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Exercise Sheet 5

(1) Ideal fluid

Consider the phase-space distribution function of local thermal equilibrium

$$f(\underline{x},\underline{p}) = \exp\{-\beta(\underline{x})[\underline{u}(\underline{x})\cdot\underline{p} - \mu(\underline{x})]\}\Big|_{p^0=E_p=\sqrt{\vec{p}^2+m^2}}. \tag{1}$$

The particle-number current density and the energy-momentum tensor are defined by

$$J^\mu(\underline{x}) = \frac{g}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{E_p} p^\mu f(\underline{x},\underline{p}), \tag{2}$$

$$T^{\mu\nu}(\underline{x}) = \frac{g}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{E_p} p^\mu p^\nu f(\underline{x},\underline{p}). \tag{3}$$

(a) Calculate these fields in the local rest frame, where $\underline{u}^*(\underline{x}^*) = (1,0,0,0)^{(\mathrm{T})}$ and define the scalar fields $n(\underline{x})$ (the particle-number density in the local rest frame), $\epsilon(\underline{x})$ (the internal-energy density in the local rest frame), and $P(\underline{x})$ (the pressure in the local rest frame).

Solutions: In the local rest frame

$$f = \exp[-\beta(E_p - \mu) = \exp(\mu\beta)\exp[-\beta(\sqrt{m^2+\vec{p}^2})]. \tag{4}$$

From symmetry $E_p = E(-\vec{p}) = E(+\vec{p})$ and rotational symmetry, i.e., $E_p = E(|\vec{p}|)$ we have

$$\underline{J}^* = \begin{pmatrix} n \\ \vec{0}\end{pmatrix}, \quad n = \frac{g}{(2\pi)^3}\exp(\beta\mu)\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p}\exp(-\beta E_p). \tag{5}$$

For the same reason $T^{\mu\nu} = \mathrm{diag}(\epsilon, P, P, P)$ with

$$\epsilon = \frac{g}{(2\pi)^3}\exp(\mu\beta)\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p} E_p\exp(-\beta E_p), \tag{6}$$

$$P = \frac{g}{(2\pi)^3}\exp(\beta\mu)\frac{1}{3}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p}\frac{\vec{p}^2}{E_p}\exp(-\beta E_p). \tag{7}$$

Note: By using

$$\frac{\partial}{\partial\vec{p}}\exp(-\beta E_p) = -\beta\frac{\vec{p}}{E_p}\exp(-\beta E_p) \tag{8}$$

one finds, integrating by parts,

$$\begin{aligned} P &= -\frac{g}{(2\pi)^3}\frac{\exp(\beta\mu)}{\beta}\frac{1}{3}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p}\,\vec{p}\cdot\frac{\partial}{\partial\vec{p}}\exp(-\beta E_p) \\ &= +\frac{g}{(2\pi)^3}\exp(\beta\mu)\frac{1}{3\beta}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p}\underbrace{\left(\frac{\partial}{\partial\vec{p}}\cdot\vec{p}\right)}_{=3}\exp(-\beta E_p) \stackrel{(5)}{=} \frac{n}{\beta} = nT, \end{aligned} \tag{9}$$

which is the ideal-gas equation (with $n = N/V$ we get the more familiar form $pV = N\cancel{k_{\mathrm{B}}}T$, where we have used that in our natural units we have set $k_{\mathrm{B}} = 1$).

(b) Use the fact that $\underline{u}$ is a four-vector field to derive the expressions for $J^\mu(\underline{x})$ and $T^{\mu\nu}(\underline{x})$ in the lab frame, where the fluid element at $\underline{x}$ moves with four-velocity $\underline{u}(\underline{x}) = \gamma_v(1,\vec{v})^{\mathrm{T}}$ to show that

$$J^\mu = n u^\mu, \quad T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - P\eta^{\mu\nu}. \tag{10}$$

