  are the structure constants of the gauge group  $SU(3)$ ,

$$[\hat{T}_c^a, \hat{T}_c^b] = i f^{abc} \hat{T}_c^c. \quad (7)$$

- (a) Consider the approximate chiral symmetries of the usual QCD with  $\hat{T}_f = \mathbb{1}$ , i.e.,  $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$  with  $N_f \in \{1, 2\}$ , which are exact global symmetries in the case that the quark flavors under consideration are massless. The associated currents are the “vector-isovector/isoscalar” and “axial-vector-isovector/isoscalar” ones:

$$\vec{j}_V^\mu = \bar{\psi} \vec{t} \gamma^\mu \psi, \quad \vec{j}_A^\mu = \bar{\psi} \vec{t} \gamma^\mu \gamma_5 \psi, \quad (8)$$

$$j_V^\mu = \bar{\psi} \gamma^\mu \psi, \quad j_A^\mu = \bar{\psi} \vec{t} \gamma^\mu \gamma_5 \psi. \quad (9)$$

The mass term breaks the conservation of the axial-vector currents already at the classical level<sup>1</sup>. Calculate  $\partial_\mu \vec{j}_A^\mu$  and  $\partial_\mu j_A^\mu$  taking into account the mass term by considering the local infinitesimal transformations (as shown in the lecture),

$$\delta \psi' = -i \delta \vec{\alpha}(\underline{x}) \cdot \hat{t} \gamma_5 \psi \Rightarrow \delta \bar{\psi}' = -i \delta \vec{\alpha}(\underline{x}) \cdot \hat{t} \bar{\psi} \gamma_5, \quad (10)$$

$$\delta \psi' = -i \delta \alpha(\underline{x}) \gamma_5 \psi \Rightarrow \delta \bar{\psi}' = -i \alpha(\underline{x}) \cdot \bar{\psi} \hat{t} \gamma_5, \quad (11)$$

where we have used that for the Lie algebras  $\mathfrak{su}(N_f)$  the generators are hermitian,  $\hat{t}^\dagger = \hat{t}$ .

**Solution:** We start with the axial-isovector current. For the evaluation of the explicit symmetry breaking on the classical level, it is sufficient to consider the free Lagrangian of the fermions, including the mass term,

$$\mathcal{L}_0 = \bar{\psi} (i \not{\partial} - \hat{M}) \psi. \quad (12)$$

The term with the couplings to the gluon field is invariant. Then using (10) we find for the variation of the Lagrangian,

$$\delta \mathcal{L}_0 = \vec{j}_A^\mu \partial_\mu \delta \alpha - 2 \bar{\psi} \hat{M} \hat{t} \psi \alpha. \quad (13)$$

Integrating this over  $\underline{x}$  we find for the variation of the action

$$\delta S_0 = \int_{\mathbb{R}^4} d^4 \underline{x} \delta \vec{\alpha} \cdot \left( -\partial_\mu \vec{j}_A^\mu + i \bar{\psi} \gamma_5 \left\{ \hat{t}, \hat{M} \right\} \right) \psi. \quad (14)$$

Using this in the path-integral formalism for the evaluation of the transformation properties of the generating functional  $Z$ , as detailed in the lecture, we get

$$\partial_\mu \langle \Omega | \vec{j}_A^\mu | \Omega \rangle = i \langle \Omega | \bar{\psi} \gamma_5 \left\{ \hat{t}, \hat{M} \right\} \psi | \Omega \rangle + \text{anomaly term}. \quad (15)$$

For the axial isoscalar current we have to use (11). We can just use (15) substituting  $\mathbb{1}_2$  instead of  $\hat{t}$ , leading to

$$\partial_\mu \langle \Omega | j_A^\mu | \Omega \rangle = 2i \langle \Omega | \bar{\psi} \gamma_5 \hat{M} \psi | \Omega \rangle + \text{anomaly term}. \quad (16)$$

- (b) Calculate the anomaly terms (5) for both axial currents.

**Solution:** We can immediately use (5) with  $\hat{T}_f = \mathbb{1}$ . The relevant traces for the axial isovector and the axial isoscalar anomalies

$$\text{tr}(\hat{T}_c^a \hat{T}_c^b) \text{tr} \hat{t} = 0, \quad (17)$$

$$\text{tr}(\hat{T}_c^a \hat{T}_c^b) \text{tr} \mathbb{1} = \frac{N_f}{2} \delta^{ab}. \quad (18)$$

Thus

$$\partial_\mu \langle \Omega | \vec{j}_A^\mu(\underline{x}) | \Omega \rangle_{\text{an}} = 0, \quad (19)$$

$$\partial_\mu \langle \Omega | j_A^\mu(\underline{x}) | \Omega \rangle_{\text{an}} = -\frac{g_s^2 N_f}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a, \quad (20)$$

i.e., the axial isovector symmetry is not anomalously broken, while the axial isoscalar symmetry has an anomaly.

<sup>1</sup>Also the vector currents are only conserved if  $\hat{M} \propto \mathbb{1}$ , i.e., if all flavors have the same mass. In QCD isospin or  $SU(3)_{\text{flavor}}$  vector currents are also not conserved at the same level as the chiral symmetry is broken since the u, d, (s)-quark masses are of the same order as their mass difference.

