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Exercise Sheet 4: Solutions

(1) Master equations for summational invariants

In the manuscript [Hee15] we have derived the general master equation for an arbitrary phase-space function $\psi(\underline{x},\underline{p})$,

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}p_1^\mu f_1\psi_1-\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}f_1\left(p_1^\mu\frac{\partial\psi_1}{\partial x^\mu}+mK_1^\mu\frac{\partial\psi_1}{\partial p_1^\mu}\right)\\ =\frac{1}{4}\frac{g}{(2\pi\hbar)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1'}{E_1'}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2'}{E_2'}(\psi_1'+\psi_2'-\psi_1-\psi_2)\\ \times W(p_1',p_2'\leftarrow p_1,p_2)f_1f_2,\end{aligned} \tag{1}$$

where the notation f_1 for phase-space functions means $f(\underline{x},\underline{p}_1)$. Further we have shown that for summational invariants, which fulfill $\psi_1'+\psi_2'=\psi_1+\psi_2$ under the condition of four-momentum conservation, $\underline{p}_1+\underline{p}_2=\underline{p}_1'+\underline{p}_2'$, which is ensured in the collision integral by the corresponding factor $W(p_1',p_2'\leftarrow p_1,p_2)\propto\delta^{(4)}(\underline{p}_1+\underline{p}_2-\underline{p}_1'-\underline{p}_2')$, i.e., for summational invariants the right-hand side vanishes. Further we have shown that ψ is a summational invariant if and only if there are fields A and B_μ such that

$$\psi(\underline{x},\underline{p})=A(\underline{x})+B_\mu(\underline{x})p^\mu. \tag{2}$$

Apply (1) to the following summational invariants and interpret the results physically:

(a) $\psi=1$
Solutions: Setting $\psi=1=\text{const}$ in (1) gives

$$\frac{\partial}{\partial x^\mu}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}p_1^\mu f_1=0. \tag{3}$$

This is a local conservation law for the particle number with the current-density vector,

$$J^\mu(\underline{x})=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{E}p_1^\mu f(\underline{x},\underline{p}). \tag{4}$$

(b) $\psi=p^\alpha$
Solutions: With this ψ (1) reads

$$\frac{\partial}{\partial x^\mu}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}p_1^\mu p_1^\alpha f_1-\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}mK_1^\alpha f_1=0. \tag{5}$$

This can be interpeted as the equation of motion of a fluid element since

$$T^{\mu\alpha}(\underline{x})=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{E}p^\mu p^\alpha f(\underline{x},\underline{p}) \tag{6}$$

is the energy-momentum tensor of the fluid, and (5) thus reads

$$\frac{\partial}{\partial x^\mu} T^{\mu\alpha}(\underline{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{E} m K^\alpha(\underline{x},\underline{p}) f(\underline{x},\underline{p}). \tag{7}$$

For α^0 we get the continuum-mechanical version of the "energy-work theorem". $T^{00}(\underline{x}) = \varepsilon(\underline{x})$ is the energy density and $T^{m0}(\underline{x}) = j_E^m$ ($m \in \{1,2,3\}$) the energy-current density. The meaning of the right-hand side of (7) becomes clear by remembering that $p_\mu K^\mu = EK^0 - \vec{p} \cdot \vec{K} = 0$ and thus

$$\frac{m}{E} K^0 = \frac{m}{E} \frac{\vec{p}}{E} \cdot \vec{K} = \frac{1}{\gamma} \vec{v} \cdot \vec{K} = \vec{v} \cdot \vec{F}, \tag{8}$$

where $\vec{F}$ is the usual non-covariant force in the (1+3)-form of the equation of motion of a relativistic point particle

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{p} = \vec{F} = \frac{\mathrm{d}\tau}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}\tau} \vec{p} = \frac{1}{\gamma} \vec{K}. \tag{9}$$

Thus for $\alpha = 0$ the right-hand side is the power density due to the motion of the fluid cell in the external field.

This matches with the interpretation of (7) for the spatial components for $\alpha = a \in \{1,2,3\}$: $T^{0a} = g^a$ is the momentum density[1]. In this case the right-hand side of (7) becomes the density of the external-force field

$$f^a = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} F^a(\underline{x},\underline{p}) f(\underline{x},\underline{p}). \tag{10}$$

(c) $\psi = x^\alpha p^\beta - x^\beta p^\alpha$.

