Einführung in die Theoretische
Kern- und Elementarteilchenphysik 2

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Exercise Sheet 11

$\pi^+ \rightarrow \mu^+ + \nu_\mu$

We combine our knowledge about the strong interactions of pions (here charged pions) and PCAC as well as the description of the weak interactions. For these low-energy (here the relevant "electroweak energy scale" is $M_{\mathrm{W}} = (80.2625 \pm 0.0077)\,\mathrm{GeV}$ [**?**]) the Fermi-theory like current-current coupling approximation is fully sufficient. The relevant piece of the Lagrangian reads (on the quark level)

$$\mathscr{L} = \frac{G_{\mathrm{F}}}{\sqrt{2}} \left(\overline{u}[\gamma^\lambda(1-\gamma_5)]d'\right)\left(\overline{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu\right) + \text{h.c.} \tag{1}$$

Note that here we write the weak charged quark current in terms of the flavor eigenstates of the down-like quarks (d', s', b') with the CKM-quark-mixing matrix (CKM=Cabibbo, Kobayashi, and Maskawa)

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \hat{V} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \quad \hat{V}^\dagger \hat{V} = \mathbb{1}_3. \tag{2}$$

For the decay of the charged pion, $\pi^+ \rightarrow \mu + \nu_\mu$ we need the invariant matrix element

$$S_{fi} = \left\langle \mu^+(p_1), \nu_\mu(p_2) \middle| \mathrm{S} \middle| \pi^+(p) \right\rangle = -\mathrm{i}(2\pi)^4 \delta^{(4)}(P_f - P_i)\mathscr{M}_{fi}. \tag{3}$$

The relation of the charged-pion fields to the real pion fields in the σ-model discussed in Lect. 9 is $\pi^\pm = (\pi^1 \mp \mathrm{i} p^2)/\sqrt{2}$ the corresponding axialvector isovector currents are

$$J_{\mathrm{A}}^{\pm\mu} = J_{\mathrm{A}}^{1\mu} \mp \mathrm{i} J_{\mathrm{A}}^{2\mu}. \tag{4}$$

The PCAC relation for the real pion fields defines the decay constant $F_\pi \simeq 92\,\mathrm{MeV}$ via

$$\left\langle \Omega \middle| J_{\mathrm{A}}^{a\lambda}(x=0) \middle| \pi^b(p) \right\rangle = \mathrm{i}\delta_{ab} p^\lambda F_\pi. \tag{5}$$

(a) Show that the the relation for the *positively charged* pion is

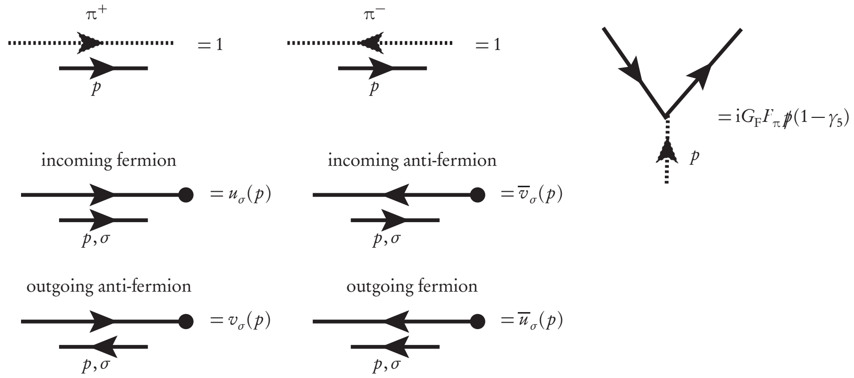
$$\left\langle \Omega \middle| J_{\mathrm{A}}^{+\mu}(x=0) \middle| \pi^+(p) \right\rangle = \mathrm{i}\delta_{ab} p_\mu \sqrt{2} F_\pi. \tag{6}$$

(b) Now use that for the weak current of the pion only the axial piece with $\overline{u}$- and d-fields with the γ_5-matrix in this current contributes (why?) and that this leads to the effective Lagrangian

$$\mathscr{L}_{\mathrm{eff}} = G_{\mathrm{F}} F_\pi (\partial_\lambda \pi^-) V_{ud} \left(\overline{\mu}\gamma^\lambda(1-\gamma_5)\nu_\mu\right) + \text{h.c.} \tag{7}$$

where μ and ν_μ are the Dirac spinor describing muons and muon neutrinos, respectively.

