Einführung in die Theoretische Kern- und Elementarteilchenphysik 2

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Exercise Sheet 10

Linear σ model

We start from the QCD Lagrangian with two quark flavors $\psi = (u, d)^{\mathrm{T}}$,

$$\mathscr{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \overline{\psi}(\mathrm{i}\not{\partial} - m)\psi, \tag{1}$$

where we have approximated the mass term by setting $m_{\mathrm{u}} = m_{\mathrm{d}} = m$, where $m = (m_{\mathrm{u}} + m_{\mathrm{d}})/2 \simeq 3.43\,\mathrm{MeV}$[1].

(a) First we determine the transformation properties of the fields, representing the lightest scalar isoscalar and pseudoscalar isovector particles, i.e., the σ-meson and the three pions, by identifying the corresponding (real scalar or pseudoscalar) fields as

$$\Phi_0 = \overline{\psi}\psi, \quad \pi^a = \mathrm{i}\overline{\psi}\gamma_5 \tau^a \psi, \tag{2}$$

where the $\tau^a \in \mathbb{C}^{2\times 2}$ are the Pauli matrices acting in flavor space.

Determine, how these fields transform under infinitesimal vector and axial isovector transformations[2], given that the quark fields transform as

$$\mathrm{V}: \quad \delta\psi = -\frac{\mathrm{i}}{2}\delta\alpha^a \tau^a \psi, \quad \delta\overline{\psi} = +\frac{\mathrm{i}}{2}\delta\alpha^a \overline{\psi}\tau^a, \tag{3}$$

$$\mathrm{A}: \quad \delta\psi = -\frac{\mathrm{i}}{2}\delta\beta^a \gamma_5 \tau^a \psi, \quad \delta\overline{\psi} = -\frac{\mathrm{i}}{2}\delta\beta^a \overline{\psi}\gamma_5 \tau^a. \tag{4}$$

Show that the resulting transformations lead to the o(4) symmetry[3]

$$\delta(\Phi_0^2 + \vec{\pi}^2) = 0. \tag{5}$$

Some helpful formulae

$$\begin{gathered} \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad \gamma_5^2 = \mathbb{1}, \\ \left[\tau^a, \tau^b\right] = \mathrm{i}\epsilon^{abc}\tau^c, \quad \left\{\tau^a, \tau^b\right\} = 2\delta^{ab}. \end{gathered} \tag{6}$$

In our simplified version of two-flavor QCD by setting the u- and d-quark masses equal, of course (1) is invariant under the vector isovector transformations $\mathrm{SU}(2)_{\mathrm{V}}$.

(b) Next we consider the mesonic part of the linear σ model [GML60], including the chiral-symmetry breaking term. We introduce the four-dimensional SO(4) vector of real scalar fields, $\tilde{\Phi} = (\Phi_0, \vec{\pi})$. Then we write down the Lagrangian (for a renormalizable) field theory, which is approximately symmetric under the SO(4) representation of the chiral symmetry found above:

$$\mathscr{L}_{\mathrm{M}} = \frac{1}{2}(\partial_\mu \tilde{\Phi}) \cdot (\partial^\mu \tilde{\Phi} - V(\tilde{\Phi}) \tag{7}$$

with the potential,

$$V = \frac{\lambda}{4}\left(\tilde{\Phi}^2 - v_0^2\right)^2 - \epsilon\Phi_0, \tag{8}$$

where we have used the scalar-product notation $\tilde{\Phi} \cdot \tilde{\Phi} = \tilde{\Phi}^2 = \Phi_0^2 + \vec{\pi}^2$.

Show that the ground state is uniquely given by $(F_\pi, 0, 0, 0)^{\mathrm{T}}$ and determine v_0^2 in terms of F_π and ϵ[4].

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[1] According to [N+24] the light-quark masses are $m_{\mathrm{u}} = (2.16 \pm 0.07)$ MeV, $m_{\mathrm{d}} = (4.70 \pm 0.07)$ MeV, and $m_{\mathrm{s}} = (93.5 \pm 0.8)$ MeV (but here we consider only the u- and d-quarks).

[2] together spanning the $\mathrm{su}(2)_{\mathrm{L}} \oplus \mathrm{sU}(2)_{\mathrm{R}}$ "=Lie algebra of the isovector part of the chiral symmetry, which is only slightly broken by the quark-mass term in (1).

[3] Of course this o(4) Lie algebra is a representation, acting on the meson fields, of the original $\mathrm{su}(2)_{\mathrm{L}} \oplus \mathrm{su}(2)_{\mathrm{R}}$ Lie algebra, acting on the quark fields.

[4] That the vacuum expectation value of the Φ_0 field should be the pion-decay constant $F_\pi \simeq 93$ MeV is dictated by phenomenology, e.g., the $\pi^0 \to \gamma + \gamma$ decay rate, discussed in the previous exercise.

