Einführung in die Theoretische Kern- und Elementarteilchenphysik 2

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SoSe 2026
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Exercise Sheet 12

Non-Abelian gauge theories

In 1954 Yang and Mills [YM54] generalized the Abelian gauge invariance of QED to non-Abelian gauge symmetries, in the quoted paper especially to SU(2) isospin symmetry. Here we consider a general SU(N) group for N Dirac fields with the *same* mass, written in a column vector $(\psi_1, \cdots, \psi_N)^{\mathrm{T}}$. The Lagrangian for the free fields reads

$$\mathscr{L}_0 = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi. \tag{1}$$

(a) Show that the free Lagrangian is invariant under global SU(N)-transformations,

$$\psi'(x) = \hat{U}\psi(x), \quad \overline{\psi}' = \overline{\psi}\hat{U}^\dagger. \tag{2}$$

(b) We define the infinitesimal generators of the group as $\hat{T}^a$, such that the global transformation reads

$$\hat{U}(\vec{\chi}) = \exp(-\mathrm{i}g\hat{T}^a\chi^a). \tag{3}$$

with real parameters $\vec{\chi} = (\chi^a)$. What follows for an infinitesimal transformation from unitarity and unimodularity,

$$\hat{U}\hat{U}^\dagger = \mathbb{1}, \quad \det\hat{U} = 1? \tag{4}$$

Hint: For the latter constraint, show first, that for an arbitrary infinitesimal matrix $\delta\hat{M}$

$$\det(\mathbb{1} + \delta\hat{M}) = 1 + \operatorname{tr}\hat{M}. \tag{5}$$

(c) Now we want to generalize the symmetry to a local gauge invariance, i.e., we make $\chi = \chi(\underline{x})$ in (3). Then, as in electrodynamics, we attempt to define a gauge-covariant derivative D_μ by introducing vector fields $\hat{A}_\mu = A_\mu^a\hat{T}^a$, $A_\mu^a \in \mathbb{R}$, and defining

$$\mathrm{D}_\mu\psi(\underline{x}) = [\partial_\mu + \mathrm{i}g\hat{A}_\mu(\underline{x})]\psi(\underline{x}). \tag{6}$$

Derive the gauge-transformation rule for $\hat{A}_\mu$ such that with

$$\psi'(\underline{x}) = \hat{U}(\underline{x})\psi(x) \tag{7}$$

the covariant derivative transforms in the same way,

$$\mathrm{D}'_\mu\psi'(\underline{x}) = \hat{U}(\underline{x})\mathrm{D}_\mu\psi(\underline{x}). \tag{8}$$

(d) Show that the transformation rule is compatible with $A_\mu^a \in \mathbb{R}$.

(e) Calculate the commutator $\left[\mathrm{D}_\mu, \mathrm{D}_\nu\right] = \mathrm{i}g\hat{F}_{\mu\nu} = \mathrm{i}gF_{\mu\nu}^a T^a$ with $F_{\mu\nu}^a \in \mathbb{R}$ by applying it to an arbitrary field ψ, which transforms under local gauge transformations as given in (6).

Note: The commutation relations of the generators are defined via the structure constants $f^{abc} \in \mathbb{R}$,

$$\left[\hat{T}^a, \hat{T}^b\right] = \mathrm{i}f^{abc}\hat{T}^c. \tag{9}$$

(f) Using the fact that the generators $\hat{T}^a$ of the Lie algebra su(N) can be chosen such that

$$\operatorname{tr}(\hat{T}^a\hat{T}^b) = \frac{1}{2}\delta^{ab}\mathbb{1}, \tag{10}$$

show that the Lagrangian

$$\mathscr{L} = -\frac{1}{2}\operatorname{tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}) + \overline{\psi}(\mathrm{i}D\!\!\!\!/ - m)\psi = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \overline{\psi}(\mathrm{i}D\!\!\!\!/ - m)\psi \tag{11}$$

is gauge invariant.

(g) How about a mass term

$$\mathscr{L}_M = M^2 \operatorname{tr}(\hat{A}_\mu \hat{A}^\mu) = \frac{1}{2} M^2 A^a_\mu A^{a\mu}? \tag{12}$$

(h) Why must the coupling g for all fields ψ the same (universality of the interaction strength in non-Abelian gauge theories)?