(c) use the resulting Feynman rules,



where the fermions can be $\mu^\pm$ or ν_μ and $\overline{\nu}_\mu$, to evaluate the invariant matrix element $\mathcal{M}_{fi}$ for the charged-pion decay $\pi^+ \to \mu^+ + \nu_\mu$.

Hint: Use that the momenta of the particles are on-shell and $p = p_1 + p_2$ (where p_1 is the four-momentum of the muon and p_2 the four-momentum of the neutrino, which is assumed to be massless.

(d) Evaluate the unpolarized matrix element, i.e., $\overline{|\mathcal{M}_{fi}|^2}$ which is summed over the spins of the outgoing muon and neutrino. Then you can use

$$\not{p}_1 v_{\mu,\sigma}(p_1) = -m_\mu v_{\mu,\sigma}, \quad \overline{u}_{\nu_\mu,\sigma}(p_2)\not{p}_2 = 0. \tag{8}$$

Useful "Diracology": Spin sums

$$\sum_\sigma u_\sigma(p)\overline{u}_\sigma(p) = \not{p} - m, \quad \sum_\sigma v_\sigma(p)\overline{v}_\sigma(p) = \not{p} + m, \tag{9}$$

where m is the mass of the particle/antiparticle described by u/v. Also the following traces over Dirac-matrix expressions are needed:

$$\mathrm{tr}(\not{p}_1\not{p}_2) = 4p_1 \cdot p_2, \quad \mathrm{tr}\,\gamma^\mu = \mathrm{tr}\,\gamma_5 = 0 = \mathrm{tr}(\gamma_5\gamma^\mu) = \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0. \tag{10}$$

(e) Evaluate the unpolarized matrix element, i.e., $\overline{|\mathcal{M}_{fi}|^2}$ which is summed over the spins of the outgoing muon and neutrino. Then you can use

$$\not{p}_1 v_{\mu,\sigma}(p_1) = -m_\mu v_{\mu,\sigma}, \quad \overline{u}_{\nu_\mu,\sigma}(p_2)\not{p}_2 = 0. \tag{11}$$

Useful "Diracology": Spin sums

$$\sum_\sigma u_\sigma(p)\overline{u}_\sigma(p) = \not{p} - m, \quad \sum_\sigma v_\sigma(p)\overline{v}_\sigma(p) = \not{p} + m, \tag{12}$$

where m is the mass of the particle/antiparticle described by u/v. Also the following traces over Dirac-matrix expressions are needed:

$$\mathrm{tr}(\not{p}_1\not{p}_2) = 4p_1 \cdot p_2, \quad \mathrm{tr}\,\gamma^\mu = \mathrm{tr}\,\gamma_5 = 0 = \mathrm{tr}(\gamma_5\gamma^\mu) = \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0. \tag{13}$$

(f) Finally use the formula for the differential decay width (see Sect. 49.4.2 of [?])

$$\mathrm{d}\Gamma = \frac{1}{64\pi^2}\frac{m_\pi^2 - m_\mu^2}{m_\pi^3}\overline{|\mathcal{M}_{fi}|^2}\mathrm{d}\Omega \tag{14}$$

to derive the total decay width and the lifetime of the charged pion (the branching ratio to this decay mode is very close to 100%). For comparison: the empirical value for the lifetime is $\tau_{\pi^+} = 2.6033 \cdot 10^{-8}$ s. The small difference to the above derived theoretical result is due to higher-order corrections.

(g) Calculate the branching ratio $\Gamma_{\pi^+ \to e^+ + \nu_e}/\Gamma_{\pi^+ \to e^+ + \nu_e}$.

Values of the constants: $G_F = 1.16637908 \cdot 10^{-5}\mathrm{GeV}^{-2}$, $F_\pi = 92.2$ MeV, $|V_{ud}| = 0.9740$, $m_{\pi^+} = 139.5704$ MeV, $m_\mu = 105.6584$ MeV, $m_e = 0.511$ MeV.