Institut für Theoretische Physik
Goethe University Frankfurt
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PD Dr. Hendrik van Hees

Exercise Sheet 3

(1) Black-Body radiation

Consider the Planck spectrum of black-body radiation. In the rest frame of the thermal source the distribution function is given by a Bose-Einstein distribution function of photons

$$f^*(\underline{k}^*, T) = 2f_{\text{B}}(E^*, T) = \frac{2}{\exp(E^*/T) - 1}, \tag{1}$$

where T is the Lorentz-invariant temperature and $\underline{k}^*$ is the wave-four-vector of the photon as measured Four-vector in the rest frame of the source.

(a) Calculate the distribution function when the black-body radiation is measured in an inertial reference frame, where the thermal source moves with a constant velocity $\vec{v} = v\vec{e}_1$ as a function of the angle θ between $\vec{k}$ and $\vec{v}$ and show that for each θ the spectrum is described by a Bose-Einstein distribution with an "effective temperature" $T_{\text{eff}}(\theta)$.

Solutions: Since the phase-space distribution function is a scalar field and with $\underline{u}^* = (1,0,0,0)$ and $\underline{u} = \gamma(1, v, 0, 0)$ the four-velocity of the heat bath in its rest frame and the "lab frame", respectively, we have

$$f(\underline{k}, T) = f^*(\underline{k}^*, T) = 2f_{\text{B}}(\underline{u}^* \cdot \underline{k}^*, T) = 2f_{\text{B}}(\underline{u} \cdot \underline{k}, T). \tag{2}$$

Now

$$\underline{u} \cdot \underline{k} = \gamma(k^0 - vk^1) = \gamma E(1 - v\cos\theta), \tag{3}$$

where we have used that $k^0 = E = |\vec{k}|$ and $\vec{v} \cdot \vec{k} = v|\vec{k}|\cos\theta = \omega v \cos\theta$. Writing out the Bose-Einstein distribution function gives

$$f(\underline{k}, T) = \frac{2}{\exp\left[\frac{\gamma E(1-v\cos\theta)}{T}\right] - 1} = \frac{2}{\exp(E/T_{\text{eff}}(\theta)) - 1}, \tag{4}$$

i.e., the "effective temperature" is

$$T_{\text{eff}}(\theta) = \frac{\sqrt{1-v^2}}{1 - v\cos\theta} T. \tag{5}$$

For $\theta = 0$ we measure the radiation from a source which approaches the detector headon. The corresponding effective temperature is

$$T_{\text{eff}}(0) = \frac{\sqrt{1-v^2}}{1-v} T = \sqrt{\frac{1+v}{1-v}} T > T, \tag{6}$$

i.e., we see a "blue-shifted" black-body spectrum in comparison to the observer in the source's rest frame, as expected from the Doppler effect, to be discussed in the next part of this problem.

For $\theta = \pi$ we have a source which recedes from the detector, leading to the red-shifted black-body spectrum with $T_{\text{eff}}(\pi) = T\sqrt{(1-v)/(1+v)}$. For all other angles we have red or blue shifts in between these extremes.

Contrary to the non-relativistic Doppler effect there's also a Doppler effect when the radiation is coming from a direction perpendicular to the velocity of the source, which is due to relativistic time dilation (see also the next part of the problem).

(b) To interpret the result as a relativistic Doppler effect for photons, discuss the Lorentz-transformation properties of an electromagnetic plane-wave mode.

Solutions: It is most easy to discuss the Lorentz transformation of the electromagnetic field in terms of its four-potential. Choosing the Lorenz gauge $\partial_\mu A^\mu = 0$ we can write with $\vec{k} = E\vec{n}$, $\vec{n}^2 = 1$,

$$\underline{A}(\underline{x}) = \begin{pmatrix} \vec{n} \cdot \vec{A}_0 \\ \vec{A}_0 \end{pmatrix} \exp[-E(t - \vec{n} \cdot \vec{x})]. \tag{7}$$