(c) Set now

$$\tilde{\Phi}(\underline{x}) = \begin{pmatrix} F_\pi + \sigma(\underline{x}) \\ \vec{\pi}(\underline{x}) \end{pmatrix} \tag{9}$$

and write the Lagrangian (7) in terms of the fields σ and $\vec{\pi}$. Show in that way that the model parameters are related to the σ and pion masses by

$$m_\sigma^2 = \frac{1}{2}\lambda F_\pi^2 + \frac{\epsilon}{4F_\pi}, \quad m_\pi^2 = \frac{\epsilon}{f\pi}. \tag{10}$$

Determine the value of the "quark condensate", $-\left\langle \Omega \middle| \overline{\psi}\psi \middle| \Omega \right\rangle$ from the Gell Mann-Oakes-Renner relation (from the "PCAC analysis" given in the lecture)[5]

$$f_\pi^2 m_\pi^2 = -m \left\langle \Omega \middle| \overline{\psi}\psi \middle| \Omega \right\rangle. \tag{11}$$

(d) Now consider the nucleon part of the Lagrangian of the Gell-Mann Levy model with only the nucleons, p and n, (with parity +1), using the hadronic isodoublet $\Psi = (p, n)^{\mathrm{T}}$ with the Dirac spinors p and n being the proton and nucleon fields,

$$\mathscr{L}_{\text{nucl}} = \overline{\Psi} \mathrm{i} \not{\partial} \Psi - g_{\pi NN} \left[\Phi_0 \overline{\Psi}\Psi + \mathrm{i}\vec{\pi} \cdot \overline{\Psi}\gamma_5 \vec{\tau}\psi \right]. \tag{12}$$

Why is it chirally symmetric and how get the nucleons their mass? How do the vector and axialvector isovector currents look like? What is the nucleon mass in this minimal linear-σ model?

Hint: To determine the currents, it is most convenient to vary the action with infinitesimal symmetry transformations for *local* transformations, i.e., using space-time-dependent infinitesimal group parameters, $\delta\vec{\alpha}(\underline{x})$ and $\delta\vec{\beta}(\underline{x})$, and noting that under these transformations only $\mathscr{L}_{\text{kin}}$ and $\mathscr{L}_{\text{SB}}$ with

$$\mathscr{L}_{\text{kin}} = \frac{1}{2}(\partial_\mu \tilde{\Phi}) \cdot (\partial_\mu \tilde{\Phi}) + \mathrm{i}\overline{\Psi} \not{\partial} \Psi, \quad \mathscr{L}_{\text{SB}} = \epsilon \Phi_0 \tag{13}$$

are *not* invariant under the transformations and that then the variation of $\mathscr{L}$ under the vector and axialvector isovector transformation reads

$$\delta_{\mathrm{V}} \mathscr{L} = (\partial_\mu \delta\vec{\alpha}) \cdot \vec{j}_{\mathrm{V}}^{\mu}, \tag{14}$$

$$\delta_{\mathrm{A}} \mathscr{L} = (\partial_\mu \delta\vec{\beta}) \cdot \vec{j}_{\mathrm{A}}^{\mu} + \delta \mathscr{L}_{\text{SB}}. \tag{15}$$

Note: The result shows that there is a discrepancy with the empirically better prediction of the Goldberger-Treiman relation from the PCAC argument and QCD given in the lecture,

$$g_{\pi NN} = \frac{g_{\mathrm{A}} m_N}{F_\pi}, \quad m_N = \frac{m_{\mathrm{p}} + m_{\mathrm{n}}}{2} \simeq 939\,\text{MeV}, \quad g_{\mathrm{A}} \simeq 1.27. \tag{16}$$

For a possible way to cure this with (non-renormalizable) derivative meson-nucleon couplings see [DM00]. A very recent paper on this topic within a chiral doubler (mirror-assignment) model, where the discrepancy between (16) without derivative couplings is even worse than in the here discussed Gell Mann-Levy model is [KLS26].

References

[DM00] V. Dmitrasinovic and F. Myhrer, Pion nucleon scattering and the nucleon sigma term in an extended linear sigma model, Phys. Rev. C **61**, 025205 (2000), `https://doi.org/10.1103/PhysRevC.61.025205`.

[GML60] M. Gell-Mann and M. Levy, The axial vector current in beta decay, Nuovo Cim. **16**, 705 (1960), `https://dx.doi.org/10.1007/BF02859738`.

[KLS26] C. Kummer, S. Leupold and L. von Smekal, Kinetic mixing and axial charges in the parity-doublet model, Phys. Rev. D **113**, 116010 (2026), `https://doi.org/10.1103/qc6c-wg6h`.

[N+24] S. Navas et al. (Particle Data Group), Review of particle physics, Phys. Rev. D **110**, 030001 (2024), `https://doi.org/10.1103/PhysRevD.110.030001`.

[5]The (average) pion mass $m_\pi = (m_{\pi^\pm} + m_{\pi^0})/2 \simeq 137\,\text{MeV} \simeq 140\,\text{MeV}$.