Einführung in die Theoretische Kern- und Elementarteilchenphysik 2

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Exercise Sheet 8

(1) Spontaneously broken U(1) symmetry

We start with the Lagrangian of a charged scalar field, $\Phi \in \mathbb{C}$

$$\mathscr{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) + \mu^2 \Phi^* \Phi - \frac{g^2}{2} (\Phi^* \Phi) = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - V(\Phi^* \Phi). \tag{1}$$

Note that the mass term has the “wrong sign”, which means that we have spontaneous symmetry breaking, i.e., the ground-state solution is $\Phi = v = \text{const}$ with $v \neq 0$. The theory has obviously the usual global U(1) symmetry under the transformations

$$\Phi'(\underline{x}) = \exp(-\mathrm{i}\alpha)\Phi(\underline{x}), \quad \Phi'^*(\underline{x}) = \exp(+\mathrm{i}\alpha)\Phi^*(\underline{x}), \quad \alpha = \text{const} \in \mathbb{R}. \tag{2}$$

(a) Derive the equations of motion for Φ by applying the Euler-Lagrange equations to Φ^*[1]. Derive v, which we choose by convention to be real with $v > 0$.

(b) Define

$$\Phi(\underline{x}) = v + \frac{\chi(\underline{x})}{\sqrt{2}}, \quad \chi(\underline{x}) = \chi_1(\underline{x}) + \mathrm{i}\chi_2(\underline{x}), \quad \chi_1, \chi_2 \in \mathbb{R}. \tag{3}$$

and write the Lagrangian in terms of the real fields χ_1 and χ_2. Interpret the result if you think of the model as the classical version of a bosonic QFT, which can be canonically quantized in the usual way[2].

(c) As an alternative way to write the complex field in terms of two real fields now use

$$\Phi(\underline{x}) = \left(v + \frac{R(\underline{x})}{\sqrt{2}}\right) \exp[\mathrm{i}\varphi(\underline{x})], \quad \Phi^*(\underline{x}) = \left(v + \frac{R(\underline{x})}{\sqrt{2}}\right) \exp[-\mathrm{i}\varphi(\underline{x})], \quad R, \varphi \in \mathbb{R} \tag{4}$$

and express the Lagrangian in terms of the fields.

(d) Explain the occurence of a massless Nambu-Goldstone boson from the shape of the potential V, when interpreted in terms of the two real field-degrees of freedom in terms of the “non-linear formulation” (4).

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(2) Higgsed U(1) gauge symmetry

Start again from the above Lagrangian (1), but now “gauge the U(1) symmetry” making it a local symmetry by introducing a gauge field, $A^\mu(\underline{x}) \in \mathbb{R}$ and application of the “principle of minimal substitution”, i.e., define the gauge-covariant derivative

$$\mathrm{D}_\mu = \partial_\mu + \mathrm{i}qA_\mu, \quad q \in \mathbb{R}, \tag{5}$$

and substitute ∂_μ by D_μ in the Lagrangian, i.e.,

$$\mathscr{L} = (\mathrm{D}_\mu \Phi)^* (\mathrm{D}_\mu \Phi) - V(\Phi^* \Phi). \tag{6}$$

(a) Show that under the *local* gauge transformation

$$\Phi'(\underline{x}) = \exp[-\mathrm{i}q\alpha(\underline{x})]\Phi(\underline{x}), \quad \Phi'^*(\underline{x}) = \exp[+\mathrm{i}q\alpha(\underline{x})], \quad A'_\mu(\underline{x}) = A_\mu(\underline{x}) + \partial_\mu \alpha(\underline{x}) \tag{7}$$

the gauge-covariant derivatives of the fields transform as

$$\mathrm{D}'_\mu \Phi' = \exp(-\mathrm{i}q\alpha)\mathrm{D}_\mu \Phi, \quad (\mathrm{D}'_\mu \Phi')^* = \exp(+\mathrm{i}q\alpha)(\mathrm{D}_\mu \Phi)^* \tag{8}$$

and that thus the gauged Lagrangian (6) is invariant under the *local* gauge transformation (7).

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[1] Then the equations ones for Φ^* are just the conjugate complex equations.

[2] You do not need to work out the quantized formulation.

(b) Show that this is also true for the Lagrangian extended by a "kinetic term" for the gauge field,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\Phi)^*(\mathrm{D}_\mu\Phi) - V(\Phi^*\Phi), \tag{9}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10}$$

is the usual field-strength tensor.

(c) As in Problem 1, the stable ground state is given by the solution $\Phi = \Phi^* = v = \text{const} \in \mathbb{R}$. Now write Φ in terms of (4) and express the Lagrangian (9) in terms of the gauge-transformed fields,

$$\tilde{\Phi}(\underline{x}) = \exp\left[-\mathrm{i}q\frac{\varphi(\underline{x})}{q}\right]\Phi(\underline{x}), \quad \tilde{\Phi}^*(\underline{x}) = \exp\left[+\mathrm{i}q\frac{\varphi(\underline{x})}{q}\right]\Phi^*(\underline{x}), \quad \tilde{A}_\mu(\underline{x}) = A_\mu + \frac{1}{q}\partial_\mu\varphi(\underline{x}). \tag{11}$$

Hint: Thanks to the gauge invariance of $\mathscr{L}$ you do not need to perform the somewhat cumbersome algebra explicitly. It's sufficient to make use of the advantage, that the Lagrangian looks the same for the fields $\tilde{\Phi}$, $\tilde{\Phi}^*$, $\tilde{A}_\mu$, and $\tilde{D}_\mu$ as the Lagrangian written with the fields and gauge-covariant derivatives without the tilde!

(d) What's the "particle content" of the theory now? Are there as many (physical) field-degrees of freedom as in the original formulation?

(e) **Riddle:** Why is there no Nambu-Goldstone boson in this model?

Hint: Is there still a degeneracy of the ground state as in the case of the only global U(1) symmetry of Problem 1?

Note: There is a close relation to the theory of superconductivity and the above applied "Higgs mechanism". In fact, the "Higgs mechanism" has been discovered by P. W. Anderson in 1962 in this context of superconductivity (of course in terms of the non-relativistic Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity in metals at low temperatures [BCS57b, BCS57a], where the "condensing Bose fields" are "Cooper-paired electrons") [And63]. The corresponding condensed-matter application of a spontaneously broken global symmetry are superfluids which can either consist of bosons (like ^{4}He), where these bosons form a condensate at low temperatures or fermions (like ^{3}He), where again the Cooper pairs are the "condensing" boson-like degrees of freedom.

For more about the interesting relation between superconductivity and the Higgs mechanism, see [Wei86, Wei96, Ran12].

References

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