Solutions: Because in the local rest frame $\underline{u}^* = (1,0,0,0)^{\mathrm{T}}$ we have

$$\underline{J}^* = \begin{pmatrix} n \\ 0 \\ 0 \\ 0 \end{pmatrix} = n\underline{u}^* \Rightarrow \underline{J} = n\underline{u}, \tag{11}$$

$$\begin{aligned} T^{*\mu\nu} &= \epsilon u^{*\mu} u^{*\nu} + P(u^{*\mu}u^{*\nu} - \eta^{\mu\nu}) = (\epsilon + P)u^{*\mu}u^{*\nu} - P\eta^{\mu\nu} \\ \Rightarrow T^{\mu\nu} &= (\epsilon + P)u^\mu u^\nu - P\eta^{\mu\nu} \end{aligned} \tag{12}$$

where we have used that n is a scalar field, i.e., $n(\underline{x}) = n^*(\underline{x}^*)$. This also holds for ϵ and P. Also $\eta^{*\mu\nu} = \eta^{\mu\nu}$, because $\eta^{\mu\nu}$ are invariant tensor components under Lorentz transformations.

(c) Use the (rotation-free) Lorentz boost with velocity $\vec{v} = \vec{u}/u^0$ (note that $u^0 = \gamma_v$, and $\vec{u} = \gamma_v \vec{v}$),

$$\hat{\Lambda} = (\Lambda^\mu{}_\nu) = \begin{pmatrix} u^0 & \vec{u}^{\mathrm{T}} \\ \vec{u} & \mathbb{1}_3 + (u^0 - 1)\vec{n} \otimes \vec{n} \end{pmatrix} \quad \text{with} \quad \vec{n} = \vec{u}/|\vec{u}| = \vec{v}/|\vec{v}|, \tag{13}$$

for which

$$\underline{u} = \hat{\Lambda}\underline{u}^*, \quad \underline{x} = \hat{\Lambda}\underline{x}^*, \quad \ldots \tag{14}$$

to verify your finding in (b).

Solutions: For the number-density current we simply have

$$\underline{J} = n\hat{\Lambda}\underline{u}^* = n\underline{u}. \tag{15}$$

For the energy-momentum tensor we find

$$T^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma [(\epsilon + p)u^{*\rho}u^{*\sigma} - P\eta^{\rho\sigma}] = (\epsilon + p)u^\mu u^\nu - P\eta^{\mu\nu}. \tag{16}$$

(d) Use the continuity equation (particle-number conservation) and the ideal-fluid equation of motion,

$$\partial_\mu J^\mu = \partial_\mu (n u^\mu) = 0, \quad \partial_\mu T^{\mu\nu} = 0 \Rightarrow u_\nu \partial_\mu T^{\mu\nu} = 0, \tag{17}$$

to verify that the entropy is (locally) conserved,

$$\partial_\mu (s u^\mu) = 0. \tag{18}$$

Hint: Use the Gibbs-Duhem relation for the enthalpy, H,

$$H = U + pV = TS + \mu N \tag{19}$$

and

$$\mathrm{d}H = \mathrm{d}U + p\mathrm{d}V + V\mathrm{d}p = T\mathrm{d}S + p\mathrm{d}V + \mu\mathrm{d}N \tag{20}$$

and

$$h = \frac{H}{V} = \frac{U + pV}{V} = \epsilon + p, \quad s = \frac{S}{V}, \quad n = \frac{N}{V} \tag{21}$$

to prove that

$$s\mathrm{d}T + n\mathrm{d}\mu + \mathrm{d}P \tag{22}$$

and with that

$$\mathrm{d}\left(\frac{h}{n}\right) = T\mathrm{d}\left(\frac{s}{n}\right) + \frac{1}{n}\mathrm{d}P. \tag{23}$$

Also note that

$$u_\nu u^\nu = 1 \Rightarrow \partial_\mu (u_\nu u^\nu) = 2u_\nu \partial_\mu u^\nu = 0. \tag{24}$$

Solutions: With (10) we have

$$\begin{aligned}
u_\nu \partial_\mu T^{\mu\nu} &= u_\nu \partial_\mu \left[n u^\mu \frac{h}{n} u^\nu - P \eta^{\mu\nu} \right] \\
&= u^\mu \partial_\mu \left(\frac{h}{n} \right) + h u^\mu u_\nu \partial_\mu u^\nu - u^\mu \partial_\mu P \\
&= n u^\mu \left[\partial_\mu \left(\frac{h}{n} \right) - \frac{1}{n} \partial_\mu P \right] = 0.
\end{aligned} \tag{25}$$