- (c) Argue how many pseudo-Goldstone bosons *really* occur for  $N_f \in \{1, 2\}$  when the chiral symmetry is spontaneously broken to the symmetry group of the vacuum state,  $SU(N_f)_V \times U(1)_V$  (in addition to the explicit breaking by the quark-mass terms).

**Note:** This is only an exact symmetry (isospin for the two-flavor and isospin+hypercharge for the three-flavor model) if the quark masses are equal. In nature also this symmetry is violated on the same level as chiral symmetry itself, which is broken by introducing quark masses in the way as in the Standard Model.

**Solution:** The original symmetry (on the classical level) is  $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$ . The  $U(1)_A$  symmetry is anomalously broken according to (20). Thus the symmetry group  $G$  of the quantized theory is only  $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ . Its dimension is  $\dim G = 2 \dim SU(N_f) + 1 = N_f(N_f - 1) + 1$ . In the chiral limit with  $\hat{M} = 0$  the symmetry group of the ground state is  $H = SU(N_f)_V \times U(1)_V$  with  $\dim H = N_f(N_f - 1)/2 + 1$  i.e., in the chiral limit we have  $N_f(N_f - 1)$ . In nature these are for  $N_f = 2$  the three pions. For  $N_f = 3$  we have again the 3 pions, the 4 kaons, and the  $\eta$ . If the  $U(1)_A$  symmetry were not anomalously broken, we'd have another pseudoscalar isoscalar particle, which would be the  $\eta'$  meson, but with  $m_{\eta'} \simeq 958$  MeV it's not at all particularly light. The pion masses are around 140 MeV, the kaon masses around 497 MeV, and the  $\eta$  mass is about 548 MeV. Note that in the  $N_f = 3$  case the masses are not that small due to the pretty large strange-quark mass of about 95 MeV. The up and down quark masses are pretty low compared to that (about 2.2 MeV and 4.7 MeV). Also note that the mass difference between the u- and d-quarks are of the same order of a few MeV as the masses themselves, i.e., isospin symmetry is as much broken as chiral symmetry!

### ABJ anomaly in QED of quarks

- (d) We consider QED of the two lightest quark flavors ( $u, d$ ), i.e., we set now  $\psi = (u, d)^T$  with  $u$  and  $d$  come with  $N_c = 3$  colors each. The relevant part of the QED Lagrangian reads

$$\mathcal{L} = \bar{\psi}(i\not{D} - \hat{M})\psi, \quad D_\mu = \partial_\mu + ie\hat{Q}A_\mu, \quad (21)$$

where  $\hat{Q} = \text{diag}(2/3, -1/3)$  is the charge matrix of the quarks (acting in flavor space). Now consider the two-flavor axial-vector isovector current,

$$\vec{j}_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\hat{t}\psi, \quad \hat{t} = \frac{1}{2}\hat{\sigma} \quad (22)$$

with the Pauli matrices  $\hat{\sigma}$  acting also in flavor space. Use the general “anomaly formula” (5) to calculate the anomaly of this current (don't forget the “hidden” color degrees of freedom; for QED the color matrix is of course  $\mathbb{1}$ , i.e., there is an additional factor  $\text{tr}_{\text{col}} \mathbb{1} = N_c = 3$ ).

**Solution:** The relevant traces in (5) in this case are

$$\text{tr}_{\text{flavor}}(\hat{t}^a \hat{Q}^2) \text{tr}_{\text{color}} \mathbb{1} = \frac{N_c}{6} \delta^{a3}, \quad (23)$$

i.e., only the isospin-3 component of the axial current (22) is anomalous, i.e., only the neutral pion gets a coupling to the em. field via an anomaly (although its electric charge is 0!). The charged pions just couple in the normal way to the photon field through the gauge-covariant derivative in (22). The anomaly for the isospin-3 component of the axial current is

$$\partial_\mu \langle \Omega | \vec{j}_5^{a\mu}(x) | \Omega \rangle_{\text{an}} = -\frac{N_c e^2}{96\pi^2} \delta^{a3} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (24)$$

- (e) (Challenging extra question) Within the within the PCAC formalism (see next lecture) the relation of the pion fields  $\vec{\pi}$  (isospin-1 vector) to the axial isovector current are given by the “PCAC-relation”

$$\vec{j}_5^\mu = f_\pi \partial^\mu \vec{\pi}. \quad (25)$$

Note that the charge states of the pions are given by  $\pi^\pm = (\pi_1 \pm i\pi_2)/\sqrt{2}$  and  $\pi^0 = \pi_3$ . How does the corresponding anomaly term in the effective QED for pions look like and which of the pions can decay to two photons?

