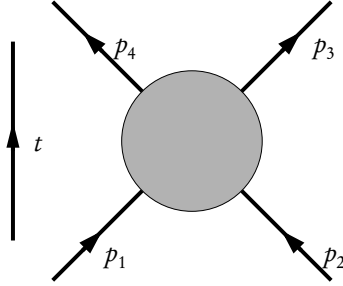


Exercise Sheet 2

(1) Scattering kinematics for $2 \rightarrow 2$ scattering



We consider the scattering process $A + B \rightarrow C + D$ as depicted in the above Feynman diagram, where the incoming particles have four-momenta \underline{p}_1 and \underline{p}_2 and the outgoing particles \underline{p}_3 and \underline{p}_4 . We work in natural units with $\hbar = c = 1$. Then the particle momenta are “on shell”, i.e., $\underline{p}_j^2 = E_j^2 - \vec{p}_j^2 = m_j^2$ with the invariant masses, m_j . Also energy-momentum conservation holds in the scattering process, $\underline{p}_1 + \underline{p}_2 = \underline{p}_3 + \underline{p}_4$.

Except the four invariant masses there are three Mandelstam variables, which characterize the scattering kinematics in a Lorentz-invariant way, defined by

$$s = (\underline{p}_1 + \underline{p}_2)^2 = (\underline{p}_3 + \underline{p}_4)^2, \quad (1)$$

$$t = (\underline{p}_1 - \underline{p}_3)^2 = (\underline{p}_2 - \underline{p}_4)^2, \quad (2)$$

$$u = (\underline{p}_1 - \underline{p}_4)^2 = (\underline{p}_2 - \underline{p}_3)^2. \quad (3)$$

(a) Prove that the Mandelstam variables are not independent, but that the equation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \quad (4)$$

holds.

Solutions: Using the definitions (1-3), the on-shell conditions, and energy-momentum conservation, we get

$$\begin{aligned} s + t + u &= (\underline{p}_1 + \underline{p}_2)^2 + (\underline{p}_1 - \underline{p}_3)^2 + (\underline{p}_1 - \underline{p}_4)^2 \\ &= (m_1^2 + m_2^2 + 2\underline{p}_1 \cdot \underline{p}_2) + (m_1^2 + m_3^2 - 2\underline{p}_1 \cdot \underline{p}_3) + (m_1^2 + m_4^2 - 2\underline{p}_1 \cdot \underline{p}_4) \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\underline{p}_1 \cdot [\underline{p}_2 - (\underline{p}_3 + \underline{p}_4)] \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\underline{p}_1 \cdot [\underline{p}_2 - (\underline{p}_1 + \underline{p}_2)] \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2. \end{aligned} \quad (5)$$

(b) Now consider the kinematics in the “**lab frame**”¹, where particle A is at rest, $\underline{p}_1 = (m_1, 0, 0, 0)^T$ and particle B moves with a four-momentum $\underline{p}_2 = (E_2, 0, 0, p_2)^T$. The **relative velocity** is defined as the (magnitude of the) three-velocity of particle B in this lab frame, where particle A is at rest, i.e., $v_{\text{rel}} = p_2/E_2$. Show that

$$v_{\text{rel}} = \frac{\sqrt{(\underline{p}_1 \cdot \underline{p}_2)^2 - m_1^2 m_2^2}}{\underline{p}_1 \cdot \underline{p}_2}. \quad (6)$$

¹This terminology originates from the early days of accelerator experiments, where one could only manage fixed-target experiments, i.e., one had a beam of particles of kind B , which were scattered with particles of kind A at rest in some fixed target like some metal foil, as in Rutherford’s seminal “gold-foil experiment”, where α particles from some radioactive sample have been scattered by gold atoms within a gold foil.

Further show that one can express the energies in three-momenta of the particles in the incoming and outgoing channel as

$$E_2^{(\text{lab})} = \frac{s - m_1^2 - m_2^2}{2m_1}, \quad (7)$$

$$P_2^{(\text{lab})} = \sqrt{(E_2^{(\text{lab})})^2 - m_2^2} = \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{2m_1}, \quad (8)$$

$$E_3^{(\text{lab})} = \frac{m_1^2 + m_3^2 - t}{2m_1}, \quad (9)$$

$$E_4^{(\text{lab})} = \frac{m_1^2 + m_4^2 - u}{2m_1}, \quad (10)$$

$$P_3^{(\text{lab})} = \frac{\sqrt{[(m_1 + m_3)^2 - t][(m_1 - m_3)^2 - t]}}{2m_1}, \quad (11)$$

$$P_4^{(\text{lab})} = \frac{\sqrt{[(m_1 + m_4)^2 - u][(m_1 - m_4)^2 - u]}}{2m_1}. \quad (12)$$

Solutions: We have

$$\begin{aligned} \underline{p}_1 \cdot \underline{p}_2 &= m_1 E_2^{(\text{lab})} = E_1^{(\text{lab})} E_2^{(\text{lab})} \Rightarrow (\underline{p}_1 \cdot \underline{p}_2)^2 - m_1^2 m_2^2 = E_1^{(\text{lab})2} P_2^{(\text{lab})2} \\ \Rightarrow v_{\text{rel}}^2 &= \frac{P_2^{(\text{lab})2}}{E_2^{(\text{lab})2}} = \frac{(\underline{p}_1 \cdot \underline{p}_2)^2 - m_1^2 m_2^2}{(\underline{p}_1 \cdot \underline{p}_2)^2}. \end{aligned} \quad (13)$$