In terms of the field components in the rest frame we have

$$\underline{A}^*(\underline{x}^*) = \hat{\Lambda}\underline{A}(\underline{x}) = \Lambda\underline{A}(\Lambda^{-1}\underline{x}^*). \tag{8}$$

with

$$\hat{\Lambda} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

To discuss the Doppler effect we only need to look at the argument of the exponential, i.e., the phase factor,

$$\underline{k}^* \cdot \underline{x}^* = \underline{k} \cdot \underline{x} = \underline{k} \cdot \hat{\Lambda}^{-1}\underline{x}^* = (\hat{\Lambda}\underline{k}) \cdot \underline{x}, \tag{10}$$

which means that

$$\underline{k}^* = E^* \begin{pmatrix} 1 \\ \vec{n}^* \end{pmatrix} = \hat{\Lambda}\underline{k} = E\hat{\Lambda}\begin{pmatrix} 1 \\ \vec{n} \end{pmatrix} = E \begin{pmatrix} \gamma(1 - v\cos\theta) \\ \gamma(n_1 - v) \\ n_2 \\ n_3 \end{pmatrix}. \tag{11}$$

From the time component we get

$$E = \frac{E^*}{\gamma(1 - v\cos\theta)} = \frac{\sqrt{1-v^2}}{1 - v\cos\theta} E^*. \tag{12}$$

For $\theta = 0$ we get the maximal possible blue shift, $E_{\theta=0} = E^* \sqrt{(1+v)/(1-v)}$ and for $\theta = \pi$ the maxmal possible red shift $E_{\theta=\pi} = E^* \sqrt{(1-v)/(1+v)}$. The relativistic Doppler effect is a combination of the corresponding non-relativistic Doppler effect due to the relative motion between the source and the observer and the relativistic time dilation. For that reason there's also a "transverse Doppler effect", $E_{\theta=\pi/2} = \sqrt{1-v^2}E^*$, which is entirely due to time dilation.

Remark: From the temporal components of (12) one obtains also the aberration effect for light. By choice of the spatial coordinate system we can assume without loss of generality $\vec{n} = (\cos\theta, \sin\theta, 0)$ with $\theta \in [0, \pi]$, Then

$$\vec{n}^* = \begin{pmatrix} \cos\theta^* \\ \sin\theta^* \\ 0 \end{pmatrix} = \frac{E}{E^*} \begin{pmatrix} \gamma(\cos\theta - v) \\ \sin\theta \\ 0 \end{pmatrix} = \frac{1}{1 - v\cos\theta} \begin{pmatrix} \cos\theta - v \\ \sqrt{1-v^2}\sin\theta \\ 0 \end{pmatrix} \tag{13}$$

and thus, because also $\theta^* \in [\theta, 0]$ (why?) we find uniquely

$$\theta^* = \arccos\left(\frac{\cos\theta - v}{1 - v\cos\theta}\right), \tag{14}$$

which describes the change of the observed direction between light source and observer in the frame, where the source moves with velocity v, in comparison to the direction as observed in the rest frame of the source, which is known as the aberration effect.

(2) Unitarity of the S matrix and the Boltzmann collision term

In the lecture we have derived the Boltzmann equation as

$$\frac{p_1^\mu}{m}\frac{\partial f_1}{\partial x^\mu} = \frac{1}{2}\frac{g}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1'}{E_1'}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2'}{E_2'}\left[\underbrace{f_{1'}f_{2'}W(\underline{p}_1,\underline{p}_2 \leftarrow \underline{p}_1',\underline{p}_2')}_{\text{“gain”}} - \underbrace{f_1 f_2 W(\underline{p}_1',\underline{p}_2' \leftarrow \underline{p}_1,\underline{p}_2)}_{\text{“loss”}}\right], \tag{15}$$

where $f_1 = f(\underline{x},\underline{p}_1)$ etc.; g is the degeneracy factor, and the factor $1/2$ takes into account the quantum-mechanical indistinguishability of the particles.

The invariant transition-probability rate W for elastic scattering is related to the invariant matrix elements from quantum field theory by

$$W(p_1',p_2' \leftarrow p_1,p_2) = \frac{\overline{|\mathscr{M}_{fi}|^2}}{16(2\pi)^6}(2\pi)^4\delta^{(4)}(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2), \tag{16}$$

where the invariant matrix element $\mathscr{M}_{fi}$ is related to the corresponding S-matrix element by

$$S_{fi} = \mathbb{1} - \mathrm{i}(2\pi)^4\delta^{(4)}(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2)T_{fi}, \quad T_{fi} = \frac{\mathscr{M}_{fi}}{\prod_{k\in\{f,i\}}[(2\pi)^3 2E_k]^{1/2}}. \tag{17}$$

To simplify the following consideration of the unitary of the S-matrix assume the “box regularization”, so that the four-momentum conserving δ distribution becomes a Kronecker-δ symbol.

