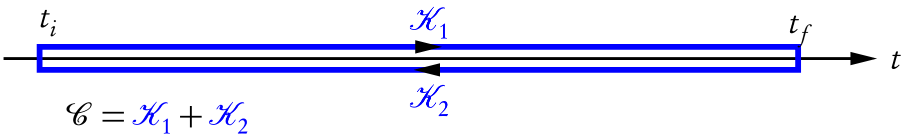
Exercise Sheet 13

Free real-time propagators for a charged scalar field

As we have seen in the lecture, for the evaluation of N-point real-time Green's functions one only needs the real-time propagators along the real branches of the Schwinger-Keldysh contour,



i.e., the matrix propagators,

$$\mathrm{i}G^{12}(x) = \left\langle \mathbf{\Phi}^{\dagger}(0)\mathbf{\Phi}(x) \right\rangle, \tag{1}$$

$$\mathrm{i}G^{21}(x) = \left\langle \mathbf{\Phi}(x)\mathbf{\Phi}^{\dagger}(0) \right\rangle \tag{2}$$

$$\mathrm{i}G^{11}(x) = \left\langle \mathscr{T}_c \mathbf{\Phi}(x)\mathbf{\Phi}^{\dagger}(0) \right\rangle, \tag{3}$$

$$\mathrm{i}G^{22}(x) = \left\langle \mathscr{T}_a \mathbf{\Phi}(x)\mathbf{\Phi}^{\dagger}(0) \right\rangle, \tag{4}$$

where $\mathbf{\Phi}$ is a free charged scalar field with the Lagrangian

$$\mathscr{L} = (\partial_\mu \Phi)^* \partial^\mu \Phi - m^2 \Phi^* \Phi. \tag{5}$$

It has a conserved charge from global U(1) symmetry,

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\, \Phi^*(x) \overleftrightarrow{\partial_t} \Phi(x), \tag{6}$$

and the average in (1-4) is taken with respect to the grand-canonical statistical operator,

$$\langle \mathbf{A} \rangle = \frac{1}{Z} \mathrm{Tr}[\exp(-\beta \mathbf{H} + \alpha \mathbf{Q})\mathbf{A}], \quad \alpha = \beta\mu, \quad \beta > 0, \quad -m \leq \mu \leq m. \tag{7}$$

The free fields have a plane-wave decomposition,

$$\mathbf{\Phi}(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{[(2\pi)^3 2E_p]^{3/2}} \left[\mathbf{a}(\vec{p})\exp(-\mathrm{i}x\cdot p) + \mathbf{b}^{\dagger}(\vec{p})\exp(\mathrm{i}x\cdot p)\right]_{p^0=E_p}, \quad E_p = \sqrt{m^2+\vec{p}^2}. \tag{8}$$

In the following we will also need the Lorentz-invariant "commutator function",

$$\mathrm{i}\Delta(x) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \underbrace{2\pi\sigma(p^0)\delta(p^2-m^2)}_{\tilde{\Delta}(p)} \exp(-\mathrm{i}p\cdot x), \tag{9}$$

which occurs in the commutator of free-field operators,

$$\left[\mathbf{\Phi}(x), \mathbf{\Phi}^{\dagger}(0)\right] = \mathrm{i}\Delta(x)\mathbb{1}. \tag{10}$$

Also note that due to the space-time translation invariance of the equilibrium state the contour Green's functions obey

$$\mathrm{i}G_{\mathscr{C}}(x,y) := \left\langle \mathscr{T}_{\mathscr{C}} \mathbf{\Phi}(x)\mathbf{\Phi}^{\dagger}(y) \right\rangle = \mathrm{i}G_{\mathscr{C}}(x-y). \tag{11}$$

The $\mathscr{T}$ symbols stand for ordering according to the time arguments in operator products: $\mathscr{T}_c$ is the "causal time ordering", referring to the upper branch $\mathscr{K}_1$, where operator products are ordered with *increasing time arguments* from right to left; $\mathscr{T}_a$ is the "anti-causal time ordering", referring to the upper branch $\mathscr{K}_2$, where operator products are ordered with *decreasing time arguments* from right to left, and $\mathscr{T}_{\mathscr{C}}$ for the Keldysh-contour ordering, i.e., with $\mathscr{T}_c$ ($\mathscr{T}_a$) if all time arguments are on $\mathscr{K}_1$ ($\mathscr{K}_2$), and otherwise all time arguments on $\mathscr{K}_2$ are considered to be "later" than all times on $\mathscr{K}_1$.

