Einführung in die Theoretische Kern- und Elementarteilchenphysik 2

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Exercise Sheet 9

1. Chiral anomalies in the standard model

QCD-like non-Abelian gauge theory

We consider a generalization of the chiral anomaly (Adler-Bell-Jackiw anomaly) in QED to more general cases. We start with a general non-Abelian gauge theory suited to analyse the possible anomalies of the (approximate) chiral symmetries of QCD. We consider a slightly more general case and consider "quarks" with N_f flavors. For the QCD in the standard model we have 6 flavors, ordered in 3 families, each family containing an up- and a downlike quark, (u,d), (c,s), and (t,b) (called up, down, charm, strange, top, and bottom), of which only u, d, s have small enough masses such that the mass terms can be treated as a perturbation of the massless case, giving rise to the said approximate chiral symmetries. In this case there is no danger that the gauged symmetry group, $\mathrm{SU}(3)_{\mathrm{col}}$, itself is anomalously broken, because the QCD Lagrangian does not contain γ_5 and thus the symmetry transformations are all unitary, and the path-integral measure of the quark fields is invariant under the transformation.

We consider only the relevant part of the Lagrangian, containing the kinetic terms as well as the coupling of the quarks to the gauge field (with the corresponding particles being the gluons). We add the possibility that the gluons couple to a linear combination of the flavors. Writing $\psi = (u,d,c,s,t,b)^{\mathrm{T}}$ with $u,\ldots,b$ Dirac spinor fields. In addition each quark field carries a color index, i.e., $u \equiv u_c$ with $c \in \{r,g,b\}$. The generators of the gauge group, which describes the local color symmetry are $\hat{T}_c^a = \hat{\lambda}^a/2$ with $a \in \{1,2,\ldots 8\}$ and $\hat{\lambda}^a$ being the Gell-Mann matrices. Further $\hat{T}_f$ is a matrix acting in flavor space, mixing the different quark flavors. In Nature, of course, $\hat{T}_f = \mathbb{1}$ since the strong interaction is "flavor blind". Our Lagrangian thus reads

$$\mathscr{L} = \overline{\psi}(\mathrm{i}\not{D} - \hat{M}\psi, \tag{1}$$

where $\hat{M} = \mathrm{diag}(m_u, m_d, m_c, m_s, m_t, m_b)$ is the mass matrix of the quarks, and the gauge-covariant derivative reads

$$\mathrm{D}_\mu = \partial_\mu + \mathrm{i} g_s \hat{T}_c^a \hat{T}_f G_\mu^a \tag{2}$$

with G_μ^a ($a \in \{1,2,\ldots,8\}$) being the gluon fields.

We consider global chiral symmetries of the kind

$$\psi' = \psi - \mathrm{i}\delta\vec{\alpha}\hat{\vec{t}}\gamma_5\psi, \tag{3}$$

where the $\hat{\vec{t}}$ act in flavor space (in QCD in the light quark sector it's either an su(2)-Lie algebra acting on the (u,d)-isospin doublet or an su(3)-Lie algebra acting on (u,d,s) (Gell-Mann's "eightfold way"), or just $\mathbb{1}$ for the axial iso-scalar current. The corresponding currents, which are conserved in the classical theory in the massless limit $\hat{M} = 0$ are

$$\vec{j}_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\vec{t}\psi. \tag{4}$$

Then the same analysis as the one shown in the lecture for the case of the $\mathrm{U}(1)_{\mathrm{em}}$ gauge theory, QED, can be used also in our non-Abelian case. The space-time and Dirac trace from the Fujikawa regulator is the same as in the QED calculation. With the various matrices in color and flavor space we simply get another trace over these matrices, which factor in the product over the traces over color and flavor matrix products, respectively. The anomaly finally reads

$$\partial_\mu \left\langle \Omega \middle| \vec{j}_5^\mu(\underline{x}) \middle| \Omega \right\rangle_{\mathrm{an}} = -\frac{g_s^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \, \mathrm{tr}_{\mathrm{col}} \, \hat{T}_c^a \hat{T}_c^b \, \mathrm{tr}_{\mathrm{flav}} \left(\hat{\vec{t}} \hat{T}_f^2 \right) G_{\mu\nu}^a G_{\rho\sigma}^b \tag{5}$$

with the gluon field-strength tensor

$$G_{\mu\nu}^a = (\partial_\mu G_\nu^a - \partial_\nu A_\mu^c - g_s f^{abc} A_\mu^b A_\nu^c, \tag{6}$$

where the f^{abc} are the structure constants of the gauge group SU(3),

$$\left[\hat{T}_c^a, \hat{T}_c^b \right] = \mathrm{i} f^{abc} \hat{T}_c^c. \tag{7}$$

(a) Consider the approximate chiral symmetries of the usual QCD with $\hat{T}_f = \mathbb{1}$, i.e., $\mathrm{SU}(N_f)_\mathrm{L} \times \mathrm{SU}(N_f)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ with $N_f \in \{1,2\}$, which are exact global symmetries in the case that the quark flavors under consideration are massless. The associated currents are the "vector-isovector/isoscalar" and "axial-vector-isovector/isoscalar" ones:

$$\vec{j}_\mathrm{V}^\mu = \overline{\psi}\vec{t}\gamma^\mu\psi, \quad \vec{j}_\mathrm{A}^\mu = \overline{\psi}\vec{t}\gamma^\mu\gamma_5\psi, \tag{8}$$

$$j_\mathrm{V}^\mu = \overline{\psi}\gamma^\mu\psi, \quad j_\mathrm{A}^\mu = \overline{\psi}\vec{t}\gamma^\mu\gamma_5\psi. \tag{9}$$