From (20) and (19) we find

$$\mathrm{d}H = T\mathrm{d}S + V\mathrm{d}P + \mu \mathrm{d}N = T\mathrm{d}S + S\mathrm{d}T + \mu\mathrm{d}N + N\mathrm{d}\mu \;\Rightarrow\; V\mathrm{d}P = S\mathrm{d}T + N\mathrm{d}\mu = 0 \;\Rightarrow\; \mathrm{d}P = s\mathrm{d}T + n\mathrm{d}\mu. \tag{26}$$

With (19), $S/V = s$, and $N/V = n$ we get

$$h = Ts + \mu n \;\Rightarrow\; \frac{h}{n} = T\frac{s}{n} + \mu \;\Rightarrow\; \mathrm{d}\left(\frac{h}{n}\right) = \frac{s}{n}\mathrm{d}T + T\mathrm{d}\left(\frac{s}{n}\right) + \mathrm{d}\mu. \tag{27}$$

With (26) this leads to

$$\mathrm{d}\left(\frac{h}{n}\right) = T\mathrm{d}\left(\frac{s}{n}\right) + \frac{1}{n}\mathrm{d}P \;\Rightarrow\; \partial_\mu \left(\frac{h}{n}\right) = T\partial_\mu \left(\frac{s}{n}\right) + \frac{1}{n}\partial_\mu P. \tag{28}$$

With this (25) simplifies to

$$nTu^\mu \partial_\mu \left(\frac{s}{n}\right) = 0. \tag{29}$$

Deviding by T and with the continuity equation we can indeed write this as

$$\partial_\mu \left(n u^\mu \frac{s}{n} \right) = \partial_\mu (s u^\mu) = 0. \tag{30}$$

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(2) Bjorken flow

The Bjorken flow is an exact solution of relativistic hydrodynamics, which describes roughly the expansion of a hot and dense fireball produced in ultra-relativistic heavy-ion collisions [Bjo83]. It uses the fact that in ultra-relativistic collions of nucleons (and thus also nuclei) in the center-momentum frame the produced hadron-number density n (and thus also ϵ and P) in the mid-rapidity region are independent of spatial rapidity, η [Fey69], where

$$\underline{x} = \begin{pmatrix} \tau \cosh \eta \\ x \\ y \\ \tau \sinh \eta \end{pmatrix}, \quad \tau = \sqrt{t^2 - z^2}, \tag{31}$$

with the z-direction the beam direction of the colliding nuclei (in the center-momentum rame).

Then one has $n = n(\tau)$, $\epsilon = \epsilon(\tau)$, and $P = P(\tau)$, and we assume that this holds for the entire (early-time) evolution of the fireball.

The incoming nuclei are highly Lorentz contracted in z direction, and when hitting each other at $z = 0$ the "leading partons" run nearly undisturbed through each other ("color transparency"), while through the secondary collisions many new partons are produced. After a very short "formation time" (after about 0.2-1 fm/c) this hot and dense "QGP" comes to local thermal equilibrium and streams as a (nearly) perfect fluid.

The Bjorken-flow solution of the ideal-fluid dynamics makes the ansatz that the fluid four-velocity is given by $v^1 = v^2 = 0$ (i.e., one neglects the "radial flow" completely) and $v^3(\underline{x}) = z/t$. In the following we only consider the longitudinal expansion of this cylindrical fireball. The four-velocity field thus is

$$\underline{u} = \begin{pmatrix} u^0 \\ u^3 \end{pmatrix} = \gamma_v \begin{pmatrix} 1 \\ z/t \end{pmatrix}, \quad \gamma_v = \frac{1}{\sqrt{1-z^2/t^2}}. \tag{32}$$

(a) For the evaluation of the idea-fluid equations,

$$\partial_\mu (n u^\mu) = 0, \quad \partial_\mu T^{\mu\nu} = 0 \quad \text{with} \quad T^{\mu\nu} = h u^\mu u^\nu - P \eta^{\mu\nu}, \quad h = \epsilon + P, \tag{33}$$

it is convenient to introduce Milne coordinates $(q^\alpha) = (\tau, \eta)$

$$\underline{x} = \begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} \tau \cosh\eta \\ \tau \sinh\eta \end{pmatrix}. \tag{34}$$