**Solution:** The Lagrangian for the neutral pion, writing simply  $\pi$  for  $\pi_3$ , interacting with the electromagnetic field should look like (neglecting the pion mass)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) + \mathcal{L}_{\pi_0\gamma\gamma}. \quad (26)$$

The equation of motion for the pion field reads

$$\square\pi = \frac{\partial \mathcal{L}_{\pi_0\gamma\gamma}}{\partial \pi}. \quad (27)$$

On the other hand (25) and (24) we get

$$\partial_\mu j_5^{3\mu} = \frac{1}{f_\pi} \square\pi \stackrel{!}{=} -\frac{e^2 N_c}{96\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (28)$$

from which (with  $N_c = 3$ )

$$\mathcal{L}_{\pi_0\gamma\gamma} = -\frac{e^2}{32\pi^2 f_\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (29)$$

**Note:** From this one can calculate the (partial) decay width of the  $\pi_0 \rightarrow \gamma\gamma$  by just evaluating the corresponding tree-level diagram. From the Lagrangian one reads off the  $\pi_0\gamma\gamma$  vertex in the usual way:

$$i\mathcal{M}^{\nu\sigma} = \frac{ie^2}{8\pi^2 f_\pi} \epsilon^{\mu\nu\rho\sigma} k_{1\mu} k_{2\rho} \quad (30)$$

where  $\underline{k}_1$  and  $\underline{k}_2$  are the four-momenta of the two photons. The remaining four-vector indices  $\nu, \sigma$  are from the “amputated” photon fields,  $A^\alpha$ . Using the Feynman rules, including a combinatorial factor for the 2 possibilities to connect the two photon vertices to the two external points of the corresponding photon legs we get the invariant matrix element

$$\mathcal{M} = \frac{ie^2}{4\pi^2 f_\pi} \epsilon^{\mu\nu\rho\sigma} \varepsilon_{1\mu} \varepsilon_{2\nu} k_{1\rho} k_{2\sigma}. \quad (31)$$

Taking  $|\mathcal{M}|^2$  and sum over the final photon polarizations, where one can use the substitution

$$\sum_{h=\pm 1} \epsilon_h^{*\mu} \epsilon_h^\nu \rightarrow -\eta^{\mu\nu}, \quad (32)$$

which is valid because of the Ward identity for matrix elements with external photon legs due to em-current conservation and gauge invariance. Then one needs to calculate also the contraction with the two Levi-Civita tensors. The final result for the polarization-summed matrix element is

$$\overline{|\mathcal{M}|^2} = 2 \left( \frac{e^2}{4\pi^2 f_\pi} \right)^2 (\underline{k}_1 \cdot \underline{k}_2)^2, \quad (33)$$

where the factor 2 comes from the  $\epsilon$ -tensor contraction. Now from momentum conservation  $\underline{p}_\pi = \underline{k}_1 + \underline{k}_2$ , from which  $(\underline{k}_1 + \underline{k}_2)^2 = 2\underline{k}_1 \cdot \underline{k}_2 = m_\pi^2$  we finally get

$$\overline{|\mathcal{M}|^2} = \frac{\alpha^2 m_\pi^4}{2\pi^2 f_\pi^2}, \quad \alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}. \quad (34)$$

With the standard decay-width formula (including a factor 1/2 due to the identity of the two photons in the final state),

$$\Gamma = \frac{1}{2} \frac{1}{16\pi m_\pi} \overline{|\mathcal{M}|^2} = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2}. \quad (35)$$

With  $m_\pi = 134.98$  MeV,  $f_\pi = 92.2$  MeV, and  $\alpha = 1/137.036$  one gets  $\Gamma \simeq 7.76$  eV or the life time  $\tau = 1/\Gamma \simeq 8.48 \cdot 10^{-17}$  s. The PDG value is  $\tau = (8.49 \pm 0.13) \cdot 10^{-17}$  s.

**Note:** The above general anomaly formula can also be applied to the electroweak sector of the standard model (Glashow-Salam-Weinberg model, aka Quantum Flavor Dynamics). Since here the model is itself based on a gauged chiral symmetry, it is of utmost importance for its consistency that the conservation of the corresponding currents (which are of the “vector-minus-axial-vector type”), to which the gauge bosons (in the Higgsed model the  $W^\pm, Z$ , and electromagnetic gauge fields) couple, are *not* anomalously broken, which would imply that the gauge symmetry itself were broken and thus the theory obsolete. It turns out that the flavor content with the specific charges of the quarks and leptons and the additional color-degrees of freedom (with  $N_c = 3!$ ) conspires in such a way that the relevant traces over the flavor (and the factor  $N_c$  from the  $\mathbb{1}_{\text{col}}$ -matrix in the electroweak currents of the quarks) cancel out the “dangerous” anomalies. For a thorough discussion of all that (and also of effective hadronic models etc.) see [DGH22]<sup>2</sup>.

<sup>2</sup>Note that this book has a somewhat different convention concerning the Dirac matrices, particularly  $\gamma_5$  has the opposite sign than the one used in this lecture, which follows the convention in [PS95], while the 4D Levi-Civita tensor follows the same sign convention,  $\epsilon^{0123} = -\epsilon_{0123} = +1$ .

## References

- [DGH22] J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model: Second edition*, Cambridge University Press, Cambridge, New York, Melbourne (2022), <https://doi.org/10.1017/9781009291033>.
- [PS95] M. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley Publ. Comp., Reading, Massachusetts (1995).