Note: The gauge groups, occuring in the standard model, are

- the Abelian case (QED) with gauge group U(1). Then one has only one generator $\hat{T} = 1$ (i.e., simply the real number 1. For any charged fermion (and its antiparticle), described by a Dirac field, with charge number Q the gauge-invariant matter Lagrangian reads

$$\mathscr{L}_{\text{QED}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi, \quad \mathrm{D}_\mu = \partial_\mu + \mathrm{i}QeA_\mu. \tag{13}$$

- $\mathrm{SU}(3)_\mathrm{C}$ (QCD) with $\hat{T}^a_C = \hat{\lambda}^a/2$, where $\hat{\lambda}^a$ are the 8 Gell-Mann matrices acting on color space of the quarks, i.e., each quark field has 3 color components $\psi = (\psi_r, \psi_g, \psi_b)^\mathrm{T}$. There are 6 quarks $\psi = (\psi_f)$, $f \in \{\mathrm{u, d, c, s, t,b}\}$, with each ψ_f being a color triplet, gauge-covariant Lagrangian for the quarks, neglecting weak and electromagnetic interactions, then reads

$$\mathscr{L} = \overline{\psi}(\mathrm{i}\not{D} - \hat{M})\psi, \tag{14}$$

with the gauge-covariant derivative, $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}g_\mathrm{s}\hat{T}^a_\mathrm{C} G^a_\mu$, and the mass matrix $\hat{M} = \operatorname{diag}(m_\mathrm{u}, \cdots, m_\mathrm{b})$.

- $\mathrm{SU}(2)_\text{wiso} \times \mathrm{U}(1)_\text{whyper}$ (QFD). This gauge group is more special since it's a chiral gauge group, acting differently on the left- and righthanded parts of the fermion fields.

For a simplified model with only 1 family the fermion-gauge-field and keeping the neutrino massless we have a wiso-doublet of left-handed fermions,

$$\begin{pmatrix} \nu_{\mathrm{e\,L}} \\ e_\mathrm{L} \end{pmatrix}, \quad \begin{pmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{pmatrix} \tag{15}$$

and wiso-singlets

$$e_\mathrm{R}, \quad u_\mathrm{R}, \quad d_\mathrm{R}. \tag{16}$$

In the following we express all fermions in terms of Dirac fields, defining a wiso-doublet for both the leptons and the quarks,

$$\psi_\ell = \begin{pmatrix} \nu_\mathrm{e} \\ e \end{pmatrix}, \quad \psi_\mathrm{q} = \begin{pmatrix} u \\ d \end{pmatrix}. \tag{17}$$

We denote the three $\mathrm{SU}(2)_\text{wiso}$ gauge fields with $W^a_\mu \in \mathbb{R}$ and the $\mathrm{U}(1)_\text{whyper}$ gauge field with B_μ and the SU(2)-generators as $\hat{T}^a_\text{wiso} = \hat{\sigma}^a/2$ (with the Pauli matrices, $\hat{\sigma}^a$ acting on flavor space).

Further we denote the hyper-charge diagonal matrices, acting in flavor space, as $\hat{Y}_{\mathrm{L}\ell} = -\mathbb{1}_2/2$, $\hat{Y}_{\mathrm{R}\ell} = \operatorname{diag}(0, -1)$, $\hat{Y}_\mathrm{L,q} = \mathbb{1}_2/6$, and $\hat{Y}_\mathrm{R,q} = \operatorname{diag}(2/3, -1/3)$. Then the gauge-covariant piece for the fermion fields (including couplings with the gauge fields) reads

$$\mathscr{L} = \overline{\psi}_\ell(\mathrm{i}\not{\partial} + \mathrm{i}g\not{W}^a\hat{T}^a P_L + \mathrm{i}g'\not{B}(\hat{Y}_{\mathrm{L}\ell}P_L + \hat{Y}_{\mathrm{R}\ell}P_R)\psi_\ell + \overline{\psi}_\mathrm{q}(\mathrm{i}\not{\partial} + \mathrm{i}g\not{W}^a\hat{T}^a P_L + \mathrm{i}g'\not{B}(\hat{Y}_\mathrm{L,q}P_L + \hat{Y}_\mathrm{R,q}P_R)\psi_\mathrm{q}. \tag{18}$$

It is also understood that each quark comes in a color triplet.

In QFD the fermions get their mass via couplings to the Higgs field, which is a wiso-doublet $\Phi = (\Phi_1, \Phi_2)^\mathrm{T}$ with hyper-charge matrix $\hat{Y}_\mathrm{H} = \mathbb{1}/2$.

References

[YM54] C.-N. Yang and R. L. Mills, Conservation of Isotopic Spin and Isotopic Gauge Invariance, Phys. Rev. **96**, 191 (1954), `https://doi.org/10.1103/PhysRev.96.191`.