The mass term breaks the conservation of the axial-vector currents already at the classical level[1]. Calculate $\partial_\mu \vec{j}_\mathrm{A}^\mu$ and $\partial_\mu j_\mathrm{A}^\mu$ taking into account the mass term by considering the local infinitesimal transformations (as shown in the lecture),

$$\delta\psi' = -\mathrm{i}\delta\vec{\alpha}(\underline{x})\cdot\hat{\vec{t}}\gamma_5\psi \Rightarrow \delta\overline{\psi}' = -\mathrm{i}\delta\vec{\alpha}(\underline{x})\cdot\hat{\vec{t}}\overline{\psi}\gamma_5, \tag{10}$$

$$\delta\psi' = -\mathrm{i}\delta\alpha(\underline{x})\gamma_5\psi \Rightarrow \delta\overline{\psi}' = -\mathrm{i}\alpha(\underline{x})\cdot\overline{\psi}\hat{\vec{t}}\gamma_5, \tag{11}$$

where we have used that for the Lie algebras $\mathrm{su}(N_\mathrm{f})$ the generators are hermitian, $\hat{\vec{t}}^\dagger = \hat{\vec{t}}$.

(b) Calculate the anomaly terms (5) for both axial currents.

(c) Argue how many pseudo-Goldstone bosons *really* occur for $N_\mathrm{f} \in \{1,2\}$ when the chiral symmetry is spontaneously broken to the symmetry group of the vaccuum state, $\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times \mathrm{U}(1)_\mathrm{V}$ (in addition to the explicit breaking by the quark-mass terms).

Note: This is only an exact symmetry (isospin for the two-flavor and isospin+hypercharge for the three-flavor model) if the quark masses are equal. In nature also this symmetry is violated on the same level as chiral symmetry itself, which is broken by introducing quark masses in the way as in the Standard Model.

ABJ anomaly in QED of quarks

(d) We consider QED of the two lightest quark flavors (u,d), i.e., we set now now $\psi = (u,d)^\mathrm{T}$ with u and d come with $N_c = 3$ colors each. The relevant part of the QED Lagrangian reads

$$\mathscr{L} = \overline{\psi}(\mathrm{i}\not{D} - \hat{M})\psi, \quad \mathrm{D}_\mu = \partial_\mu + \mathrm{i}e\hat{Q}A_\mu, \tag{12}$$

where $\hat{Q} = \mathrm{diag}(2/3, -1/3)$ is the charge matrix of the quarks (acting in flavor space). Now consider the two-flavor axial-vector isovector current,

$$\vec{j}_5^\mu = \overline{\psi}\gamma^\mu\gamma^5\hat{\vec{t}}\psi, \quad \hat{\vec{t}} = \frac{1}{2}\hat{\vec{\sigma}} \tag{13}$$

with the Pauli matrices $\hat{\vec{\sigma}}$ acting also in flavor space. Use the general "anomaly formula" (5) to calculate the anomaly of this current (don't forget the "hidden" color degrees of freedom; for QED the color matrix is of course $\mathbb{1}$, i.e., there is an additional factor $\mathrm{tr}_\mathrm{col}\,\mathbb{1} = N_\mathrm{c} = 3$).

(e) (Challenging extra question) Within the within the PCAC formalism (see next lecture) the relation of the pion fields $\vec{\pi}$ (isospin-1 vector) to the axial isovector current are given by the "PCAC-relation"

$$\vec{j}_5^\mu = f_\pi \partial^\mu \vec{\pi}. \tag{14}$$

Note that the charge states of the pions are given by $\pi^\pm = (\pi_1 \pm \mathrm{i}\pi_2)/\sqrt{2}$ and $\pi^0 = \pi_3$. How does the corresponding anomaly term in the effective QED Lagrangian for pions look like and which of the pions can decay to two photons?

Note: The above general anomaly formula can also be applied to the electroweak sector of the standard model (Glashow-Salam-Weinberg model, aka Quantum Flavor Dynamics). Since here the model is itself based on a gauged chiral symmetry, it is of utmost importance for its consistency that the conservation of the corresponding currents (which are of the

[1] Also the vector currents are only conserved if $\hat{M} \propto \mathbb{1}$, i.e., if all flavors have the same mass. In QCD isospin or $\mathrm{SU}(3)_\mathrm{flavor}$ vector currents are also not conserved at the same level as the chiral symmetry is broken since the u, d, (s)-quark masses are of the same order as their mass difference.

"vector-minus-axial-vector type"), to which the gauge bosons (in the Higgsed model the $W^{\pm}$, Z, and electromagnetic gauge fields) couple, are *not* anomalously broken, which would imply that the gauge symmetry itself were broken and thus the theory obsolete. It turns out that the flavor content with the specific charges of the quarks and leptons and the additional color-degrees of freedom (with $N_c = 3$!) conspires in such a way that the relevant traces over the flavor (and the factor N_C from the $\mathbb{1}_{col}$-matrix in the electroweak currents of the quarks) cancel out the "dangerous" anomalies. For a thorough discussion of all that (and also of effective hadronic models etc.) see [DGH22][2].

References

[DGH22] J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model: Second edition*, Cambridge University Press, Cambridge, New York, Melbourne (2022), https://doi.org/10.1017/9781009291033.

[PS95] M. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley Publ. Comp., Reading, Massachusetts (1995).

[2]Note that this book has a somewhat different convention concerning the Dirac matrices, particularly γ_5 has the opposite sign than the one used in this lecture, which follows the convention in [PS95], while the 4D Levi-Civita tensor follows the same sign convention, $\epsilon^{0123} = -\epsilon_{0123} = +1$.