Exercise Sheet 8

(1) Spontaneously broken U(1) symmetry

We start with the Lagrangian of a charged scalar field, $\Phi \in \mathbb{C}$

$$\mathscr{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) + \mu^2 \Phi^* \Phi - \frac{g^2}{2} (\Phi^* \Phi) = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - V(\Phi^* \Phi). \tag{1}$$

Note that the mass term has the "wrong sign", which means that we have spontaneous symmetry breaking, i.e., the ground-state solution is $\Phi = v = \text{const}$ with $v \neq 0$. The theory has obviously the usual global U(1) symmetry under the transformations

$$\Phi'(\underline{x}) = \exp(-\mathrm{i}\alpha)\Phi(\underline{x}), \quad \Phi'^*(\underline{x}) = \exp(+\mathrm{i}\alpha)\Phi^*(\underline{x}), \quad \alpha = \text{const} \in \mathbb{R}. \tag{2}$$

(a) Derive the equations of motion for Φ by applying the Euler-Lagrange equations to Φ^*[1]. Derive v, which we choose by convention to be real with $v > 0$.

Solution: The Euler"=Lagrange equation reads

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^*)} = \Box \Phi = \frac{\partial \mathscr{L}}{\partial \Phi^*} = \Phi(\mu^2 - g^2 \Phi^* \Phi). \tag{3}$$

For $\Phi = \Phi^* = v = \text{const} \in \mathbb{R}$ this leads to

$$\mu^2 - g^2 v^2 = 0 \Rightarrow v = \frac{\mu}{g}. \tag{4}$$

(b) Define

$$\Phi(\underline{x}) = v + \frac{\chi(\underline{x})}{\sqrt{2}}, \quad \chi(\underline{x}) = \chi_1(\underline{x}) + \mathrm{i}\chi_2(\underline{x}), \quad \chi_1, \chi_2 \in \mathbb{R}. \tag{5}$$

and write the Lagrangian in terms of the real fields χ_1 and χ_2. Interpret the result if you think of the model as the classical version of a bosonic QFT, which can be canonically quantized in the usual way[2].

Solution: In terms of the χ_j ($j \in \{1,2\}$) the Lagrangian reads

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \chi_1)(\partial^\mu \chi_1) + \frac{1}{2}(\partial_\mu \chi_2)(\partial^\mu \chi_2) - \mu^2 \chi_1^2 - \frac{g\mu}{\sqrt{2}} \chi_1 (\chi_1^2 + \chi_2)^2 - \frac{g^2}{8} (\chi_1^2 + \chi_2^2)^2 + \frac{\mu^4}{2g^2}. \tag{6}$$

From this one reads off that the quantized theory describes one sort of particles, represented by the real scalar field χ_1 with mass $m_1^2 = 2\mu^2$, i.e., $m_1 = \mu\sqrt{2}$, and another sort of particles, represented by χ_2, with mass $m_2 = 0$. This is the Nambu"=Goldstone boson the spontaneously broken U(1) symmetry. The only subgroup that leaves the "vacuum", represented by the field representation $\Phi = \Phi^* = v$, invariant is the trivial group, i.e., the identity transformation of the field, which is a "zero-dimensional" Lie group, i.e., we have $\dim U(1) - \dim H = 1$ Nambu-Goldstone boson as confirmed explicitly by the above calculation.

(c) As an alternative way to write the complex field in terms of two real fields now use

$$\Phi(\underline{x}) = \left(v + \frac{R(\underline{x})}{\sqrt{2}}\right) \exp[\mathrm{i}\varphi(\underline{x})], \quad \Phi^*(\underline{x}) = \left(v + \frac{R(\underline{x})}{\sqrt{2}}\right) \exp[-\mathrm{i}\varphi(\underline{x})], \quad R, \varphi \in \mathbb{R} \tag{7}$$

and express the Lagrangian in terms of the fields.

Solution: The Lagrangian in terms of R and φ is given by

$$\mathscr{L} = \frac{1}{2}(\partial_\mu R)(\partial^\mu R) + \left(v + \frac{R}{\sqrt{2}}\right)^2 (\partial_\mu \varphi)(\partial^\mu \varphi) - \mu^2 R^2 + \frac{g\mu R^3}{\sqrt{2}} + \frac{g^2 R^4}{8}. \tag{8}$$

[1]Then the equations ones for Φ^* are just the conjugate complex equations.

[2]You do not need to work out the quantized formulation.

In this "non-linear representation" of the theory we see the same "particle content" as described above: there's one scalar boson, represented by the field, R, with mass $m_R = \mu\sqrt{2}$ and a massless scalar boson, represented by the field, φ. Note that now we have derivative couplings for the interaction of the R and φ fields.

