

Exercise Sheet 7

From the Langevin to the Fokker-Planck equation

The relativistic stochastic Langevin equation for the heavy-quark motion in a thermally equilibrated medium is defined in the (local) rest frame of the heat bath, by the time step,

$$dx_j = \frac{p_j}{E} dt, \quad dp_j = -\Gamma p_j dt + C_{jk}^{(\xi)} \rho_k \sqrt{dt}. \quad (1)$$

The ρ_k are uncorrelated Gaussian norml-distributed random numbers with the statistical properties

$$\langle \rho_j \rangle = 0, \quad \langle \rho_j \rho_k \rangle = \delta_{jk}. \quad (2)$$

$\Gamma = \Gamma(\vec{p})$ is the drag (or friction) coefficient and the covariance matrix $\hat{C} = \hat{C}^T = \sqrt{\hat{B}(\vec{p})}$ with the diffusion matrix \hat{B} , as derived in the lecture.

In (1) the realization of the stochastic process depends on the choice, at which momentum value between \vec{p} and $\vec{p} + d\vec{p}$ the covariance matrix of the fluctuatin force \hat{C} in (1) is evaluated, parametrized by the parameter $\xi \in [0, 1]$, i.e., one has to use

$$\Gamma_{jk}^{(\xi)} = \Gamma_{jk}(\vec{p} + \xi d\vec{p}) = \xi d\vec{p}_l \frac{\partial}{\partial p_l} \Gamma(\vec{p}). \quad (3)$$

We want to derive the Fokker-Planck equation for the phase-space distribution function of the heavy quarks that describes the statistics of the heavy-quark evolution due to the stochastic dynamics given by the Lange equation (1). By definition with the phase-space distribution function $f(t, \vec{x}, \vec{p})$ the expectation value for an arbitrary phase-space function $g(\vec{x}, \vec{p})$ at time t is given by

$$\langle g(\vec{x}, \vec{p}) \rangle_t = \int_{\mathbb{R}^3} d^3 \vec{x} \int_{\mathbb{R}^3} d^3 \vec{p} g(\vec{x}, \vec{p}) f(t, \vec{x}, \vec{p}). \quad (4)$$

The Fokker-Planck equation to be derived is

$$\partial_t f + \frac{p_j}{E} \frac{\partial}{\partial x_j} f = \left(\Gamma p_j - \xi C_{lk} \frac{\partial C_{jk}}{\partial p_l} \right) f + \frac{1}{2} \frac{\partial^2}{\partial p_j \partial p_k} (C_{jl} C_{kl} f). \quad (5)$$

To prove it, follow the following steps to calculate the expression for the time derivative $d_t \langle g(\vec{x}, \vec{p}) \rangle_t$.

(a) Expand

$$g[\vec{x}(t+dt), \vec{p}(t+dt)] - g(\vec{x}, \vec{p}) = g(\vec{x} + d\vec{x}, \vec{p} + d\vec{p}) - g(\vec{x}, \vec{p}), \quad (6)$$

with $d\vec{x}$ and $d\vec{p}$ given by (1) in powers of dt up to order dt (note that the next order to be omitted is $\mathcal{O}(dt^{3/2})$ due to the \sqrt{dt} dependence of fluctuating-force contribution).

Solution: With (1) the expansion reads

$$\begin{aligned} g(\vec{x} + d\vec{x}, \vec{p} + d\vec{p}) - g(\vec{x}, \vec{p}) &= dx_j \frac{\partial g}{\partial x_j} + dp_j \frac{\partial g}{\partial p_j} + dp_j dp_k \frac{1}{2} \frac{\partial^2}{\partial p_j \partial p_k} g + \mathcal{O}(dt^{3/2}) \\ &= dt \frac{p_j}{E} \frac{\partial g}{\partial x_j} + \left(-dt \Gamma p_j + \sqrt{dt} C_{jk} \rho_k + dt \xi \frac{\partial C_{jk}}{\partial p_l} C_{lm} \rho_k \rho_m \right) \frac{\partial g}{\partial p_j} \\ &\quad + dt \frac{1}{2} \frac{\partial^2 g}{\partial p_j \partial p_k} C_{jl} C_{km} \rho_l \rho_m + \mathcal{O}(dt^{3/2}). \end{aligned} \quad (7)$$

(b) Take the expectation value of this expression,

$$\langle g(\vec{x} + d\vec{x}, \vec{p} + d\vec{p}) - g(\vec{x}, \vec{p}) \rangle \quad (8)$$

using the statistical properties of the fluctuating force, defined by (2).

Solution: With (2) we get

$$\begin{aligned} \langle g(\vec{x} + d\vec{x}, \vec{p} + d\vec{p}) - g(\vec{x}, \vec{p}) \rangle = & dt \frac{p_j}{E} \frac{\partial g}{\partial x_j} + dt \left(-\Gamma p_j + \xi \frac{\partial C_{jk}}{\partial p_l} C_{lk} \right) \frac{\partial g}{\partial p_j} \\ & + dt \frac{1}{2} C_{jl} C_{kl} \frac{\partial^2 g}{\partial p_j \partial p_k} + \mathcal{O}(dt^{3/2}). \end{aligned} \quad (9)$$

(c) Calculate the expectation value, on the other hand, by using the phase-space distribution function, f :

$$\langle g(\vec{x} + d\vec{x}, \vec{p} + d\vec{p}) - g(\vec{x}, \vec{p}) \rangle = \int_{\mathbb{R}^3} d^3 \vec{x} \int_{\mathbb{R}^3} d^3 \vec{p} f(t, \vec{x}, \vec{p}) [\langle g(\vec{x} + d\vec{x}, \vec{p} + d\vec{p}) - g(\vec{x}, \vec{p}) \rangle]. \quad (10)$$

Use integration by parts to shuffle all partial derivative wrt. to \vec{x} and \vec{p} to the distribution function.

Solution: After the said integration by parts the expectation value becomes

$$\begin{aligned} \langle g(\vec{x} + d\vec{x}, \vec{p} + d\vec{p}) - g(\vec{x}, \vec{p}) \rangle = & dt \int_{\mathbb{R}^3} d^3 \vec{x} \int_{\mathbb{R}^3} d^3 \vec{p} g(\vec{x}, \vec{p}) \left[-\frac{p_j}{E} \frac{\partial f}{\partial x_j} + \left(\Gamma p_j - \xi \frac{\partial C_{jk}}{\partial p_l} C_{lk} \right) \frac{\partial f}{\partial p_j} \right. \\ & \left. + \frac{1}{2} \frac{\partial^2}{\partial p_j \partial p_k} (C_{jl} C_{kl} f) \right] \end{aligned} \quad (11)$$

(d) Divide the found expression by dt and take the limit $dt \rightarrow 0$ to get

$$\frac{d}{dt} \langle g(\vec{x}, \vec{p}) \rangle_t = d_t \int_{\mathbb{R}^3} d^3 \vec{x} \int_{\mathbb{R}^3} d^3 \vec{p} f(t, \vec{x}, \vec{p}) g(\vec{x}, \vec{p}) = \int_{\mathbb{R}^3} d^3 \vec{x} \int_{\mathbb{R}^3} d^3 \vec{p} g(\vec{x}, \vec{p}) \partial_t f(t, \vec{x}, \vec{p}). \quad (12)$$

By comparing this with the result from the previous step, conclude the validity of the Fokker-Planck equation (5).

Solution: Setting the right-hand side of (12) equal to the said limit, and because this phase-space integral equation must hold for *arbitrary* phase-space functions, g , we finally get the Fokker-Planck equation (5) after shuffling the first term in the square bracket in (11) to the left-hand side.