(a) Using the following relations (see Lecture)

$$\mathbf{\Phi}(t,\vec{x}) = \exp(\mathrm{i}t\mathbf{H})\mathbf{\Phi}(0,\vec{x})\exp(-\mathrm{i}\mathbf{H}t), \tag{12}$$
$$\exp(-\alpha\mathbf{Q})\mathbf{\Phi}(x)\exp(\alpha\mathbf{Q}) = \exp(\alpha)\mathbf{\Phi}(x), \tag{13}$$
$$\exp(-\alpha\mathbf{Q})\mathbf{\Phi}^\dagger(x)\exp(\alpha\mathbf{Q}) = \exp(-\alpha)\mathbf{\Phi}^\dagger(x), \tag{14}$$

$[\mathbf{H},\mathbf{Q}] = 0$, and $\langle\mathbf{AB}\rangle = \langle\mathbf{BA}\rangle$, prove the **KMS** condition,

$$G^{21}(t,\vec{x}) = \exp(-\alpha)G^{12}(t+\mathrm{i}\beta,\vec{x}). \tag{15}$$

For which *complex* time arguments are G^{21} and G^{12} well defined?

(b) In the following we use the usual definitions of the Fourier transformations between space-time and energy-momentum representation,

$$F(x) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{F}(p)\exp(-\mathrm{i}p\cdot x), \quad \tilde{F}(p) = \int_{\mathbb{R}^4} \mathrm{d}^4 x F(x)\exp(+\mathrm{i}p\cdot x). \tag{16}$$

Show that the KMS condition (15) translates to

$$\tilde{G}^{21}(p) = \exp(\beta p^0 - \alpha)\tilde{G}^{12}(p) \tag{17}$$

in the energy-momentum representation of the Green's functions.

(c) Show that

$$\tilde{G}^{21}(p) - \tilde{G}^{12}(p) = \tilde{\Delta}(p) = -2\pi\mathrm{i}\sigma(p^0)\delta(p^2-m^2). \tag{18}$$

(d) Use this relation and the KMS condition (17) to show that

$$\tilde{G}^{21}(p) = \left[1+f_{\mathrm{B}}(p^0-\mu)\right]\tilde{\Delta}(p), \quad \tilde{G}^{12}(p) = f_{\mathrm{B}}(p^0-\mu)\tilde{\Delta}(p). \tag{19}$$

(e) With help of

$$\tilde{\Theta}(p^0) = \int_{\mathbb{R}} \mathrm{d}x^0 \exp[\mathrm{i}x^0(p^0+\mathrm{i}0^+)]\Theta(t) = \frac{\mathrm{i}}{p^0+\mathrm{i}0^+} \tag{20}$$

and the convolution theorem

$$\tilde{C}(p^0) = \int_{\mathbb{R}^3} \frac{\mathrm{d}k^0}{2\pi}\tilde{A}(p-k^0)\tilde{B}(k^0), \tag{21}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are the Fourier transforms of $A(t)$, $B(t)$, and $C(t) = A(t)B(t)$, that the retarded propagator,

$$G_{\mathrm{ret}}(x) = -\mathrm{i}\Theta(t)\left\langle\left[\mathbf{\Phi}(x),\mathbf{\Phi}^\dagger(0)\right]\right\rangle = \Theta(t)\Delta(x), \tag{22}$$

in the energy-momentum representation translates to

$$\tilde{G}_{\mathrm{ret}}(p) = \frac{1}{p^2-m^2+\mathrm{i}\sigma(p^0)0^+}. \tag{23}$$

(f) Finally show that

$$\tilde{G}^{11}(p) = \tilde{G}_{\mathrm{ret}}(p) + \tilde{G}^{12}(p) = \frac{1}{p^2-m^2+\mathrm{i}0^+\sigma(p^0)} - 2\pi\mathrm{i}\delta(p^2-m^2)\sigma(p_0)f_{\mathrm{B}}(p^0-\mu), \tag{24}$$
$$\tilde{G}^{22}(p) = \tilde{G}^{21}(p) - \tilde{G}_{\mathrm{ret}}(p) = -\frac{1}{p^2-m^2+\mathrm{i}0^+\sigma(p^0)} - 2\pi\mathrm{i}\delta(p^2-m^2)\sigma(p_0)\left[1+f_{\mathrm{B}}(p^0-\mu)\right]. \tag{25}$$