For simplicity we write for the box-regularized S matrix

$$\hat{S} = \mathbb{1} - \mathrm{i}\hat{T} \tag{18}$$

(a) Use the unitarity of the S-matrix,

$$\hat{S}^\dagger\hat{S} = \hat{S}\hat{S}^\dagger = \mathbb{1}, \tag{19}$$

to derive the relation for the T-matrix elements,

$$\hat{T}^\dagger\hat{T} = \hat{T}\hat{T}^\dagger. \tag{20}$$

Solutions: Using (18) in (19) we find

$$\hat{S}\hat{S}^\dagger = (\mathbb{1}-\mathrm{i}\hat{T})(\mathbb{1}+\mathrm{i}\hat{T}^\dagger) = \mathbb{1} - \mathrm{i}(\hat{T}-\hat{T}^\dagger) + \hat{T}\hat{T}^\dagger = \mathbb{1} \Rightarrow \hat{T}\hat{T}^\dagger = \mathrm{i}(\hat{T}-\hat{T}^\dagger). \tag{21}$$

On the other hand

$$\hat{S}^\dagger\hat{S} = (\mathbb{1}+\mathrm{i}\hat{T}^\dagger)(\mathbb{1}-\mathrm{i}\hat{T}) = \mathbb{1} - \mathrm{i}(\hat{T}-\hat{T}^\dagger) + \hat{T}^\dagger\hat{T} \Rightarrow \hat{T}^\dagger\hat{T} = \mathrm{i}(\hat{T}-\hat{T}^\dagger) \stackrel{(21)}{=} \hat{T}\hat{T}^\dagger, \tag{22}$$

which proves (20)

(b) Show that this implies, now written again in the infinite-volume limit and for the integral over the two-particle final states

$$\int \mathrm{d}^3\vec{p}_1' \int \mathrm{d}^3\vec{p}_2'\delta^{(4)}(P_f-P_i)|T_{fi}|^2 = \int \mathrm{d}^3\vec{p}_1' \int \mathrm{d}^3\vec{p}_2'\delta^{(4)}(P_f-P_i)|T_{if}|^2, \tag{23}$$

relevant for elastic scattering[1].

Solutions: In the regularized version $\hat{T}$ includes the Kronecker"=δ ensuring the conservation of the discrete four-momenta. This implies that for the transition *rates* in the infinite-volume limit there's only one of the δ distributions to be included in $|S_{fi}|^2$, which shows that (20) in the infinite-volume limit implies (23).

[1]This relation is known as the “weak principle of detailed balance”

(c) Use this in the loss term of (15) to show that the collision term simplifies to

$$\frac{p_1^\mu}{m}\frac{\partial f_1}{\partial x^\mu} = \frac{1}{2}\frac{g}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}_2 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_1'}{E_1}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2'}{E_2} W(p_1', p_2' \leftarrow p_1, p_2)(f_{1'}f_{2'} - f_1 f_2). \tag{24}$$

Solutions: Writing (23) in terms of W gives

$$\int \frac{\mathrm{d}^3\vec{p}_1'}{E_1'}\int \frac{\mathrm{d}^3\vec{p}_2'}{E_2'} W(p_1', p_2' \leftarrow p_1, p_2) = \int \frac{\mathrm{d}^3\vec{p}_1'}{E_1'}\int \frac{\mathrm{d}^3\vec{p}_2'}{E_2'} W(p_1, p_2 \leftarrow p_1', p_2'), \tag{25}$$

i.e., in the loss term in (15) we can write

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_1'}{E_1'}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_2'}{E_2'} f_1 f_2 W(\underline{p}_1', \underline{p}_2' \leftarrow \underline{p}_1, \underline{p}_2) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_1'}{E_1'}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_2'}{E_2'} f_1 f_2 W(\underline{p}_1, \underline{p}_2 \leftarrow \underline{p}_1', \underline{p}_2'). \tag{26}$$

Using this in (15) indeed simplifies the collision term to (24).