(d) Explain the occurence of a massless Nambu-Goldstone boson from the shape of the potential V, when interpreted in terms of the two real field-degrees of freedom in terms of the "non-linear formulation" (7).

Solution: The potential V is "mexican-hat shaped", i.e., the minimum is at $|\Phi|^2 = \mu^2/g^2 > 0$. Now in terms of the parametrization of the Φ-field (7)

$$|\Phi|^2 = \left(v + \frac{R}{\sqrt{2}}\right)^2, \tag{9}$$

i.e., the potential does not depend on φ. Thus a shift $\varphi \to \varphi + \alpha$ for $R = 0$ does not change the energy, i.e., one moves along the rim of the mexican hat, which does not need energy, which means in terms of the quantized theory that the corresponding field excitations indeed describe massless particles.

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(2) Higgsed U(1) gauge symmetry

Start again from the above Lagrangian (1), but now "gauge the U(1) symmetry" making it a local symmetry by introducing a gauge field, $A^\mu(\underline{x}) \in \mathbb{R}$ and application of the "principle of minimal substitution", i.e., define the gauge-covariant derivative

$$\mathrm{D}_\mu = \partial_\mu + \mathrm{i} q A_\mu, \quad q \in \mathbb{R}, \tag{10}$$

and substitute ∂_μ by D_μ in the Lagrangian, i.e.,

$$\mathscr{L} = (\mathrm{D}_\mu \Phi)^* (\mathrm{D}_\mu \Phi) - V(\Phi^* \Phi). \tag{11}$$

(a) Show that under the *local* gauge transformation

$$\Phi'(\underline{x}) = \exp[-\mathrm{i} q \alpha(\underline{x})] \Phi(\underline{x}), \quad \Phi'^*(\underline{x}) = \exp[+\mathrm{i} q \alpha(\underline{x})], \quad A'_\mu(\underline{x}) = A_\mu(\underline{x}) + \partial_\mu \alpha(\underline{x}) \tag{12}$$

the gauge-covariant derivatives of the fields transform as

$$\mathrm{D}'_\mu \Phi' = \exp(-\mathrm{i} q \alpha) \mathrm{D}_\mu \Phi, \quad (\mathrm{D}'_\mu \Phi')^* = \exp(+\mathrm{i} q \alpha) (\mathrm{D}_\mu \Phi)^* \tag{13}$$

and that thus the gauged Lagrangian (11) is invariant under the *local* gauge transformation (12).

Solution: using (12) we get

$$\mathrm{D}'_\mu \Phi' = (\partial_\mu + \mathrm{i} q A_\mu + \mathrm{i} q \partial_\mu \alpha)[\exp(-\mathrm{i} q \alpha) \Phi] = \exp(-\mathrm{i} q \alpha)(\partial_\mu + \mathrm{i} a A_\mu) \Phi = \exp(-\mathrm{i} q \alpha) \mathrm{D}_\mu \Phi. \tag{14}$$

Taking the conjugate complex of this equation, leads to the 2$^{\text{nd}}$ equation in (13).

(b) Show that this is also true for the Lagrangian extended by a "kinetic term" for the gauge field,

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\mathrm{D}_\mu \Phi)^* (\mathrm{D}_\mu \Phi) - V(\Phi^* \Phi), \tag{15}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{16}$$

is the usual field-strength tensor.

Solution: The gauge transformation of A_μ according to (12) leads to

$$F'_{\mu\nu} = \partial_\mu (A_\nu + \partial_\nu \alpha) - \partial_\nu (A_\mu + \partial_\mu \alpha) = \partial_\mu A_\nu - \partial_\nu A_\mu + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \alpha = F_{\mu\nu}, \tag{17}$$

i.e., $F_{\mu\nu}$ is gauge invariant.

(c) As in Problem 1, the stable ground state is given by the solution $\Phi = \Phi^* = v = \text{const} \in \mathbb{R}$. Now write Φ in terms of (7) and express the Lagrangian (15) in terms of the gauge-transformed fields,

$$\tilde{\Phi}(\underline{x}) = \exp\left[-\mathrm{i}q\frac{\varphi(\underline{x})}{q}\right]\Phi(\underline{x}), \quad \tilde{\Phi}^*(\underline{x}) = \exp\left[+\mathrm{i}q\frac{\varphi(\underline{x})}{q}\right]\Phi^*(\underline{x}), \quad \tilde{A}_\mu(\underline{x}) = A_\mu + \frac{1}{q}\partial_\mu\varphi(\underline{x}). \tag{18}$$

Hint: Thanks to the gauge invariance of $\mathscr{L}$ you do not need to perform the somewhat cumbersome algebra explicitly. It's sufficient to make use of the advantage, that the Lagrangian looks the same for the fields $\tilde{\Phi}$, $\tilde{\Phi}^*$, $\tilde{A}_\mu$, and $\tilde{D}_\mu$ as the Lagrangian written with the fields and gauge-covariant derivatives without the tilde!