Show that because of

$$\partial_\mu = \frac{\partial q^\alpha}{\partial x^\mu} \frac{\partial}{\partial q^\alpha} \tag{35}$$

one has

$$(\partial_\mu) = \begin{pmatrix} \partial_t \\ \partial_z \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} \tau\cosh\eta & -\tau\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} \partial_\tau \\ \partial_\eta \end{pmatrix} = \begin{pmatrix} \cosh\eta\partial_\tau - (1/\tau)\sinh\eta\partial_\eta \\ -\sinh\eta\partial_\tau + (1/\tau)\cosh\eta\partial_\eta \end{pmatrix}. \tag{36}$$

Hint: Defining the matrices

$$U^\mu{}_\alpha = \frac{\partial x^\mu}{\partial q^\alpha} \quad \text{and} \quad T^\alpha{}_\mu = \frac{\partial q^\alpha}{\partial x^\mu} \tag{37}$$

one has $\hat{U}\hat{T} = \hat{T}\hat{U} = \mathbb{1}_2$.

Solutions: We calculate the Jacobi matrix of the coordinate transformation (34),

$$\hat{U} = \left(\frac{\partial x^\mu}{\partial q^\alpha} \right) = \begin{pmatrix} \cosh\eta & \tau\cosh\eta \\ \sinh\eta & \tau\cosh\eta \end{pmatrix} \tag{38}$$

its inverse is

$$\hat{T} = \frac{1}{\tau} \begin{pmatrix} \tau\cosh\eta & -\tau\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix}. \tag{39}$$

From this we get

$$\begin{pmatrix} \partial_t \\ \partial_z \end{pmatrix} = \hat{T}^{\mathrm{T}} \begin{pmatrix} \partial_\tau \\ \partial_\eta \end{pmatrix} = \begin{pmatrix} \cosh\eta\partial_\tau - \sinh(\eta)/\tau\partial_\eta \\ -\tau\sinh\eta\partial_\tau + \cosh(\eta)/\tau\partial_\eta \end{pmatrix}. \tag{40}$$

(b) Show that the above defined fluid four-velocity is given by

$$\underline{u} = \begin{pmatrix} u^0 \\ u^3 \end{pmatrix} = \begin{pmatrix} \cosh\eta \\ \sinh\eta \end{pmatrix}. \tag{41}$$

Solutions: With $v^3 = z/\tau$ and (34) we get

$$\gamma_v = \frac{1}{\sqrt{1-z^2/t^2}} = \frac{1}{\sqrt{1-\sinh^2\eta/\cosh^2\eta}} = \cosh\eta \tag{42}$$

and from that

$$\underline{u} = \begin{pmatrix} u^0 \\ u^3 \end{pmatrix} = \gamma_v \begin{pmatrix} 1 \\ v^3 \end{pmatrix} = \begin{pmatrix} \cosh\eta \\ \sinh\eta \end{pmatrix}. \tag{43}$$

(c) Show that

$$\partial_\mu u^\mu = \frac{1}{\tau}. \tag{44}$$

Solutions: With (40) we find

$$\partial_\mu u^\mu = \partial_t u^0 + \partial_z u^3 = -\frac{1}{\tau}\sinh\eta\partial_\eta\cosh\eta + \frac{1}{\tau}\cosh\eta\partial_\eta\sinh\eta = \frac{1}{\tau}(\cosh^2\eta - \sinh^2\eta) = \frac{1}{\tau}. \tag{45}$$

(d) What follows for the (proper) density of conserved quantities like $n = n(\tau)$ (particle-number density) and $s(\tau)$ (entropy density) by using the continuity equation

$$\partial_\mu(n u^\mu) = 0, \quad \partial_\mu(s u^\mu) = 0. \tag{46}$$

Note: The ideal-fluid dynamics are only valid for some initial proper time, $\tau_0 > 0$, which we take to be the "formation time" of the locally equilibrated fluid.