The Mandelstam variable s is

$$\begin{aligned} s &= (\underline{p}_1 + \underline{p}_2)^2 = (m_1 + E_2^{(\text{lab})})^2 - P_2^{(\text{lab})2} \\ &= m_1^2 + E_2^{(\text{lab})2} + 2m_1 E_2^{(\text{lab})} - P_2^{(\text{lab})2} = m_1^2 + m_2^2 + 2m_1 E_2^{(\text{lab})} \\ \Rightarrow E_2 &= \frac{s - m_1^2 - m_2^2}{2m_1}. \end{aligned} \quad (14)$$

and with that

$$\begin{aligned} P_2^{(\text{lab})2} &= E_2^{(\text{lab})2} - m_2^2 = \frac{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{4m_1^2} \\ &= \frac{[s - (m_1^2 + m_2^2 + 2m_1 m_2)][s - (m_1^2 + m_2^2 - 2m_1 m_2)]}{4m_1^2} \\ &= \frac{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}{4m_1^2}. \end{aligned} \quad (15)$$

Similarly we have

$$t = (\underline{p}_1 - \underline{p}_3)^2 = m_1^2 + m_3^2 - 2m_1 E_3^{(\text{lab})} \Rightarrow E_3^{(\text{lab})} = \frac{m_1^2 + m_3^2 - t}{2m_1} \quad (16)$$

and

$$P_3^{(\text{lab})2} = E_3^{(\text{lab})2} - m_3^2 = \frac{[(m_1 + m_3)^2 - t][(m_1 - m_3)^2 - t]}{4m_1^2}. \quad (17)$$

The analogous expressions for u follow by making all labels “3” to labels “4” in the previous two equations:

$$u = (\underline{p}_{-1} - \underline{p}_{-4})^2 = m_1^2 + m_4^2 - 2m_1 E_4^{(\text{lab})} \Rightarrow E_4^{(\text{lab})} = \frac{m_1^2 + m_4^2 - t}{2m_1} \quad (18)$$

and

$$P_4^{(\text{lab})2} = E_3^{(\text{lab})2} - m_4^2 = \frac{[(m_1 + m_4)^2 - t][(m_1 - m_4)^2 - t]}{4m_1^2}. \quad (19)$$

(c) Now consider the “**center-momentum system**”, where the total three-momentum in the incoming (and thus also in the outgoing) channel vanishes,

$$\underline{p}_{-1}^{(\text{cm})} = \begin{pmatrix} E_1^{(\text{cm})} \\ \vec{p}_{-1}^{(\text{cm})} \end{pmatrix}, \quad \underline{p}_{-2}^{(\text{cm})} = \begin{pmatrix} E_2^{(\text{cm})} \\ \vec{p}_{-2}^{(\text{cm})} \end{pmatrix}, \quad \underline{p}_{-3}^{(\text{cm})} = \begin{pmatrix} E_3^{(\text{cm})} \\ \vec{p}'^{(\text{cm})} \end{pmatrix}, \quad \underline{p}_{-4}^{(\text{cm})} = \begin{pmatrix} E_4^{(\text{cm})} \\ -\vec{p}'^{(\text{cm})} \end{pmatrix}. \quad (20)$$

Calculate the Mandelstam variable s and interpret the physical meaning of \sqrt{s} . Further show that for the magnitudes of the three-momentum in the cm frame

$$P^{(\text{cm})} = \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{2\sqrt{s}}, \quad (21)$$

$$P'^{(\text{cm})} = \frac{\sqrt{[s - (m_3 + m_4)^2][s - (m_3 - m_4)^2]}}{2\sqrt{s}}. \quad (22)$$

What follows from the physical constraints that $P^{(\text{cm})} \geq 0$ and $P'^{(\text{cm})} \geq 0$.

Solutions: In the CM frame we have

$$s = (E_1^{(\text{cm})} + E_2^{(\text{cm})})^2 = (E_3^{(\text{cm})} + E_4^{(\text{cm})})^2 \Rightarrow \sqrt{s} = E_{\text{cm}}, \quad (23)$$

i.e. \sqrt{s} is the total collision energy of the two particles in the center-momentum frame.

On the other hand

$$\begin{aligned} s &= m_1^2 + m_2^2 + 2(E_1^{(\text{cm})} + P^{(\text{cm})})^2 \\ &= m_1^2 + m_2^2 + 2[E_1^{(\text{cm})}(\sqrt{s} - E_1^{(\text{cm})}) + P^{(\text{cm})2}] \\ &= m_2^2 - m_1^2 + 2E_1^{(\text{cm})}\sqrt{s} \\ \Rightarrow E_1^{(\text{cm})} &= \frac{s - m_2^2 + m_1^2}{2\sqrt{s}}. \end{aligned} \quad (24)$$

For the center-mass momentum in the incoming channel we get

$$\begin{aligned} P^{(\text{cm})2} &= E_1^{(\text{cm})2} - m_1^2 = \frac{(s - m_2^2 + m_1^2)^2 - 4s m_1^2}{4s} \\ &= \frac{(s - 2\sqrt{s} m_1 + m_1^2 - m_2^2)(s + \sqrt{s} m_1 + m_1^2 - m_2^2)}{4s} \\ &= \frac{[(\sqrt{s} - m_1)^2 - m_2^2][(\sqrt{s} - m_1)^2 - m_2^2]}{4s} \\ &= \frac{(\sqrt{s} - m_1 - m_2)(\sqrt{s} - m_1 + m_2)(\sqrt{s} + m_1 - m_2)(\sqrt{s} + m_1 + m_2)}{4s} \\ &= \frac{[s - (m_1 + m_2)^2][\sqrt{s} - (m_1 - m_2)^2]}{4s}. \end{aligned} \quad (25)$$

For the center-mass momentum in the outgoing channel we only have to make $m_1 \rightarrow m_3$ and $m_2 \rightarrow m_4$ in the previous equation, which proves (22).