Solution: Writing Φ in terms of (7) in the first equation of (18) gives

$$\tilde{\Phi}(\underline{x}) = v + \frac{R(\underline{x})}{\sqrt{2}}. \tag{19}$$

Since $v, R\ in \mathbb{R}$ by complex conjugation one finds $\tilde{\Phi}^* = \Phi$. Thus

$$\tilde{D}_\mu\tilde{\Phi} = \frac{1}{\sqrt{2}}\partial_\mu R + \mathrm{i}qA_\mu\left(v + \frac{R}{\sqrt{2}}\right) \tag{20}$$

and by complex conjugation,

$$(\tilde{D}_\mu\tilde{\Phi})^* = \frac{1}{\sqrt{2}}\partial_\mu R - \mathrm{i}qA_\mu\left(v + \frac{R}{\sqrt{2}}\right). \tag{21}$$

Plugging this into the Lagrangian (using $\tilde{F}_{\mu\nu} = F_{\mu\nu}$), yields (using $v = \mu/g$)

$$\mathscr{L} = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}(\partial_\mu R)(\partial^\mu R) + q^2\left(\frac{\mu}{g} + \frac{R}{\sqrt{2}}\right)^2\tilde{A}_\mu\tilde{A}^\mu - \mu^2R^2 - \frac{g\mu}{\sqrt{2}}R^3 - \frac{g^2R^4}{8} + \frac{\mu^4}{2g^2}. \tag{22}$$

The third term now includes the mass term

$$\mathscr{L}_A^{(m)} = \frac{q^2\mu^2}{g^2}\tilde{A}_\mu\tilde{A}^\mu = \frac{m_A^2}{2}\tilde{A}_\mu\tilde{A}^\mu \Rightarrow m_A = \frac{\sqrt{2}q\mu}{g}. \tag{23}$$

(d) What's the "particle content" of the theory now? Are there as many (physical) field-degrees of freedom as in the original formulation?

Solution: Now we have a scalar "R-particle" with mass $m_R = \mu\sqrt{2}$ and a vector boson with mass m_A described by $\tilde{A}_\mu$.

In the original formulation we had a complex scalar field Φ (i.e., 2 real scalar-field degrees of freedom) and a massless vector field A_μ, which has 2 physical degrees of freedom (two helicity ± 1 polarization degrees of freedom), i.e. overall we have 4 real field-degrees of freedom.

In the "broken case" we have only one massive scalar "R-boson" and a massive $\tilde{A}_\mu$-vector meson. Since a massive vector boson has 3 polarization degrees of freedom (corresponding to the three spin-z eigenvalues, $\pm 1, 0$) we have overall the same number of physical field-degrees of freedom, i.e., 1+3=4.

(e) **Riddle:** Why is there no Nambu-Goldstone boson in this model?

Hint: Is there still a degeneracy of the ground state as in the case of the only global U(1) symmetry of Problem 1?

Solution: Now there is no degeneracy of the ground state of the model anymore since a local gauge transformation does not lead to a new physical situation, since all physical observables of the theory must be gauge-invariant quantities, i.e., they are *not* changed under a gauge transformation. This implies that there is only 1 vacuum state and thus there are no massless excitations (i.e., no massless Nambu-Goldstone bosons) left in the model. Note that the particular choice of the fields above is just a particular choice of "gauge fixing", which is particularly convenient to figure out the "particle content" of the theory. In terms of the quantized theory (particularly in the more general case of a non-Abelian gauge symmetry) that's called the "unitary gauge", because it has only physical degrees of freedom in the (non-gauge-fixing part of the) Lagrangian.

Note: There is a close relation to the theory of superconductivity and the above applied "Higgs mechanism". In fact, the "Higgs mechanism" has been discovered by P. W. Anderson in 1962 in this context of superconductivity (of course in terms of the non-relativistic Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity in metals at low temperatures [BCS57b, BCS57a], where the "condensing Bose fields" are "Cooper-paired electrons") [And63]. The corresponding condensed-matter application of a spontaneously broken global symmetry are superfluids which can either consist of bosons (like ^{4}He), where these bosons form a condensate at low temperatures or fermions (like ^{3}He), where again the Cooper pairs are the "condensing" boson-like degrees of freedom.

For more about the interesting relation between superconductivity and the Higgs mechanism, see [Wei86, Wei96, Ran12].

References

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