Solutions: For J^μ we find, again using (40) and (44)

$$\partial_\mu J^\mu = \partial_\mu[n(\tau)u^\mu] = u^\mu\partial_\mu n + n\partial_\mu u^\mu = \cosh^2\eta n'(\tau) - \sinh^2\eta n'(\tau) + n(\tau)/\tau = n'(\tau) + n(\tau)/\tau = 0. \tag{47}$$

This implies

$$\frac{n'}{n} = -\frac{1}{\tau} \Rightarrow \ln\left(\frac{n}{n_0}\right) = -\ln\left(\frac{\tau}{\tau_0}\right) \Rightarrow n(\tau) = n_0\frac{\tau_0}{\tau}. \tag{48}$$

Since also entropy is locally conserved and $S^\mu = s u^\mu$ we also have

$$s(\tau) = s_0\frac{\tau_0}{\tau}. \tag{49}$$

(e) Now also consider the equation

$$u_\nu\partial_\mu T^{\mu\nu} = 0 \tag{50}$$

under the assumption of a constant speed of sound c_s

$$P = c_s^2\epsilon. \tag{51}$$

What follows for ϵ, P, and the temperature T (as a function of τ)?

Solutions: We start with evaluating (50), using (12) with $\epsilon + P = h = (c_s^2+1)\epsilon$, (24), and (44):

$$\begin{aligned} u_\nu\partial_\mu T^{\mu\nu} &= u_\nu[(\partial_\mu h)u^\mu u^\nu + h(\partial_\mu u^\mu)u^\nu + h u^\mu\partial_\mu u^\nu - \partial^\nu P \\ &= u^\mu\partial_\mu(h-P) + \frac{h}{\tau} \\ &= u^\mu\partial_\mu\epsilon + \frac{h}{\tau} \\ &= \epsilon'(\tau) + \frac{c_s^2+1}{\tau}\epsilon = 0 \end{aligned} \tag{52}$$

with the solution

$$\epsilon(\tau) = \epsilon_0\left(\frac{\tau_0}{\tau}\right)^{c_s^2+1}. \tag{53}$$

Fpr the pressure we immediately get

$$P(\tau) = \frac{1}{c_s^2}\epsilon = P_0\left(\frac{\tau_0}{\tau}\right)^{c_s^2+1}. \tag{54}$$

From (9) we have $P = nT$ and thus

$$T(\tau) = \frac{P(\tau)}{n(\tau)} = T_0 \left(\frac{\tau_0}{\tau}\right)^{c_s^2}. \tag{55}$$

Note: The assumption on the equation of state is valid for an ultrarelativistic gas, where we set $m \simeq 0$, because then we can easily evaluate (6) and (8), because and thus

$$\epsilon = 3P = \frac{g}{(2\pi)^3} \exp(\beta\mu) 4\pi \int_0^\infty \mathrm{d}p\, p^3 \exp(-\beta p). \tag{56}$$

To evaluate the integral we note that

$$\int_0^\infty \mathrm{d}p\, p^3 \exp(-\beta p) = -\frac{\mathrm{d}^3}{\mathrm{d}\beta^3} \int_0^\infty \mathrm{d}p \exp(-\beta p) = -\frac{\mathrm{d}^3}{\mathrm{d}\beta^3} \left(\frac{1}{\beta}\right) = \frac{6}{\beta^4} = 6T^4 \tag{57}$$

and thus

$$\epsilon = 3P = \frac{3g}{\pi^2} \exp(\beta\mu) T^4. \tag{58}$$

For the invariant particle-number density we get from (5)

$$n = \frac{g}{(2\pi)^3} \exp(\beta\mu) 4\pi \int_0^\infty \mathrm{d}p\, p^2 \exp(-\beta p) = \frac{g}{\pi^2} T^3. \tag{59}$$

In this case we indeed have $c_s^2 = \partial P / \partial \epsilon = 1/3$.

Of course, already for $m \neq 0$ the equation of state becomes considerably more complicated. Of course also realistic equations of state of strongly interacting matter are much more complicated and its determination is one of the most interesting problems investigated in heavy-ion collisions as well as the physics of neutron stars, where the equation of state is determining the mass-radius relation for a given total mass and particularly which is the mass limit for a neutron star (see, e.g., [SB20]).

References

[Bjo83] J. D. Bjorken, Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region, Phys. Rev. D **27**, 140 (1983), `https://doi.org/10.1103/PhysRevD.27.140`.

[Fey69] R. P. Feynman, Very high-energy collisions of hadrons, Phys. Rev. Lett. **23**, 1415 (1969), `https://doi.org/10.1103/PhysRevLett.23.1415`.

[SB20] J. Schaffner-Bielich, *Compact Star Physics*, Cambridge University Press, Cambridge, New York, Melbourne (2020), `https://doi.org/10.1017/9781316848357`.