Solutions: Here the integral under the four-divergence in (1) is

$$J^{\mu\alpha\beta}(\underline{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{E} p^\mu (x^\alpha p^\beta - x^\beta p^\alpha) f(\underline{x},\underline{p}). \tag{11}$$

For $(\alpha,\beta) = (a,b)$ with $a, b \in \{1,2,3\}$ and $\mu = 0$ we obviously get the density of the angular momentum and with $\mu = m \in \{1,2,3\}$ the corresponding angular-momentum-current denity. For these components (1) reads

$$\partial_\mu J^{\mu ab} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{E} m[x^a K^b(\underline{x},\underline{p}) - x^b K^a(\underline{x},\underline{p})] = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}[x^a F^b(\underline{x},\underline{p}) - x^b F^a(\underline{x},\underline{p})], \tag{12}$$

obviously the right-hand side is the density of the torque on the fluid cell due to the external force.

For $\alpha = a \in \{1,2,3\}$ and $\mu = \beta = 0$ we get

$$J^{0a0}(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}(x^a E - t p^a) f(\underline{x},\underline{p}) = x^a \varepsilon(\underline{x}) - t g^a(\underline{x}), \tag{13}$$

where ε is the energy density and g^a the momentum density defined above. Thus, in this case (1) describes the motion of the "center of energy"[2]:

$$\partial_\mu J^{\mu a0}(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}[x^a \vec{v} \cdot \vec{F}(\underline{x},\underline{p}) - t F^a(\underline{x},\underline{p})]. \tag{14}$$

[1]Note that because of the symmetry of the energy-momentum tensor $T^{0a} = T^{a0}$ in our natural units with $c = 1$ the momentum density equals the energy-current density.

[2]This is in analogy to the motion of "center of mass" in Newtonian mechanics. It is an example for the fact that in relativistic mechanics the "measure of inertia" is the total energy rather than mass in Newtonian mechanics. Of course in our covariant definition of relativistic energy the energy of a particle at rest is m, i.e., the inertia due to the mass of the particle is included in its on-shell energy $E = \sqrt{m^2 + \vec{p}^2}$.

(2) Equilibrium distributions

The collision term, taking into account Fermi-Dirac statistics, reads

$$\begin{aligned}\mathscr{C}[f_1] = \frac{1}{2}\frac{g}{(2\pi\hbar)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1{}'}{E_1'}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2{}'}{E_2'} \\ \times W(p_1', p_2' \leftarrow p_1, p_2)[f_1' f_2'(1-f_1)(1-f_2) - f_1 f_2(1-f_1')(1-f_2')],\end{aligned} \tag{15}$$

where the factors $(1-f)$ take "Pauli blocking" into account, i.e., that a particle cannot be scattered into a state, which is already occupied. We have also seen from the H-theorem, that the equilibrium distribution $f^{(\mathrm{eq})}$ is determined by the vanishing of the collision term, i.e., the vanishing of the square bracket under the collision integral.
Show that this condition implies that for the equilibrium distribution

$$\phi = -\ln\left(\frac{f}{1-f}\right) \tag{16}$$

is a summational invariant, for which (2) must hold. In this case set

$$A(\underline{x}) = -\beta(\underline{x})\mu(\underline{x}), \quad B^\mu(\underline{x}) = \beta(\underline{x}) u_\mu(\underline{x}). \tag{17}$$

Which equilibrium distribution follows and what is the physical meaning of $\beta(\underline{x})$ and $\mu(\underline{x})$?
Solutions: With

$$\phi = \beta(u_\mu p^\mu - \mu) = \beta(\underline{u}\cdot\underline{p} - \mu) \tag{18}$$

and with (16) we get

$$\frac{f}{1-f} = \exp\left[\beta(\underline{u}\cdot\underline{p}-\mu)\right] \Rightarrow f(\underline{x},\underline{p}) = \frac{1}{\exp\left\{\beta(\underline{x})\left[\underline{u}(\underline{x})\cdot\underline{p} - \mu(\underline{x})\right]\right\}+1}. \tag{19}$$

This describes a situation, where for the fluid cell at $\underline{x}$ the distribution function is given by a Fermi-distribution function. In order to have a finite particle density we must have $\underline{u}$ time like and $\beta > 0$. Normalizing $uvecu$ such that $\underline{u}\cdot\underline{u} = 1$, we can interpret $\underline{u}$ as the four-velocity of the fluid element. Then $\underline{u}\cdot\underline{p}$ is the energy of the fluid element in its rest frame and $\beta = 1/T$ is the inverse temperature and μ the chemical potential of the fluid cell. This describes a situation, where the fluid is in local thermal equilibrium. Of course everything is valid also for β, u, and $\mu = \text{const}$, which describes a fluid in global thermal equilibrium.

References

[Hee15] H. van Hees, Introduction to relativistic transport theory (2015), https://itp.uni-frankfurt.de/~hees/publ/kolkata.pdf.